Connected sums and Euler characteristic invariants of G-manifolds

The purpose of this note is to construct some examples which are mentiond in another paper. The methods and conclusions are well known to workers in the area, but examples of this type do not seem to be easily accessible in the literature. The standard background reference is the book, *Introduction to Compact Transformation Groups*, by G. Bredon. Some additional references for equivariant connected sums are given at the end of the document.

Example 1. *Let q be a positive integer. Then there is a compact smooth manifold M and a smooth semifree action of S^1 on M with the following properties:*

(i) *The Euler characteristic of M is zero.*

(ii) *If F is the fixed point set of the action, then F has two components F_1 andF_2 such that the Euler characteristic of F_1 is q and the Euler characteristic of F_2 is $-q$.*

Results in Bredon's book imply that the Euler characteristic of F is equal to zero, so the examples show that one cannot extend such a conclusion to the components of the fixed point set. One can obtain examples of finite group actions from Example 1 by restricting to the nontrivial finite (hence cyclic) subgroups of S^1.

Our constructions are based upon taking suitable equivariant connected sums. For Example 1, we do this along fixed point sets. The idea is very similar to the usual connected sum: Suppose that we have N_1 and N_2 with fixed points p_1 and p_2 such that the representations of the group G on the tangent spaces at p_1 and p_2 are equivalent. Then there are closed disk neighborhoods of p_1 and p_2 which are equivalent to the unit disk in some orthogonal G-representation V; to construct the connected sum, we cut out the interiors of these disks and join their boundary spheres by a tube. As in the case of ordinary connected sums, it is necessary to be careful about choices of orientations; the references at the end of this document explain how this can be done (and why it must be done).

One feature of connected sums which is important for our purposes is the following Euler characteristic formula for closed even-dimensional manifolds (the odd-dimensional case is not interesting because all Euler characteristics vanish):

$$\chi\left(M^{2n} \# N^{2n}\right) \;=\; \chi\left(M^{2n}\right) \;+\; \chi\left(N^{2n}\right) \;-\; 2$$

We shall use the following generalization of the preceding formula repeatedly:

$$\chi\left(\#_{i=1}^{k}\; M_i^{2n}\right) \;=\; \left(\sum_{i=1}^{k} \chi\left(M_i^{2n}\right)\right) \;-\; 2(k-1)$$

Construction of examples. Let X be a closed oriiented smooth $2m$-manifold ($m \geq 0$) and take $M_0 = X \times S(\mathbb{C}^n \times \mathbb{R})$, where $n \geq 2$ and $S(\cdots)$ denotes the unit sphere for the standard linear action of S^1 on $\mathbb{C}^n \times \mathbb{R}$ by scalar multiplication on the complex coordinates and the trivial action on the real coordinates. Now form the connected sum of M_0 with q copies of

$$P \;=\; S(\mathbb{C} \times \mathbb{R}^{2m-1}) \times S(\mathbb{C}^{n-1} \times \mathbb{R}^3)$$

equivariantly along one component of the fixed point set of M_0; as before, the action of S^1 is by scalar multiplication on the complex coordinates and the trivial action on the real coordinates. By constructionthe fixed point set of the action on P is $S^{m-2} \times S^2$. Then the Euler characteristic of M_0 is zero and the fixed point set has two components, one of which is X and the other of which is a connected sum of X with X with q copies of $S^{m-2} \times S^2$. The Euler characteristics of these components are $+q$ and $-q$ respectively.■

Example 2. *Let p be a prime. Then there is a compact smooth $\mathbb{Z}_p$ manifold M^{2n} such that $\chi(M^{2n}) = 0$ but the Euler characteristic of the fixed point set is nonzero.*

It is well known that the Euler characteristic of M and the fixed point set are congruent mod p (this follows from Chapter 3 of Bredon's book), and in fact the Euler characteristic of the fixed point set in our example is equal to $2p$.

Construction of examples. Let p be a prime, and let $N = S(\mathbb{C}^n \times \mathbb{R}^{2n+1})$ where $\mathbb{Z}_p$ acts on $\mathbb{C}^n$ by scalar multiplication on the complex coordinates and acts trivally on the real coordinates. Take an equivariant smooth embedding

$$D(\mathbb{R}^{2m} \times \mathbb{C}^n) \ \amalg \ \mathbb{Z}_p \times D^{2(m+n)} \longrightarrow N$$

(where $\amalg$ denotes disjoint union), and denote the image of this embedding by E. Form the $\mathbb{Z}_p$-manifold M by gluing two copies of $N - \mathrm{Int}\,(E)$ equivariantly along the boundary (this can also be described as $\partial\,(N - \mathrm{Int}\,(E) \times [0,1])$ with the corners rounded). If M_0 is this manifold, then the fixed set of M_0 is S^{2n}. Furthrmore, since $\chi\,(N - \mathrm{Int}\,(E)\,) = 1$ it follows that $\chi(M_0) = 2 - 2p$. Finally, take M to be an equivariant connected sum of M_0 with $p-1$ copies of the manifold P in Example 1. Then $\chi(M) = 0$ but the Euler characteristic of the fixed point set is $2p$.■

REFERENCES

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