Smoothable submanifolds of a smooth manifold

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The purpose of this note is to describe a result from geometric topology which is well-known to workers in the area but difficult to locate in the literature. Our discussion will be somewhat informal, the goal being mainly to explain how the result can be extracted from the literature.

Question. *Suppose we have a (second countable) smooth manifold N^n and a topological submanifold $M^m \subset N^n$, where $m < n$. Can one describe necessary and sufficient conditions under which M^m can be approximated (within its homeomorphism type) by a smooth submanifold of N^n?*

The main result on this question is essentially answered by the work of R. Kirby and L. Siebenmann on triangulations and smoothings of topological manifolds [15]; a corresponding result for *piecewise smooth* submanifolds, with no dimensional restrictions, is contained in earlier work of R. Lashof and M. Rothenberg [23], and the proof of the result for topological submanifolds is similar to the argument in [23]. A discussion of cases not covered by the main result appears in Section 5.

To expedite the discussion, we shall make two simply stated assumptions that are not particularly restrictive.

- **1:** We shall assume that *the submanifold M is a closed subset of N*. The motivation for this hypothesis is discussed in the third paragraph of Section 1.
- **2:** We shall assume that M is *connected*. In general, M may have up to $\aleph_0$ components, but since N is normal and M is closed in N, one can use the Tietze Extension Theorem for continuous real valued mappings to construct a continuous real valued function on N which maps each component of M to a distinct positive integer. One can then take the inverse images of pairwise disjoint neighborhoods for the individual integers to obtain have pairwise disjoint open neighborhoods for the separate components, and then one can restrict attention to each of these neighborhoods individually.

For similar reasons, we shall also assume that *the boundaries of M and N are empty*; some remarks about the bounded case appear in Section 4.

1. The main result

If N^n is a smooth manifold and M^m is a smooth submanifold (both without boundary), then by the (smooth) **Tubular Neighborhood Theorem** (see [2], [10] or [21]), then M^m has a *tubular neighborhood* given by a vector bundle (*i.e.*, a **vector bundle neighborhood**). Specifically, there is an open neighborhood U of M^m in N^n and an $(n-m)$-dimensional vector bundle ξ over M with total space $E(\xi)$ such that the pair (U, M) is homeomorphic — in fact, *diffeomorphic* — to the pair $\big(E(\xi)$, zero section$\big)$; the total space $E(\xi)$ has a canonical smooth structure which is determined up to a suitable notion of equivalence by the vector bundle ξ and the smooth structure on M.

The existence of a topological vector bundle neighborhood implies that the embedding of M in N is **locally flat** (see [34], p. 33). Since it is possible to construct uncountable families of inequivalent manifold embeddings that are not locally flat for almost all choices of m and n

(compare [34], Chapter 2), it follows immediately that many topological submanifolds cannot be smoothable.

Local flatness implies that M is a *locally closed* subset of N; *i.e.*, for every point $x \in M$ there is an open neighborhood U of $x \in N$ such that $U \cap M$ is a closed subset of U. A standard exercise in point set topology states that a subset of a space is locally closed if and only if it is the intersection of an open subset and a closed subset, and hence if M is locally flat in N then there is an open subset $N_0 \subset N$ such that $M \subset N_0$ and M is closed in N_0. Thus if we replace N by the open subset N_0, then we can adjust things so that M is a closed subset of the ambient manifold. Therefore every smoothly embedded submanifold S of a smooth manifold X is closed in some open subset $X_0 \subset X$, so there is no real loss of generality in assuming that the submanifold is a closed subset.

Although local flatness is a necessary condition for smoothability, the condition by itself is usually not sufficient to imply that a submanifold is smoothable. For example, if M^m is a compact manifold (without boundary) which does not admit a smooth structure (*e.g.*, the 10-dimensional manifold constructed in [16]), then one can construct a locally flat embedding of M^m in $\mathbb{R}^n$ for n sufficiently large (see [26] for a strong global version of this result), but the results of [26] show that M^m cannot have a vector bundle neighborhood.

The main result on the question about smoothable submanifolds is essentially a converse to the smooth Tubular Neighborhood Theorem:

Theorem 1. *Let $n, m \geq 5$, let N^n be a smooth n-manifold, and let $M^m \subset N^n$ be a topological m-manifold that is embedded in N^n. Then there is a smooth structure on M^m such that the inclusion of M in N is isotopic to a smooth embedding if and only if M has a topological vector bundle neighborhood.*

We have already noted that the proof of this result is formally parallel to the earlier result of Lashof and Rothenberg on smoothing piecewise smooth submanifolds [23]; the main difference is that the latter depends crucially on the Cairns-Hirsch smoothability theorem for piecewise linear manifolds [11], and in the proof of Theorem 1 we shall substitute the Product Structure Theorem of Kirby and Siebenmann ([15], Essay I, Section 5) for the Cairns-Hirsch Theorem.

Although local flatness does not in general imply the existence of a vector bundle neighborhood for a topological submanifold, there are reasonable analogs of tubular neighborhoods for locally flat submanifolds. For $n - m \geq 3$ the basic results are described in [33] and [12] (see also [9]), and in the remaining cases with $n - m \leq 2$ vector bundle neighborhoods always exist (*e.g.*, see [14]). If we combine the latter with Theorem 1, we obtain following conclusion:

Theorem 2. *Let $k = 1$ or 2, let $m \geq 5$, let N^{m+k} be a smooth $(n + k)$-manifold, and let M^m be a locally flat m-dimensional submanifold of N^{m+k}. Then there is a smooth structure on M^m such that the inclusion of M in N is isotopic to a smooth embedding.*

In Section 5 we shall give examples to show that Theorem 2 does not extend to the cases where $m = 3$ or 4, and it also does not extend to the case where $m = 2$ and $n = 4$; in contrast, the analog of Theorem 2 is valid when $n \leq 3$, and this will also be discussed in Section 5.

2. Smoothing vector bundles

Although there are many treatments of vector bundles in textbooks and other publications, such accounts usually emphasize one of two basic categories — the smooth and topological categories — with very little (if anything) said about the relationship between smooth and topological vector bundles over a smooth manifold. Since the proof of Theorem 1 requires an explicit understanding of this relationship, we shall review the necessary facts here. The key ideas appear in Steenrod's classic book ([35], Section 6.7, pp. 25–28), so we shall limit ourselves to stating the crucial points in the form that is useful for our purposes.

In order to simplify the discussion we shall limit our attention to real vector bundles, but one can also treat complex vector bundles similarly by substituting $\mathbb{C}$ for $\mathbb{R}$ (and "unitary" for "orthogonal") throughout; everything also goes through for quaternionic vector bundles, but at some points one must phrase things more carefully in order to compensate for the noncommutativity of the quaternions.

COMPARING CATEGORIES OF VECTOR BUNDLES. The classical examples of vector bundles in differential geometry are tangent bundles and various sorts of tensor bundles over a smooth manifold, and these are smooth vector bundles, at least if the manifold is smooth of class $\mathcal{C}^2$ (*e.g.*, see [21]). On the other hand, for many purposes in topology it is more convenient to consider *continuous* and *topological* vector bundles as in M. F. Atiyah's classic set of lecture notes [1]. It is straightforward to check that every smooth vector bundle has an underlying topological vector bundle, just as smooth manifolds have underlying topological manifolds. For our purposes the following converse relationship is fundamentally important:

Theorem 3. *Let M be a smooth manifold, and let q be a positive integer.*
(i) If ξ is a continuous q-dimensional vector bundle over M, then there is a smooth q-dimensional vector bundle ξ' and a vector bundle isomorphism $\varphi : \xi' \to \xi$; in other words, if $E(\xi')$ and $E(\xi)$ are the total spaces and π and π' are the projections, then there is a homeomorphism $E(\varphi) : E(\xi') \to E(\xi)$ such that $\pi\varphi = \pi'$, and for each $x \in M$ the map φ defines a vector bundle isomorphism from the vector space $\xi'_x = \pi'^{-1}[\{x\}]$ to $\xi_x = \pi^{-1}[\{x\}]$.
(ii) If ξ and ξ' are smooth q-dimensional vector bundles over M and $\varphi : \xi' \to \xi$ is a continuous vector bundle isomorphism, then φ is isotopic to a smooth vector bundle isomorphism; in other words, there is a homotopy $\Phi : E(\xi') \times [0, 1] \to E(\xi)$ such that $\Phi|E \times \{0\}$ is given by φ, for each $t \in [0, 1]$ the map $\Phi|E \times \{t\}$ is a vector bundle isomorphism, and $\Phi|E \times \{1\}$ is a diffeomorphism.

VECTOR BUNDLES AND PRINCIPAL BUNDLES. Recall that in both the smooth and topological categories, there is a 1–1 correspondence between isomorphism classes of q-dimensional (real) vector bundles over a given base B and principal $\mathbf{GL}(q, \mathbb{R})$-bundles over B. Given a vector bundle $\pi : E \to B$, the corresponding $\mathbf{GL}(q, \mathbb{R})$-bundle is called the **bundle of q-frames**, and it consists of all ordered q-tuples $(\mathbf{v}_1, \cdots, \mathbf{v}_q)$ where $\mathbf{v}_j \in E$ for all j such that the following hold:

1: $\pi(\mathbf{v}_1) = \cdots = \pi(\mathbf{v}_q)$ (call this common point b).
2: In the vector space $E_b = \pi^{-1}[\{b\}]$, the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_q$ are linearly independent and hence form a basis for E_b.

Conversely, if we start with a principal $\mathbf{GL}(q, \mathbb{R})$-bundle, then the corresponding vector bundle is merely the associated fiber bundle with fiber $\mathbb{R}^q$, where $\mathbf{GL}(q, \mathbb{R})$ acts on $\mathbb{R}^q$ in the usual way by invertible linear transformations. Further details on this correspondence can be found in Section 4.4 of [37] (in particular, see Proposition 4.11 on page 31).

By the preceding discussion, the proof of Theorem 3 reduces to proving a similar result for smooth and topological principal $\mathbf{GL}(n,\mathbb{R})$-bundles over a smooth manifold. We shall analyze this relationship between the two types of bundles using standard results from bundle theory. The classical formulation of bundle theory in the 1949–1950 Séminaire Henri Cartan ([5], Exposés 5–8bis) is particularly useful for our purposes.

The next step in the process is to observe that isomorphism classes of principal $\mathbf{GL}(n,\mathbb{R})$-bundles correspond bijectively to isomorphism classes of principal O_q-bundles, where as usual $O_q \subset \mathbf{GL}(q,\mathbb{R})$ is the orthogonal group. Furthermore, if we are given two principal O_q-bundles and an isomorphism of their extensions to principal $\mathbf{GL}(q,\mathbb{R})$-bundles, then this isomorphism can be deformed to second isomorphism which is an extension of a principal O_q-bundle isomorphism.

By the discussion thus far, Theorem 3 will be true if we can prove the following more general result:

Theorem 4. *Let M be a smooth manifold, and let G be a compact Lie group.*
(*i*) *If ξ is a topological principal G-bundle over M, then there is a smooth principal G-bundle bundle ξ' and a principal bundle isomorphism $\varphi : \xi' \to \xi$; in other words, if $E(\xi')$ and $E(\xi)$ are the total spaces and π and π' are the projections, then there is a homeomorphism $E(\varphi) : E(\xi') \to E(\xi)$ such that $\pi\varphi = \pi'$, and for each $x \in M$ the map φ defines a G-equivariant bijection from the fiber $\xi'_x = \pi'^{-1}[\{x\}]$ to $\xi_x = \pi^{-1}[\{x\}]$.*
(*ii*) *If ξ and ξ' are smooth principal G-bundles bundles over M and $\varphi : \xi' \to \xi$ is a continuous principal bundle isomorphism, then φ is isotopic to a smooth principal bundle isomorphism; in other words, there is a homotopy $\Phi : E(\xi') \times [0,1] \to E(\xi)$ such that $\Phi|E \times \{0\}$ is given by φ, for each $t \in [0,1]$ the map $\Phi|E \times \{t\}$ is a principal bundle isomorphism, and $\Phi|E \times \{1\}$ is a diffeomorphism.*

Sketch of proof. Given a compact Lie group G and a positive integer n, one has the usual sort of n-universal principal G-bundle $p_n : E_n \to B_n$ such that if M^m is a manifold whose dimension m is sufficiently small with respect to m, then principal G-bundles over M are classified up to isomorphism by homotopy classes of maps from M into B_n. A closer examination of the construction in [5] yields two stronger conclusions. First, an isomorphism of principal G-bundles over M determines a homotopy of classifying maps. Second, an isotopy of principal G-bundle isomorphisms determines a homotopy of homotopies of classifying maps.

The assertions in the previous paragraph are true for both the topological and smooth categories. In order to check this for the smooth category, it is necessary to find an n-universal principal G-bundle which is smooth. However, thes can be done fairly directly as in Steenrod's book [35]: For each compact Lie group G there is a smooth injective homomorphism $\rho : G \to O_p$ for some $p > 0$, and one can then take the n-universal bundle to be the smooth principal bundle

$$G \longrightarrow O_{n+p+1}/O_{n+1} \longrightarrow O_{n+p+1}/O_{n+1} \times \rho(G)$$

where the first arrow represents the composite

$$G \to O_p \to O_{n+1} \times O_p \to O_{n+1+p} \to O_{n+p+1}/O_{n+1} \ .$$

To prove the theorem, take a topological principal bundle ξ over M, and choose a continuous mapping f from M to the base space of a smooth n-universal bundle B_n, where $n > m$ such that ξ is isomorphic to the pullback of the universal bundle γ via f. Standard results on smooth approximations to continuous functions (*e.g.*, see the first half of [28]) show that f is homotopic to a smooth map f_1. By construction the pullback bundle $f_1^*\gamma$ is a smooth bundle, and as noted before the homotopy from f to f_1 defines a topological principal bundle isomorphism from

$\xi \cong f^*\gamma$ to $f_1^*\gamma$. This proves the first part. To prove the second part, note that in this case the topological isomorphism of smooth principal bundles from $\xi_1 \cong f_1^*\gamma$ and and $\xi_2 \cong f_2^*\gamma$ (where f_i is smooth) will determine a continuous homotopy $H : M \times [0,1] \to B_n$ from the smooth map f_1 to the smooth map f_2. Constructing the desired isotopy amounts to constructing a relative homotopy $K : M \times [0,1] \times [0,1]$ from H to a smooth homotopy H' such that the homotopy is fixed on $M \times \{0,1\} \times [0,1]$; such a relative homotopy will define a relative isotopy from the original continuous isomorphism of principal bundles to a smooth isomorphism. In this case, the existence of a suitable relative homotopy follows from a relative version of the results on approximating continuous mappings by smooth ones.■

Theorems 3 and 4 have the following important consequence that we shall need in the next section:

Theorem 5. *Let M be a smooth manifold, let q be a positive integer, and let ξ be a continuous q-dimensional vector bundle over M. Then there is a canonical smooth structure on the total space $E(\xi)$ such that the embedding of the zero section is a smooth embedding.*■

3. Proof of the main results

We have noted that Theorem 2 follows from Theorem 1 and the result of Kirby-Siebenmann, so we shall concentrate on the proof of Theorem 1. Furthermore, since we know the "only if" implication is true, it will suffice to restrict our attention to the "if" implication.

Suppose now that we have $n > m \geq 5$, and we are given a smooth manifold N^n with an embedded topological submanifold M^m such that M has a topological vector bundle neighborhood. Let ξ be the vector bundle such that M has an open neighborhood U in N which is homeomorphic to $E(\xi)$ with M corresponding to the zero section. The smooth structure on N determines a smooth structure on U and hence on $E(\xi)$. Let $\pi : E(\xi) \to M$ be the projection map.

The next step in the proof is to show that M is smoothable, and the argument closely resembles the corresponding part of [23]. We know that for some positive integer p there is a p-dimensional inverse vector bundle ω^p over M such that the Whitney sum $\xi \oplus \omega$ is a trivial vector bundle and hence $E(\xi \oplus \omega) \cong M \times \mathbb{R}^{q+p}$. For every vector bundle β over M there is a standard vector bundle identity

$$E(\pi^*\beta) \;\cong\; E(\xi \oplus \beta)$$

and if we apply this to ω we see that $E(\pi^*\omega)$ is homeomorphic to $M \times \mathbb{R}^{q+p}$.

Since we have a smooth structure on $E(\xi)$, the results of the preceding section imply that

$$E(\pi^*\beta) \;\cong\; E(\xi \oplus \beta) \;\cong\; M \times \mathbb{R}^{q+p}$$

also has a smooth structure. Since $m \geq 5$, we can apply the Product Structure Theorem in [15] (see Essay I, Theorem 5.1, p. 31) to conclude that M is smoothable.

Choose an arbitrary smooth manifold M_0 which is homeomorphic to M. The results of Section 2 yield a smooth manifold V_0 which is homeomorphic to $E(\xi)$, and we shall need some insight into the relationship between the smooth structure carried by V_0 and the original smooth structure which arises from the homeomorphism $E(\xi) \cong U$ and the smooth structure which U inherits from N.

By the results of [15], if W is an arbitrary smooth manifold of dimension ≥ 5, then the isotopy classes of smoothings of W are in 1–1 correspondences with homotopy classes of continuous mappings from W to a space known as TOP/O; the main thing we shall need about this space is that it has the homotopy type of a topological group, a fact which follows from geometric considerations. Furthermore, this bijection is compatible with taking total spaces of vector bundles. Specifically, if α is a vector bundle over W and $\pi_\alpha : E(\alpha) \to W$ is the projection, then the map of homotopy groups

$$\pi_\alpha^* : \ [W, \mathrm{TOP/O}] \ \longrightarrow \big[\, E(\alpha),\ \mathrm{TOP/O} \,\big]$$

sends the class of a smooth structure M_1 to the class of the associated smooth structure on the total space which is given by the results of Section 2. Since π_α is a homotopy equivalence, the map of homotopy groups is bijective.

If we apply the preceding discussion to the comparison between U and V_0, we see that the smooth structure which U inherits from N is isotopic to a smooth structure on $E(\xi)$ associated to some smooth structure M_1 on M. But this means that U is diffeomorphic to smooth tubular neighborhood for a smooth embedding of M_1. In fact, a closer examination of the situation shows that the original topological embedding of M is topologically ambient isotopic to the smooth embedding of M_1 in $U \subset N$, which is the result we wanted to prove.■

REMARK. The smooth structure on the submanifold M is not necessarily unique, and in general it depends upon the choice of a homeomorphism from a neighborhood of M to the total space of some vector bundle. For example, if $M = S^m$ and n is sufficiently large with respect to m, then different choices of the homeomorphism yield the usual smooth structure on S^n and also all possible diffeomorphism classes of exotic smooth structures (*cf.* [17]). In fact, similar examples exist for all m and n such that $m \geq 5$ and $n - m \geq 4$. For example, if $M^m = S^3 \times T^{m-3}$ and we take the standard neighborhood of M^m in $\mathbb{R}^n$ given by

$$S^3 \times T^{m-3} \times \{\mathbf{0}\} \ \subset \ S^3 \times T^{m-3} \times \mathbb{R}^{n-m} \ \subset \ S^3 \times \mathbb{R}^{n-3} \ \subset \ \mathbb{R}^n$$

then by the results of [15] one can find a homeomorphism of $S^3 \times T^{m-3} \times \mathbb{R}^{n-m}$ to itself which is the identity on $S^3 \times T^{m-3} \times \{\mathbf{0}\}$, such that the resulting smoothing of M_α is not diffeomorphic to $S^3 \times T^{m-3}$ (although M_α is homeomorphic to $S^3 \times T^{m-3}$).

4. Generalization to bounded manifolds

Similar methods yield analogs to the previous results for bounded manifolds. We shall merely summarize the results and indicate what one needs in order to prove the generalizations. Although we continue to assume that our manifolds are connected, we shall **not** assume that their boundaries are connected.

DEFAULT ASSUMPTIONS. In this section we shall only consider **properly embedded** submanifolds with boundary; in other words, we have an inclusion of pairs $(M, \partial M) \subset (N, \partial N)$ such that $M \cap \partial N = \partial M$. We shall also assume that $M - \partial M$ is locally flat in $N - \partial N$ and that for points on ∂M we have the following elaboration of local flatness which always holds for proper smooth embeddings. In this context $\mathbb{R}^k_+ \subset \mathbb{R}^k$ denotes the set of all points whose first coordinates are nonnegative.

> Each point in ∂M has an open neighborhood U in N such that $(U, U \cap M)$ is homeomorphic/diffeomorphic to $(\mathbb{R}^n_+, \mathbb{R}^m_+ \times \{\mathbf{0}\})$ with $(U \cap \partial N, U \cap \partial M)$ corresponding to $(\{0\} \times \mathbb{R}^{n-1}, \{0\} \times \mathbb{R}^{m-1} \times \{\mathbf{0}\})$.

The appropriate notion of vector bundle neighborhood for proper submanifolds with boundary requires an extra condition called *neatness with respect to the boundary* (compare [10]), and it involves collar neighborhoods for the boundaries; the latter exist by the Collar Neighborhood Theorems for smooth [21] and topological [3] manifolds. Neatness means that $(M\partial M)$

and $(N, \partial N)$ have *compatible* smooth/topological collar neighborhoods given by homeomorphisms/diffeomorphisms $c_M : \partial M \times [0,1) \to M$ and $c_N : \partial N \times [0,1) \to N$ such that $c_M(x,t) = c_N(x,t)$ for all $(x,t) \in \partial M \times [0,1)$. One can refine the proofs of the Collar Neighborhood Theorems to prove neatness under our default assumptions at the boundary (for the smooth case this is done explicitly in [10]). A neighborhood W of M in N is said to be *neat with respect to a pair of compatible collar neighborhoods* if there is an open set $W_0 \subset \partial N$ containing ∂M such that

$$W \ \cap c_N\big(\partial N \times [0,1)\big) \ = \ c_N\big(W_0 \times [0,1)\big) \ .$$

A *vector bundle neighborhood pair* for the properly embedded submanifold with boundary

$$(M, \partial M) \subset (N, \partial N)$$

will be a pair of open neighborhoods $(U, \partial U)$ of $(M, \partial M)$ in $(N, \partial N)$ such that there is a vector bundle ξ over M and a homeomorphism of pairs from $\big(E(\xi), E(\xi|\partial M)\big)$ to $(U, \partial U)$ such that $(M, \partial M) \subset (U, \partial U)$ corresponds to the pair of zero sections in $\big(E(\xi), E(\xi|\partial M)\big)$. Such a neighborhood pair is said to be *neat at the boundary* if U is neat with respect to a pair of compatible collar neighborhoods.

As noted before, the existence of neat smooth vector bundle neighborhood pairs for a properly embedded smooth submanifold is established in [10]. It is an elementary (but not quite trivial) exercise to prove the following result.

Theorem 6. *Let $n > m$, and let $(M^m, \partial M^{m-1}) \subset (N^n, \partial N^{n-1})$ be a (topologically) properly embedded submanifold with boundary which satisfy the default hypotheses, and suppose we are given compatible topological collar neighborhoods of the boundaries. Suppose further that we have a topological vector bundle neighborhood pair $(U, \partial U)$ of $(M, \partial M)$. Then there is a subneighborhood pair $(V, \partial V) \subset (U, \partial U)$ such that $(V, \partial V)$ is a neat topological vector bundle neighborhood pair for $(M, \partial M)$.*■

RELATIVE VERSIONS OF THEOREMS 1 AND 2. Compatible collar neighborhoods provide one tool needed for extending Theorems 1 and 2 to properly embedded manifolds with boundaries. Another important step in the proof involves relative versions of Theorems 1 and 2. Precise statements are given below; in each case the proof proceeds along similar lines but requires more care to ensure that nothing is changed near a closed subset, and we shall omit the details.

Theorem 7. *Let $n, m \geq 5$, let N^n be a smooth n-manifold, and let $M^m \subset N^n$ be a topological m-manifold that is embedded in N^n. Assume that F is a closed subset of M and U is an open subset of M which contains U, and suppose that U has a smooth structure for which it is a smoothly embedded submanifold. Then there is a an open subset V such that $F \subset V \subset \overline{V} \subset U$ and a smooth structure on M^m extending the given one on V such that the inclusion of M in N is isotopic to a smooth embedding, by an isotopy which fixes V, if and only if M has a topological vector bundle neighborhood which is a smooth vector bundle neighborhood over U.*■

Theorem 8. *Let $k = 1$ or 2, let $m \geq 5$, let N^{m+k} be a smooth $(n+k)$-manifold, and let M^m be a locally flat submanifold of N^n, Assume that F is a closed subset of M and U is an open subset of M which contains U, and suppose that U has a smooth structure for which it is a smoothly embedded submanifold. Then there is an open set V such that $F \subset V \subset \overline{V} \subset U$ and a smooth structure on M^m such that the smooth structure extends the given smooth structure on V and inclusion of M in N is isotopic to a smooth embedding by an isotopy which is fixed on V.*■

PROOFS OF THEOREMS 1 AND 2 FOR PROPERLY EMBEDDED SUBMANIFOLDS WITH BOUNDARY. The first step is an application of the unbounded case to the embedding of ∂M in ∂N. Using compatible collar neighborhoods, we can then construct an ambient isotopy which moves a collar neighborhood of ∂M to a properly embedded smooth submanifold of a collar neighborhood for ∂N. Suppose that $c_M : \partial M \times [0,1) \to M$ is a collar neighborhood which is mapped smoothly into a collar neighborhood of ∂N, let F be the image of $\partial M \times [0, \frac{1}{2}]$ (this is a closed subset of M) and let U be the open set defined by the image of $\partial M \times [0, \frac{3}{4})$. The relative versions of Theorems 1 and 2 imply that we can find a smoothing of the unbounded manifold $M - \partial M$ which agrees with the smoothing of the collar on the image of $\partial M \times (0, \frac{1}{2})$ and an ambient isotopy which is fixed on the image of $\partial M \times (0, \frac{1}{2}]$ and moves M to a smoothly embedded submanifold. The latter is a smooth perturbation of the original submanifold with all the required properties.■

5. Exceptional dimension pairs

We have already noted that the analogous result to [23] for smoothing piecewise differentiable submanifolds of a smooth manifold is valid for all dimension pairs (n, m) such that $n > m$, so it is natural to ask whether the conclusion to Theorem 1 holds if either $m \leq 4$ or $n \leq 4$. The answer depends upon m and also to some extend upon n. We shall assume $m > 0$ since Theorem 1 extends to cases where $m = 0$ for trivial reasons. In most cases we have chosen examples and arguments which reflect the main ideas of earlier sections; there are alternative arguments in many cases, and some do not require the full force of the machinery developed in [15].

The case $m = 4$ and $n \geq 5$. Theorem 1 DOES NOT EXTEND to this case. The work of M. Freedman [8] and S. Donaldson [6] on 4-dimensional manifolds can be used to find examples as follows: There is a smooth simply connected 4-manifold K called the *Kummer manifold* such that $H^2(K;\mathbb{Z}) \cong \mathbb{Z}^{22}$ and the cup product defines an even quadratic form on this group with signature equal to 16. The results of [8] show that K splits topologically as a connected sum $L \# 3\, S^2 \times S^2$, and that L has a positive definite even quadratic form with signature equal to 16. By the results of [15], it follows that $L \times \mathbb{R}^k$ is smoothable for all $k > 0$. However, the results of [6] imply that L itself is not smoothable.■

Note. It is not known whether there are counterexamples for which M^4 is smoothable but there is no smoothing such that the topological embedding of M^4 in the smooth manifold N^5 can be isotoped to a smooth embedding.

The case $m = 3$ and $n \geq 5$. Theorem 1 also DOES NOT EXTEND to this case, and in fact counterexamples can be constructed using the results of Kirby and Siebenmann in [15]. Their results yield a smooth manifold M^5 such that M is homeomorphic to $S^3 \times T^2$ but not diffeomorphic to it. The slice inclusion of $S^3 \cong S^3 \times \{\text{pt.}\}$ in $S^3 \times T^2$ can be combined with the homeomorphism $S^3 \times T^2 \cong M$ to yield a locally flat topological embedding of S^3 in M^5, but it cannot be deformed to a smooth embedding. If it could, then there would be an exotic smooth structure on S^3 and it would be detected by the Rochlin invariant (see [29], Section 3), but it is known that S^3 only has smooth structures that are equivalent to the standard one. — One can construct similar examples for $n > 5$ by taking the corresponding locally flat embedding of S^3 in $M \times T^k$, where $k > 0$ is arbitrary.■

A somewhat different example for $m = 3$ and $n = 5$ is due to Lashof [22], who constructed a locally flat topological embedding of S^3 in $\mathbb{R}^5$ which is not smoothable. The nonsmoothability of this embedding is related to the Rochlin invariant.■

The case $m = 3$ and $n = 4$. Theorem 1 also DOES NOT EXTEND to this case. Let $\Sigma(2, 3, 5)$ be the Poincaré homology 3-sphere obtained by taking the quotient of SO_3 by the group of orientation preserving symmetries of the regular icosahedron. It is known that the Rochlin invariant of this manifold is nonzero (again see Section 3 of [29]). On the other hand, results of [8] imply that there is an open subset of $\mathbb{R}^4$ (with the usual smooth structure) which is homeomorphic to $\Sigma(2, 3, 5) \times \mathbb{R}$. If one could approximate the corresponding topological submanifold $\Sigma \times \{0\}$ by a smooth embedding, then it would follow that the Rochlin invariant of Σ would be zero.■

Because of the complexity of 4-manifold topology, we shall also describe a more subtle example. The results of [7] and [8] show the existence of a smooth manifold M^4 which is homeomorphic but not diffeomorphic to the connected sum $\mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}^2}$, where $\overline{\mathbb{CP}^2}$ denotes $\mathbb{CP}^2$ with the opposite orientation. It follows immediately that the signatures of both manifolds are equal to -8, and one major conclusion in Donaldson's work [7] is that a manifold like the exotic example M^4 cannot be written as a smooth connected sum of smooth manifolds whose signatures are both negative; in contrast, there are obvious ways of writing the standard manifold $\mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}^2}$ as a connected sum of two 4-manifolds with negative signatures:

$$\mathbb{CP}^2 \# 9\,\overline{\mathbb{CP}^2} \;\cong\; \left(\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}^2}\right) \;\#\; \left(9 - k\,\overline{\mathbb{CP}^2}\right)$$

(where $1 \leq k \leq 8$)

Therefore, if we take the separating 3-spheres in these connected sum decompositions and consider their images in M^4 under the homeomorphism, it will follow that these 3-spheres have vector bundle neighborhoods in M^4 but are not smoothable.■

The cases $m = 1, 2$ and $n \geq 5$. In contrast to the cases discussed thus far, Theorem 1 CAN BE EXTENDED to these situations. One quick way to see this is to use the fact that the space TOP/O is 2-connected (see Essay V of [15]). If ξ is the vector bundle which yields a vector bundle neighborhood for M, then it follows that there is exactly one isotopy class of smoothings for $E(\xi)$, and by the previous results this must be given by the standard smoothing of M. Therefore we see that the smoothing of $E(\xi)$ inherited from N is isotopic to the standard vector bundle smoothing, and thus the original embedding can be deformed to a smooth embedding.■

The case $m = 2$ and $n = 4$. Theorem 1 DOES NOT EXTEND to this case, and the reasons involve many of the exceptional phenomena that one encounters in 4-manifold theory. The first point to note is that local flatness implies the existence of a vector bundle neighborhood by results from [8] (see Section 9.4). Perhaps the most easily described examples involve embedded 2-spheres in the 4-manifold $S^2 \times S^2$. Given a continuous map $f : S^2 \to S^2 \times S^2$, let f_1 and f_2 denote its projections onto the first and second factors, and let $(d_1, d_2) \in \mathbb{Z}^2$ be defined by $d_i = \deg(f_i)$. The results of [8] imply that that every pair of nonzero relatively prime pair of integers (d_1, d_2) can be realized as the degree pair for some locally flat topological embedding $f : S^2 \times S^2 \times S^2$. On the other hand, a result of K. Kuga [18] and A. Suciu [36] states that a degree pair can be realized by a *smooth* embedding if and only if (at least) one of the integers d_1 or d_2 is equal to 0 or ± 1. Many further results of this type are known (*e.g.*, see [24]), but many easily stated questions are still unanswered and much remains to be discovered. The smoothability question for locally flat surfaces in 4-manifolds is currently the least well understood of all the cases discussed in this section (numerous problems in this area appear in [13]).

The case $m = 1$ and $n = 4$. In this case, Theorem 1 CAN BE EXTENDED to embeddings of S^1, and this does not require all the machinery of [15]. The starting point is a result of M. Brown and H. Gluck [4] that every locally flat embedding of S^1 in a smooth 4-manifold is ambiently isotopic to a piecewise smooth embedding. Results of C. P. Rourke and B. J. Sanderson ([30],

[31], [32]) combined with geometrical computations from [25] then imply that the piecewise smoothly embedded curve given by [4] has a piecewise smooth vector bundle neighborhood. One can now apply the main result of [23] to conclude that the piecewise smooth embedding is piecewise smoothly ambient isotopic to a smooth embedding (actually, the conclusion in this particular situation also follows from more elementary arguments, but a reference to [23] requires the least additional discussion). Similar but more delicate arguments yield the desired result for locally flat embeddings of $\mathbb{R}$; since every connected 1-manifold without boundary is diffeomorphic to either S^1 or $\mathbb{R}$, this covers all cases.■

The cases $0 < m < n \leq 3$. In these cases, Theorem 1 CAN BE EXTENDED by special versions of results from [15] which apply to manifolds of dimension ≤ 3. The proofs in these cases are similar to the arguments in [19] and Sections 4–5 of [20].

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