Manifolds with Boundary

In nearly any program to describe manifolds as built up from relatively simple building blocks, it is necessary to look more generally at manifolds with boundaries. Perhaps the simplest example of something that should be a manifold with boundary is the standard unit n – disk D^n, whose boundary will then be the unit n – sphere S^{n-1}.

Preliminary definitions. Suppose that A is a subset of $\mathbb{R}^k$ for some k and f is a mapping from A to $\mathbb{R}^m$ for some m. We shall say that f is a smooth mapping if it extends to a smooth map on some open neighborhood of A. A partitions of unity argument shows that this is equivalent to a local extension hypothesis: For each a in A there is an open neighborhood U of a in $\mathbb{R}^k$ such that $f | U \cap A$ extends to a smooth function on U. — It is straightforward to check that this definition of smooth function does not depend upon the choice of k (in particular, if we view A as contained in $\mathbb{R}^{k+1}$ via the usual inclusion of $\mathbb{R}^k$ in $\mathbb{R}^{k+1}$, then if the statement is true for A contained in one of these Euclidean spaces, it will be true for A contained in the other. More generally, if A and B are subsets of $\mathbb{R}^k$ for some sufficiently large k, and if we say that a map from A to B is smooth if the composite map from A to $\mathbb{R}^k$ is smooth, then it is straightforward to check that the collection of all smooth maps is a subcategory of the category of continuous maps on the spaces under consideration.

A ***topological n – manifold M with boundary*** is defined to be a Hausdorff space (usually also second countable) such that every point in M has an open neighborhood which is homeomorphic to an open subset in $\mathbb{R}^n{}_+$, the set of all points whose last coordinate is nonnegative. This includes the usual defininition of manifold (without boundary), for if a point has a neighborhood homeomorphic to an open subset in $\mathbb{R}^n$, then it also has an open neighborhood homeomorphic to an open subset in the pont set theoretic $\textbf{INTERIOR}\left(\mathbb{R}^n{}_+ \text{ in } \mathbb{R}^n\right)$ of all points in $\mathbb{R}^n$ whose last coordinates are positive (verify this!).

More generally, given a topological n – manifold M with boundary, we define the ***interior*** of M, written $\mathrm{INT}(M)$, to be the set of all points which have open neighborhoods which are homeomorphic to open subsets of $\textbf{INTERIOR}\left(\mathbb{R}^n{}_+ \text{ in } \mathbb{R}^n\right)$. This is a topological n – manifold without boundary, and it is an open subset

(***Sketch of proof:*** If U is an open neighborhood of a point x in the interior and U is homeomorphic to an open subset V of $\mathbb{R}^n{}_+$, let F be the portion of U corresponding to the intersection of V and $\mathbb{R}^n{}_+$. Then one checks that $U - F$ is also an open neighborhood of x, and it lies completely in the interior.).

Before proceeding, we need the following observation.

Claim. If M is a topological manifold with boundary and x is a point of M, then exactly one of the two statements below is true:

1. The point x has an open neighborhood which is homeomorphic to an open subset in $\mathbb{R}^n$.
2. The point x has an open neighborhood base such that no set in the base is homeomorphic to an open subset in $\mathbb{R}^n$.

The complement of the interior is called the ***boundary***; it is a closed subset of M and is denoted by ∂M. Note that M is an n – manifold in the usual sense if and only if the boundary is empty.

The proof of the claim is a consequence of Brouwer's Invariance of Domain. Suppose we are given a point x such that x has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n{}_+$ and under such a homeomorphism x corresponds to a point $\mathbf{v}$ in $\mathbb{R}^n$ whose last coordinate is zero. Then the second statement clearly holds (any open neighborhood of $\mathbf{v}$ must contain points whose last coordinates are negative). We need to check that the first and second statements cannot be true simultaneously. Suppose they are, and let U be an open neighborhood of x which is homeomorphic to an open subset of $\mathbb{R}^n$, and let V be a smaller open neighborhood given by the second assertion, so that V is not homeomorphic to an open subset of $\mathbb{R}^n$. But V is open in U, and hence if U is homeomorphic to an open subset U' in $\mathbb{R}^n$ then it follows that the image V of V' is also open in $\mathbb{R}^n$ by Invariance of Domain. This is impossible since V is not open in $\mathbb{R}^n$, and therefore the first statement is false if the second is true.

Proposition. If M is a topological n – manifold with boundary, then ∂M is a topological $(n - 1)$ – manifold without boundary.

This is true because if x is in the boundary and U is an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n{}_+$, then the intersection of U and the boundary is homeomorphic to an open subset of $\mathbb{R}^{n-1}$.

Example. Clearly $\mathbb{R}^n{}_+$ is a topological n – manifold with boundary, and its boundary is $\mathbb{R}^{n-1}$. More generally, if V is a topological $(n-1)$ – manifold without boundary, then $V \times \mathbb{R}^n{}_+$ is a topological manifold with boundary, and the latter is given by $V \times \mathbb{R}^{n-1}$. Note that if $n = 1$ then the boundary is $V \times \{0\}$, which is homeomorphic to V itself. To see that D^n is a manifold with boundary, consider the radial homeomorphism from $S^{n-1} \times \mathbb{R}^n{}_+$ to $D^n \setminus \{0\}$ sending (x, t) to $e^{-t}x$. Since the center point $0 \in D^n$ has an open neighborhood in the latter which is homeomorphic to an open subset of $\mathbb{R}^n$ (the open unit disk!) and the complement of $\{0\}$ is an open set which we know is a topological n – manifold with boundary by the preceding sentence, it follows that D^n itself is also a topological n – manifold with boundary.

Exercises. **1.** Generalize the preceding argument as follows: If $M = A \cup B$ where A is a topological n – manifold without boundary which is open in M, and B is a topological n – manifold with boundary which is also open in M, then M is a topological n – manifold with boundary.

2. For every connected Hausdorff topological space X, there is at most one nonnegative integer n such that X is a topological n – manifold. As usual, we call this the ***dimension*** of M.

3. If M is a compact n – manifold with boundary, prove that ∂M is also compact. Is the converse true? Prove this or give a counterexample for which the boundary is nonempty.

SMOOTH STRUCTURES. Clearly we would like the unit disk to be an example of a smooth manifold with boundary. The first step is to define the proper analog of a smooth atlas. The coordinate charts for the smooth atlas will be continuous mappings $h_\alpha : U_\alpha \to M$, where U_α is open in $\mathbb{R}^n{}_+$ as before, and we want the transition maps to be smooth in an appropriate sense. Specifically, the transition maps are given by

$$\psi_{\beta,\alpha} : h_\alpha^{-1}(h_\beta(U_\beta)) \longrightarrow h_\beta^{-1}(h_\alpha(U_\alpha))$$

where $\psi_{\beta,\alpha}(x) = h_\beta^{-1}\big(h_\alpha(x)\big)$, and we want the composites of these maps to be smooth in the sense described at the beginning of this document; in other words, the composite of with the inclusion of the codomain in $\mathbb{R}^n$ should be smooth. We then have the following statement, which has a straightforward but somewhat messy proof.

<u>Claim. ("*Exercise*")</u> The maps $\psi_{\beta,\alpha}$ can be extended to diffeomorphisms from a neighborhood of the domain to a neighborhood of the codomain.

[Note that the maps in question are homeomorphisms because, as in the unbounded case, one knows that $\psi_{\beta,\alpha}$ and $\psi_{\beta,\alpha}$ are inverse to each other. Furthermore, they are clearly diffeomorphisms on the intersections of their domains and codomains with the complement of $\mathbb{R}^n{}_+$, and we know that $\psi_{\beta,\alpha}$ (followed by inclusion) can be extended to an open neighborhood of its domain.]

We can now lay down the foundations for a theory of smooth functions on smooth manifolds with boundary exactly as in the unbounded case, and we can also define a tangent space $\mathbf{T}(M)$ associated to a smooth manifold with boundary; over each point of M, including the boundary, one has an n – dimensional space of tangent vectors. At a point x on the boundary ∂M, the vector space $\mathbf{T}_x(M)$ contains a naturally embedded copy of the $(n-1)$ – dimensional vector space $\mathbf{T}_x(\partial M)$ with a copy of the real numbers. The set – theoretic difference $\mathbf{T}_x(M) \setminus \mathbf{T}_x(\partial M)$ splits into a disjoint union of two open half – spaces, and because of the next result one of these half – spaces is called the set of ***inward pointing tangent vectors*** at x.

<u>Proposition.</u> Let x be a point in ∂M. Then there is an open half – space H in the difference $\mathbf{T}_x(M) \setminus \mathbf{T}_x(\partial M)$ with the following property: For each $\mathbf{v}$ in H there is a smooth curve $\gamma: [0, h) \to M$ such that $\gamma(0) = x$, $\gamma'(0) = \mathbf{v}$, and $\gamma(\mathrm{t})$ lies in the interior of M for $t > 0$.

It is straightforward to check this locally (in which case the distinguished half – space corresponds to all vectors whose last coordinates are positive), and then one must check that the derivatives transition maps for a manifold with boundary preserve the half – spaces of tangent vectors with positive last coordinates.

<u>COLLAR NEIGHBORHOODS.</u> We already noted that if V is a smooth manifold without boundary, then $V \times [0, 1)$ is a manifold with boundary equal to $V \times \{0\}$. A basic result called the ***Collar Neighborhood Theorem*** states that if M is a smooth manifold with boundary, then ∂M has an open neighborhood of this type.

Collar Neighborhood Theorem. Let M be a smooth manifold with boundary. Then the boundary ∂M has an open neighborhood which is diffeomorphic to the product $\partial M \times [0, 1)$.

There is an analog of this result for topological manifolds which is basically due to M. H. Brown. Here are some references for the compact case:

M. H. Brown, *Locally flat embeddings of topological manifolds***.** Topology of 3 – manifolds and related topics (Proceedings of the University of Georgia Institute, 1961), pp. 83 – 91. Prentice – Hall, Englewood Cliffs, N.J., 1962.

R. Connelly, *A new proof of Brown's collaring theorem.* Proceedings of the American Mathematical Society **27** (1971), 180 – 182.

Proof of the Collar Neighborhood Theorem. One major step is the construction of an inward pointing vector field over ∂M. Before beginning, we note that the tangent space $\mathbf{T}(M)$ is a smooth manifold with boundary, and the boundary consists of all tangent vectors to points of ∂M (verify this!). The vector field will then be a smooth map $\mathbf{X}$ from ∂M to $\partial\mathbf{T}(M)$ which sends each point to an tangent vector over that point. Locally this is fairly straightforward to do, and we can construct a global vector field from the local ones using a smooth partition of unity. Furthermore, we can construct an extension of this vector field to all of M. This vector field has a local 1 – parameter group of integral curves (the ***integral flow***), and the integral curves with initial conditions on the boundary define a smooth map from an open neighborhood W of $\partial M \times \{0\}$ in $\partial M \times \mathbb{R}_+$ such that $\partial M \times \{0\}$ corresponds to ∂M and points of $W \setminus \partial M \times \{0\}$ are mapped into the interior of M. Furthermore, by construction the derivative map over each point of $\partial M \times \{0\}$ is an isomorphism, so that the integral flow is locally a diffeomorphism at boundary points. One can then argue as in the proof of the Tubular Neighborhood Theorem that there is a neighborhood W_0 of $\partial M \times \{0\}$ in W such that the restriction of the integral flow to W_0 is injective and hence must be a diffeomorphism. Finally, also as in the proof of the Tubular Neighborhood Theorem one can trim the neighborhood down further to obtain a subneighborhood which is diffeomorphic to $\partial M \times [0, 1)$.

The Collar Neighborhood Theorem is an extremely useful result for analyzing manifolds with boundary. For example, it leads to the following fundamental embedding theorem:

Theorem. Let M be a smooth n – manifold with boundary. Then there is a smooth embedding of M into some higher – dimensional Euclidean half – space $\mathbb{R}^q{}_+$ ***as a closed subset*** such that ∂M is smoothly embedded in $\mathbb{R}^q$ and the interior of M is mapped into the interior of $\mathbb{R}^q{}_+$.

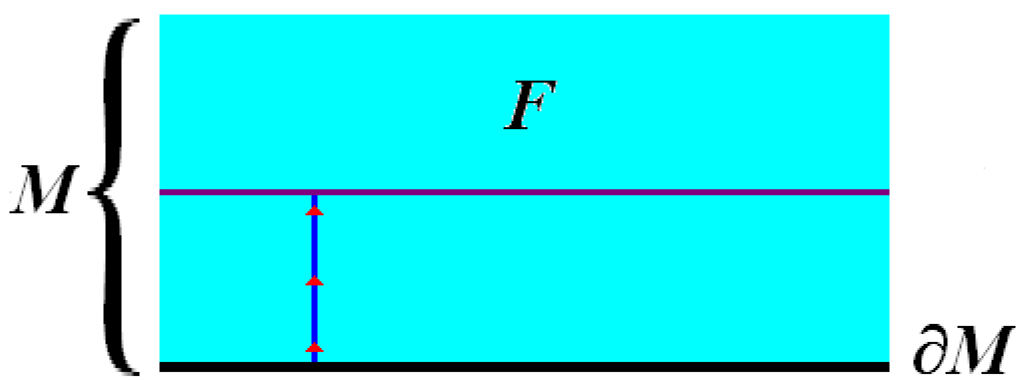
The idea is simple: One first constructs an embedding of the boundary using the methods which apply in the unbounded case, then one extends it to an open neighborhood using a collar neighborhood, and finally one extends it over the rest of the interior again using the methods from the unbounded case.

Here is another noteworthy consequence of the Collar Neighborhood Theorem.

Theorem. If M is a smooth n – manifold with boundary, then the inclusion of the interior $\mathrm{INT}(M)$ in M is a homotopy equivalence.

Proof. By standard formal properties of homotopy equivalences, it suffices to find a closed subset F of $\mathrm{INT}(M)$ and M such that F is a deformation retract of both $\mathrm{INT}(M)$ and M.

To construct F, let h be a diffeomorphissm mapping $\partial M \times [0, 1)$ onto an open neighborhood V of ∂M in M, and let F be the complement of the image of the open subset $\partial M \times [0, ½)$. Define a mapping ρ from M to F by $\rho\big(h(x, t)\big) = h(x, ½)$ if $t \leq ½$, and set ρ equal to the identity on F. Since the intersection of these sets is equal to the image of $\partial M \times \{½\}$ under h and the two definitions agree on this subset, it follows that ρ yields a well – defined continuous retraction from M to F, and likewise it defines a continuous retraction on the interior of M. To visualize this construction, one can take M to be $\partial M \times [0, 1)$ and h to be the identity; in this case F will be equal to $\partial M \times [½, 1)$.



If we let j denote the inclusion of F in either M or its interior, then there is an obvious vertical homotopy K_s from $j\rho$ to the identity of M or $\mathrm{INT}(M)$ defined by $K_s(h(x,t)) = h(x, \tfrac{1}{2}s + (1-s)t)$ if $t \leq \tfrac{1}{2}$, and K_s is the identity on F; note that if t is positive, then so is $\tfrac{1}{2}s + (1-s)t$ and therefore K_s maps the interior of M into itself (in the drawing, the retraction collapses the blue segment onto its endpoint on the purple line, and the homotopy is indicated by the red arrows). Thus in both cases the inclusion of F is a strong deformation retract, which is what we needed to prove.

FINAL REMARK. If M and N are topological manifolds with boundary, then their product $M \times N$ is also a manifold with boundary, and the latter is equal to $\partial M \times N \cup M \times \partial N$. However, if both manifolds with boundary have smooth structures and both boundaries are nonempty, then there is no "natural" smooth structure on the product; instead, one obtains objects known as ***manifolds with corners***. For example, if we take M and N to be the closed interval $[0, 1]$, then the corner structure at points of $\partial M \times \partial N$ is evident when we think of the intervals as embedded in the real line and their product as embedded in the Cartesian plane. However, it **IS** possible to make the topological product into a smooth manifold with boundary by one of several equivalent ***ad hoc*** processes called ***rounding corners*** or ***straightening angles***; for example, if we take the product of two copies of a closed interval, then the idea is to approximate the boundary curve by a smooth curve which is the same as the square's boundary off small neighborhoods of the four corner points. We shall try to avoid getting into such constructions explicitly, but there are many situations in which they are very important. A standard reference for this topic is the Appendix to the following classic paper:

A. Borel and J. – P. Serre, *Corners and arithmetic groups* (with an appendix, *Arrondissement des variétés à coins*, by A. Douady and L. Hérault), *Comment. Math. Helv.* **48** (1973), 436 – 491.