CHAPTER VI

MULTIDIMENSIONAL PROJECTIVE GEOMETRY

In this chapter we shall study coordinate projective spaces of arbitrary dimension. As in the previous chapter, we shall use concepts from linear algebra extensively. Although some portions of this chapter contain results of the previous one as special cases, most of the material involves concepts not covered earlier in these notes.

One major difference between this chapter and the previous one is that we are mainly interested in somewhat different types of results. In particular, we are interested in the geometric automorphisms of a coordinate projective space $\mathbb{FP}^n$, and the results of this chapter give a simple but complete description of them. In the final section of this chapter we shall assume that we have a field (or skew-field) of scalars $\mathbb{F}$ which has a notion of ordering with the same basic properties of the orderings of the real or rational numbers, and we shall analyze the geometrical implications of such algebraic orderings.

1. Linear varieties and bundles

Our first objective is to extend the results of Section V.1 on duality and homogeneous coordinates from $\mathbb{FP}^2$ to $\mathbb{FP}^n$, where $n \geq 3$ is arbitrary. As indicated in Theorem IV.16, if (S, Π, d) is an n-dimensional projective (incidence) space, then the "points" of the dual projective n-space (S^*, Π^*, d^*) are the hyperplanes of S. Suppose now that $S = \mathbb{FP}^n$ for some skew-field $\mathbb{F}$; by the results of Section V.1, if $n = 2$ then we can introduce homogeneous coordinates into the dual projective plane $\left(\mathbb{FP}^2\right)^*$. We shall extend this to all $n \geq 2$, showing that one can define well-behaved homogeneous coordinates for the hyperplanes of $\mathbb{FP}^n$ for all $n \geq 2$ such that most of the fundamental results from Section V.1 also extend to this more general setting.

According to Theorem III.12, a hyperplane in $\mathbb{FP}^n$ is definable by a right homogeneous linear equation

$$\sum_{i=1}^{n+1} u_i\, x_i \;=\; 0$$

where the coefficients u_i are not all zero. Furthermore, two n-tuples $(u_1, \;\cdots\;, u_{n+1})$ and $(v_1, \;\cdots\;, v_{n+1})$ define the same hyperplane if and only if there is a nonzero $k \in \mathbb{F}$ such that $u_i \;=\; k\, v_i$ for all i (compare Section V.1). This immediately yields the following analog of Theorem V.1:

THEOREM VI.1. *Let $\mathbb{F}$ be a skew-field, and let $n \geq 2$. Then the set of hyperplanes in $\mathbb{FP}^n$ is in $1-1$ correspondence with $\mathcal{S}_1(\mathbb{F}^{1,n+1})$. Furthermore, if the hyperplane H corresponds to the left 1-dimensional vector subspace $\mathbb{F}\cdot\theta$ and $X \in \mathbb{FP}^n$ is given by $\xi\cdot\mathbb{F}$, then $X \in H$ if and only if $\theta\cdot\xi \;=\; 0$.* ■

As before, if the hyperplane H corresponds to the left 1-dimensional vector subspace Ω of $\mathbb{F}^{1,n+1}$, then a set of *homogeneous coordinates* for H is any nonzero vector in Ω.

Motivated by the preceding description of hyperplanes, we define a *linear variety* in $\mathbb{FP}^n$ to be the set of all points whose homogeneous coordinates satisfy a system of linear homogeneous equations

$$\sum_{j=1}^{n+1} u_{i.j} x_j = 0 \qquad 1 \leq i \leq m .$$

The following result shows that linear varieties are the same as geometrical subspaces.

THEOREM VI.2. *Let V be a linear variety defined by a system of linear homogeneous equations as above, and suppose that the (left) row rank of the matrix $B = (u_{i,j})$ is equal to r. Then V is an r-plane in $\mathbb{FP}^n$.*

Proof. If V_0 is the solution space of the system of equations, then clearly $V = \mathcal{S}_1(V_0)$. Since the rank of B is r, then dimension of V_0 is equal to $n+1-r$ by Theorem A.10, and hence V is an $(n-r)$-plane in $\mathbb{FP}^n$.

On the other hand, assume that W is a $(k+1)$-dimensional vector subspace of $\mathbb{F}^{n+1,1}$, so that $\mathcal{S}_1(W)$ is a k-plane. Let $\mathbf{w}_1, \cdots \mathbf{w}_{k+1}$ be a basis for W, and write $\mathbf{w}_i = {}^{\mathbf{T}}(w_{1,i} \quad \cdots \quad w_{k+1,i})$. Consider the left-homogeneous system of linear equations

$$\sum_i y_i \, w_{j,i} = 0 \qquad (1 \leq i \leq k+1) .$$

Since the right column rank of the matrix $C = (w_{j,i})$ is equal to k+1, the left subspace of solutions has dimension equal to $n-k$ (again by Theorem A.10). Let $\mathbf{v}_1, \cdots \mathbf{v}_{n-k}$ be a basis for the space of solutions, and write $\mathbf{v}_i = (v_{i,1} \quad \cdots \quad v_{i,n+1})$. Then, by construction, the vector subspace W is contained in the space of solutions of the system

$$\sum_j v_{j,i} \, x_j = 0 \qquad (1 \leq j \leq n-k) .$$

On the other hand, the space of solutions W' has dimension equal to

$$(n+1) - (n-k) = k + 1 .$$

Since this is the dimension of W, we must have $W' = W$, and this proves the second half of the theorem.■

Similarly, we may define a *linear variety of hyperplanes* to be the set of all hyperplanes whose homogeneous coordinates satisfy a system of left-homogeneous linear equations

$$\sum_i u_i \, x_{i,j} = 0 \qquad (1 \leq j \leq m) .$$

If the right rank of $X = (x_{i,j})$ is r, the variety of hyperplanes is said to be $(n-r)$-dimensional. The following result shows that linear varieties of hyperplanes are also equivalent to geometrical subspaces of $\mathbb{FP}^n$.

THEOREM VI.3. *An r-dimensional linear variety of hyperplanes in $\mathbb{FP}^n$ consists of all hyperplanes containing a fixed $(n-r-1)$-plane in the terminology of Chapter* IV*, a linear bundle with the given $(n-r-1)$-plane as center). Conversely, every $(n-r-1)$-plane in $\mathbb{FP}^n$ is the center of some linear variety of hyperplanes.*

Proof. The ideas are similar to those employed in Theorem 2. Let C_0 be the span of the rows of the matrix $(\,x_{i,j}\,)$. By hypothesis, $\dim C_0 = n-r$. Thus $C = \mathcal{S}_1(C_0)$ is an $(n-r-1)$-plane in $\mathbb{FP}^n$, and every hyperplane containing it automatically belongs to the linear variety. Conversely, if $\mathbf{y}_0 \in C_0$, then we may write $\mathbf{y} \;=\; \sum_i \,\mathbf{x}_i\, r_i$ where $\mathbf{x}_i \;=\; (x_{1,i}, \;\cdots\;, x_{n+1,i})$ and $r_i \in \mathbb{F}$, so that if H lies in the variety and θ is a set of homogeneous coordinates for H then we have

$$\theta \cdot \mathbf{y} \;=\; \sum_i \,(\theta \cdot \mathbf{x}_i)\, r_i \;=\; 0\ .$$

Thus every hyperplane in the variety contains every point of C. This proves the first half of the theorem.

Now suppose that we are given an $(n-r-1)$-plane $Z \;=\; \mathcal{S}_1(Z_0)$. Let $\mathbf{z}_1, \;\cdots\;, \mathbf{z}_{n-r}$ be a basis for Z_0, and write $\mathbf{z}_j \;=\; (z_{1,j}, \;\cdots\;, z_{n+1,j})$. Consider the variety of hyperplanes defined by the system of homogeneous equations

$$\sum_j \,u_i\, z_{i,j} \;=\; 0 \qquad\qquad (1 \;\leq\; j \;\leq\; n-r)\ .$$

Since the right rank of the matrix $(\,z_{i,j}\,)$ is equal to $(n-r)$, this bundle is r-dimensional. Furthermore, its center Z' is an $(n-r-1)$-plane which contains every point of Z by the reasoning of the previous paragraph. Therefore we have $Z \;=\; Z'$.$\blacksquare$

The preceding result has some useful consequences.

Theorem VI.4. *Let $(\,\mathbb{FP}^n\,)^*$ be the set of hyperplanes in $\mathbb{FP}^n$, and let Π^* and d^* be defined as in Section* IV.3. *Then a subset $S \subset (\,\mathbb{FP}^n\,)^*$ is in Π^* if and only if it is a linear variety of hyperplanes, in which case $d^*(S)$ is the dimension as defined above.*$\blacksquare$

Theorem VI.5. (compare Theorem V.1) *The triple*

$$\big(\,(\,\mathbb{FP}^n\,)^*\,,\ \Pi^*,\ d^*\big)$$

is a projective n-space which is isomorphic to $\mathcal{S}_1(\mathbb{F}^{1,n+1})$.

Since Theorem 4 is basically a restatement of Theorem 3, we shall not give a proof. However, a few remarks on Theorem 5 are in order.

Proof of Theorem 5. By Theorem 1 we have a 1–1 correspondence between $(\,\mathbb{FP}^n\,)^*$ and $\mathcal{S}_1(\mathbb{F}^{1,n+1})$. Furthermore, the argument used to prove Theorem 2 shows that r-dimensional varieties of hyperplanes correspond to set of the form $\mathcal{S}_1(V)$, where V is an $(r+1)$-dimensional left subspace of $\mathbb{F}^{1,n+1}$ (merely interchange the roles of left and right in the proof, switch the orders of the factors in products, and switch the orders of double subscripts). But r-dimensional linear bundles correspond to r-dimensional linear varieties of hyperplanes by Theorems 3 and 4. Combining these, we see that r-dimensional linear bundles of hyperplanes correspond to r-planes in $\mathcal{S}_1(\mathbb{F}^{1,n+1})$ under the 1–1 correspondence between $(\,\mathbb{FP}^n\,)^*$ and $\mathcal{S}_1(\mathbb{F}^{1,n+1})$.$\blacksquare$

By the Coordinatization Theorem (Theorem IV.18), this result implies the first half of Theorem IV.16. On the other hand, if we interchange the roles of left and right, column vectors and row vectors, and the orders of multiplication and indices in the reasoning of this section, we find that the dual of $\mathcal{S}_1(\mathbb{F}^{1,n+1})$ is isomorphic to $\mathbb{FP}^{n+1}$. In fact, this isomorphism $h : \mathcal{S}_1(\mathbb{F}^{1,n+1})^* \to \mathcal{S}_1(\mathbb{F}^{n+1,1})$ is readily seen to have the property that the composite $h \circ f^* \circ e$ given by

$$\mathbb{FP}^n \xrightarrow[\cong]{e} (\mathbb{FP}^n)^{**} \xrightarrow{f^*} \mathcal{S}_1(F^{1,n+1})^* \xrightarrow{h} \mathbb{FP}^n$$

is the identity (here f^* is an isomorphism of dual spaces induced by f as in Exercise IV.3.4). This establishes the second half of Theorem IV.16 and allows us to state the **Principle of Duality in Higher Dimensions:**

METATHEOREM VI.6. *A theorem in projective geometry in dimension $n \geq 2$ remains true if we interchange the expressions* point *and* hyperplane*, the phrases* r-planes in an n-space *and* $(n-r-1)$-planes in an n-space*, and the words* contains *and* is contained in.■

We shall now assume that $\mathbb{F}$ is commutative. Since $\mathbb{F} = \mathbb{F}^{\mathsf{OP}}$ in this case, the dual of $\mathbb{FP}^n$ is isomorphic to $\mathbb{FP}^n$. Hence the metatheorem may be modified in an obvious way to treat statements about projective n-spaces **over fields**.

The cross ratio of four hyperplanes four hyperplanes in $\mathbb{FP}^n$ containing a common $(n-2)$-plane may be defined in complete analogy with the case $n = 2$, which was treated in Section V.2. In particular, Theorem V.14 generalizes as follows.

THEOREM VI.7. *Let H_1, H_2, H_3 be distinct hyperplanes through an $(n-2)$-plane K in $\mathbb{FP}^n$, and let $H_4 \neq H_1$ be another hyperplane through K. Let L be a line disjoint from K, and let X_i be the unique point where L meets H_i for $1 = 1, 2, 3$. Then the point $X_4 \in L$ lies on H_4 if and only if we have*

$$\mathsf{XR}(X_1, X_2, X_3, X_4) \;=\; \mathsf{XR}(H_1, H_2, H_3, H_4)\ .$$

The proof of this result is formally identical to the proof of Theorem V.1.■

EXERCISES

1. Let $\mathbb{F}$ be a field, and let $X,\ Y,\ Z \in \mathbb{FP}^3$ be noncollinear points. Suppose that homogeneous coordinates for these points are respectively given as follows:

$$\xi \;=\; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \eta \;=\; \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \qquad \zeta \;=\; \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

Prove that the plane they determine has the following homogeneous coordinates:

$$\left(\begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_{,4} \end{vmatrix},\ -\begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_{,4} \end{vmatrix},\ \begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix},\ -\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \right)$$

By Theorem A.11, not all of the four determinants vanish because ξ, η and ζ are linearly independent. To see that $X,\ ,Y,\ Z$ lie on the above hyperplane, consider the determinants of the three 4×4 matrices whose rows are given by ${}^{\mathbf{T}}\omega$, ${}^{\mathbf{T}}\xi$, ${}^{\mathbf{T}}\eta$ and ${}^{\mathbf{T}}\zeta$, where ω runs through the three vectors in the set $\xi,\ \eta,\ \zeta$.

2. Explain why four points $\mathbf{p}_i = (x_i,\, y_i,\, z_i) \in \mathbb{F}^3$ (where $1 \le i \le 4$) are coplanar if and only if the 4×4 determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

is zero, where $\mathbb{F}$ is a field. Formulate an analogous statement for n dimensions. [*Hint:* For both parts, use the properties of determinants as described in Appendix A and the characterization of hyperplanes in terms of n-dimensional vector subspaces of $\mathbb{F}^{1,n+1}$.]

3. Write out the 3-dimensional projective duals of the following concepts:

(a) A set of collinear points.

(b) A set of concurrent lines.

(c) The set of all planes through a given point.

(d) Four coplanar points, no three of which are collinear.

(e) A set of noncoplanar lines.

4. What is the 3-dimensional dual of Pappus' Theorem?

5. Let $\{A,\ B,\ C,\ D\}$ and $\{A',\ B',\ C'',D'\}$ be two triples of noncoplanar points in a projective 3-space, and assume that the lines AA', BB', CC' and DD' are concurrent. Prove that the lines of intersection

$$\begin{aligned} G &= \mathsf{plane}(ABC) \cap \mathsf{plane}(A'B'C') \\ H &= \mathsf{plane}(ABD) \cap \mathsf{plane}(A'B'D') \\ K &= \mathsf{plane}(ACD) \cap \mathsf{plane}(A'C'D') \\ L &= \mathsf{plane}(BCD) \cap \mathsf{plane}(B'C'D') \end{aligned}$$

are coplanar, and state and prove the converse.

6. Find the equations of the hyperplanes through the following quadruples of points in $\mathbb{RP}^4$.

(a)

$$\begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 4 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 3 \\ 4 \\ 0 \\ 0 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 0 \\ 5 \\ 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

2. Projective coordinate systems

Theorems V.6 and V.7, which provided particularly convenient choices for homogeneous coordinates in one or two dimensions, proved to be extremely useful in Chapter V. We shall prove a corresponding result for higher dimensions in this section; as one might expect, this result has corresponding importance in higher dimensional projective geometry. All the results of this section (except for Exercise 1) are valid if $\mathbb{F}$ is an arbitrary skew-field.

THEOREM VI.8. *Let $\{B_0, \cdots, B_r\}$ be a set of independent points in $\mathbb{FP}^n$, and let U be a point in the r-plane $B_0 \cdots B_r$ such that every proper subset of $\{B_0, \cdots, B_r, U\}$ is independent. Then homogeneous coordinates β_i and ψ can be chosen for these points may be chosen such that*

$$(\ddagger) \qquad \psi = \beta_0 + \cdots + \beta_r .$$

Furthermore, if β'_i and ψ' is another collection of homogeneous for these points such that (‡) holds, then there is a nonzero $a \in \mathbb{F}$ such that $\psi = \psi' a$ and $\beta_i = \beta'_i$ for $i = 0, \cdots, r$.

Proof. . Since the points B_i are independent, if we take arbitrary homogeneous coordinates $\widetilde{\beta_i}$ and ψ then there exist unique scalars c_i such that

$$\widetilde{\psi} = \widetilde{\beta_0}\, c_0 + \cdots + \widetilde{\beta_r}\, c_r .$$

None of the coefficients c_i can be equal to zero, for otherwise a proper subset of $\{B_0, \cdots, B_r, U\}$ would be independent, contradicting our assumption about such proper subsets. If we now take $\beta_i = \widetilde{\beta_i}\, c_i$ for each i, we then have the desired relation (‡).□

Conversely, suppose that (‡) is satisfied. If we are given arbitrary homogeneous coordinates β'_i and ψ' for the points B_i and U, then there exist nonzero scalars a and q_i such that $\psi = \psi' a$ and $\beta_i = \beta'_i\, q_i$. The new homogeneous coordinate vectors satisfy a relation of the form

$$\psi' = \beta'_0\, q_0\, a^{-1} + \cdots + \beta'_r\, q_r\, a^{-1}$$

and if (‡) is valid then all the coefficients on the right hand side must be equal to 1. In other words, we must have $b_i\, a^{-1} = 1$ for all i or equivalently $b_i = a$ for all i, which is exactly what we wanted to prove.■

Assume now that (‡) is valid, and let X be any point of the r-plane $L = B_0 \cdots B_r$. A set of homogeneous coordinates ξ for X is then a linear combination of the form $\xi = \sum_i \beta_i\, x_i$. Since ξ is defined up to multiplication by a scalar factor and the vectors β_j are defined up to multiplication by a common scalar factor, it follows that the coefficients x_i are also determined up to multiplication by a common scalar factor, and such an ordered $(r+1)$-tuple $(x_0, \cdots, x_r)$ of coefficients is called a set of *homogeneous coordinates for $X \in L$ relative to the projective coordinate system* $(B_0 \cdots B_r \,|\, U)$. It is frequently denoted by notation such as $\vec{X}(B_0 \cdots B_r \,|\, U)$. The set $\{B_0, \cdots, B_r\}$ is often called the *coordinate simplex* or *fundamental simplex*, the points B_i are said to be the *vertices* of this coordinate simplex, and the point U is often called the *unit point* because homogeneous coordinates for this point in the projective coordinate system are given by $(1, \cdots, 1)$.

The homogeneous coordinates given in the definition of projective space may be viewed as a special case of the preceding construction; specifically, if the unit vectors in $\mathbb{F}^{n+1,1}$ are given by $\mathbf{e}_i$, then the appropriate corrdinate simplex has vertices $\mathbf{e}_i \cdot \mathbb{F}$ and the corresponding unit point is $\mathbf{d} \cdot \mathbb{F}$, where $\mathbf{d} = \sum_i \mathbf{e}_i$. This is often called the *standard coordinate system.*

The next result describes the change in homogeneous coordinates which occurs if we switch from one projective coordinate system to another.

THEOREM VI.9. *Let* $(B_0 \ \cdots \ B_r \,|\, U)$ *and* $(B_0^* \ \cdots \ B_r^* \,|\, U^*)$ *be two projective coordinate systems for an* r*-plane in* $\mathbb{FP}^n$*, and let* X *be a point in this* r*-plane. Then the homogeneous coordinates* x_i *and* x_i^* *of* X *relative to these respective coordinate systems are related by the coefficients of an invertible matrix* $A = (\, a_{i,j} \,)$ *as follows:*

$$x_i^* \cdot \rho \;=\; \sum_{k=0}^{r} a_{i,k}\, x_k$$

Here ρ *is a nonzero scalar in* $\mathbb{F}$.

Proof. Suppose that the coordinate vectors are chosen as before so that $\psi = \sum_i \beta_i$ and $\psi' = \sum_i \beta_i'$. If ξ is a set of homogeneous coordinates for X, then homogeneous coordinates for ξ are defined by the two following two equations:

$$\xi \;=\; \sum_i \beta_i\, x_i \qquad\qquad \xi^* \;=\; \sum_i \beta_i^*\, x_i^*$$

Since the points lie in the same r-plane, we have

$$\beta_i^* \;=\; \sum_k \beta_k\, a_{i,k}$$

for sutiable scalars $a_{i,k}$, and the matrix A with these entries must be invertible because the set $\{B_0 \ \cdots \ B_r\}$ is independent. Straightforward calculation shows that

$$\xi \;=\; \sum_k \beta_k\, x_k \;+\; \sum_{i,k} \beta_i^*\, a_{i,k}\, x_k \;=\; \sum_i \beta_i^*\, x_i^*$$

which implies that $x_i^* = \sum_k a_{i,k}\, x_k$. These are the desired equations; we have added a factor ρ because the homogeneous coordinates are defined only up to a common factor.■

EXERCISES

1. Take the projective coordinate system on $\mathbb{RP}^3$ whose fundamental simplex points B_i have homogeneous coordinates

$$\beta_0 \;=\; \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \qquad \beta_1 \;=\; \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \qquad \beta_2 \;=\; \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \qquad \beta_3 \;=\; \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

and whose unit point U has homogeneous coordinates

$$\psi \;=\; \begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

Find the homogeneous coordinates of the point A with respect to the system $(B_0\, B_1\, B_2\, B_3 \mid U)$ where ordinary homogeneous coordinates α for A are given below; there are two parts to this exercise corresponding to the two possibilities for A.

$$\alpha \;=\; \begin{pmatrix} 2 \\ 1 \\ 4 \\ 1 \end{pmatrix}\,, \qquad \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}\,.$$

2. Let T be an invertible linear transformation on $\mathbb{F}^{n+1,1}$ with associated invertible matrix A, let T_* be the associated geometric symmetry of $\mathbb{FP}^n$, let $(B_0 \;\cdots\; B_n \mid U)$ define the standard homogeneous coordinate system, and let $X \in \mathbb{FP}^n$ have homogeneous coordinates given by $x_0, \;\cdots\;, x_n$. What are the homogeneous coordinates of X with respect to the coordinate system

$$\big(\,T_*(B_0) \;\;\cdots\;\; T_*(B_n) \mid T_*(U)\,\big) \;?$$

3. Collineations

At the beginning of Section II.6, we noted that an appropriate notion of isomorphism for figures in Euclidean space is given by certain 1–1 correspondences with special properties. If one analyzes the situation further, it turns out that the relevant class of 1–1 correspondences is given by maps which extend to *isometries* of the Euclidean n-space $\mathbb{R}^n$. Specifically, these are 1–1 mappings T from $\mathbb{R}^n$ to itself with the following two properties:

(i) If $\mathbf{x}$ and $\mathbf{y}$ are distinct points in $\mathbb{R}^n$, then T satisfies the identity

$$d(\mathbf{x}, \mathbf{y}) \;=\; d\big(\,T(\mathbf{x}),\, T(\mathbf{y})\,\big)\ ;$$

in other words, T preserves distances between points.

(ii) If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are distinct points in $\mathbb{R}^n$, then T satisfies the identity

$$\mathsf{measure}(\angle\,\mathbf{xyz}) \;=\; \mathsf{measure}\big(\,\angle\,T(\mathbf{x})T(\mathbf{y})T(\mathbf{z})\,\big)\ ;$$

in other words, T preserves angle measurements.

Further information on such mappings and closely related issues can be found in the Addendum to Appendix A and the references cited there. For our purposes here, one important point is that one can describe all such isometries of $\mathbb{R}^n$ in a very simple and explicit manner. Specifically, every such isometry of $\mathbb{R}^n \cong \mathbb{R}^{n,1}$ is given by a mapping of the form $T(\mathbf{x}) \;=\; A\mathbf{x} + \mathbf{b}$, where $\mathbf{b} \in \mathbb{R}^n \cong \mathbb{R}^{n,1}$ and A is an $n \times n$ matrix which is *orthogonal*; the latter means that ${}^{\mathbf{T}}\!A$ is equal to A^{-1} or equivalently that the rows and columns of A define orthonormal sets of vectors. In this section we shall prove similar results for symmetries of projective spaces, showing that all geometric symmetries of $\mathbb{FP}^n$ are also given by some fairly basic constructions using linear algebra.

Frequently in this section we shall use the term **collineation** to denote an isomorphism from one n-dimensional incidence space to another (assuming $n \geq 2$). This name dates back to the 19^{th} century, and at the time collineations were the first types of incidence space isomorphisms to be considered abstractly.

Algebraic automorphisms and geometric symmetries

We have seen that every invertible $(n+1) \times (n+1)$ matrix A determines a collineation f_A of $\mathbb{FP}^n$ which is defined by the formula

$$f_A(\mathbf{x} \cdot \mathbb{F}) \;=\; A\mathbf{x} \cdot \mathbb{F}\ .$$

However, for many choices of $\mathbb{F}$ there are examples which do not have this form. In particular, if $\mathbb{F}$ is the complex numbers $\mathbb{C}$ and χ denotes the map on $\mathbb{F}^{n+1,1}$ which takes a column vector with entries z_j to the column vector whose entries are the complex conjugates[1] $\overline{z_j}$, then there is a well-defined collineation g_χ on $\mathbb{CP}^n$ such that

$$g_\chi(\mathbf{x} \cdot \mathbb{C}) \;=\; \chi(\mathbf{x}) \cdot \mathbb{C}$$

that can also be defined, but it turns out that such a map is not equal to any of the maps f_A described previously. The proof that g_χ is a conjugation depends upon the fact that complex conjugation is an automorphism; *i.e.*, we have $\overline{z_1 + z_2} \;=\; \overline{z_1} + \overline{z_2}$, $\overline{z_1 \cdot z_2} \;=\; \overline{z_1} \cdot \overline{z_2}$, and conjugation is a 1–1 correspondence to $\mathbb{C}$ to itself because conjgation is equal to its own inverse.

[1]Recall that if a complex number is given by $u + v\mathbf{i}$, where $\mathbf{i}^2 = -1$, then its conjugate is equal to $a - b\mathbf{i}$.

More generally, if $\mathbb{F}$ is an arbitrary skew-field and χ is an automorphism of $\mathbb{F}$, then one can construct a similar collineation g_χ on $\mathbb{FP}^n$ that is not expressible as f_A for some A. One major objective of this section is to prove that mappings of the form f_A and g_χ for the various choices of A and χ determine **all** collineations of $\mathbb{FP}^n$. In order to simplify the arguments, for the rest of this section we shall assume that the skew-field $\mathbb{F}$ is commutative; at the end of the section we shall discuss some aspects of the noncommutative case.

In fact, one of the most important prepreties of a collineation f from one coordinate projective space to another (perhaps over a different field) is that the collineation determines an isomorphism Φ_f of the underlying fields; if the two projective spaces are identical, this isomorphism becomes an automorphism. The first result of this section establishes the relationship between collineations and field isomorphisms.

THEOREM VI.10. *Let f be a collineation from the projective space $\mathbb{FP}^n$ to the projective space $\mathbb{EP}^n$, where $\mathbb{F}$ and $\mathbb{E}$ are fields and $n \geq 2$. Then there is an isomorphism*

$$\Phi_f : \mathbb{F} \longrightarrow \mathbb{E}$$

characterized by the equation

$$\Phi_f\big(\mathsf{XR}(Y_1,\, Y_2,\, Y_3,\, Y_4)\big) \;=\; \mathsf{XR}\big(\,\Phi_f(Y_1),\, \Phi_f(Y_2),\, \Phi_f(Y_3),\, \Phi_f(Y_4)\,\big)$$

where $Y_1,\ Y_2,\ Y_3,\ Y_4$ is an arbitrary sequence of collinear points such that the first three are distinct and $Y_4 \neq Y_2$.

There are three basic steps in the proof; namely, defining a map from $\mathbb{F}$ to $\mathbb{E}$ which dependes upon some choices, showing that such a map is an isomorphism of fields, and finally showing that the map is independent of the choices that were made at the first step. The second part of the proof uses the results from Section V.5, and the third part — which is by far the longest — relies heavily on results from Chapter V on cross ratios.

Proof of Theorem 10. *Construction of a mapping from $\mathbb{F}$ to $\mathbb{E}$.* Let X_0, X_1 and U be three distinct collinear points, and let $q \in \mathbb{F}$. Then there is a unique point $Q \in X_0X_1$ such that $Q \notin P_0$ and $\mathsf{XR}(X_0, X_1, U, Q) \;=\; q$. Define $\Phi_f(q) \;=\; \mathsf{XR}\big(f(X_0), f(X_1), f(U), f(Q)\big)$. Strictly speaking, one should write this as $\Phi_{f,X_0,X_1,U}$ to indicate that it depends upon the choices of X_0, X_1 and U.

We claim that *the map $\Phi_{f,X_0,X_1,U}$ defines an isomorphism from $\mathbb{F}$ to $\mathbb{E}$.* — Since the elements of $\mathbb{E}$ are in 1–1 correspondence with the elements of $f(X_0)f(X_1) - \{f(X_0)\}$ and f maps $X_0X_1 - \{X_0\}$ bijectively to $f(X_0)f(X_1) - \{f(X_0)\}$, it follows that $\Phi_{f,X_0,X_1,U}$ is 1–1 and onto. Furthermore, to see that the latter map is an isomorphism, take another line L through X_0, coplanar points Z_0 and D, and points $A,\, B \in X_0X_1$ as in Section V.3. Let $f(X_i) = X_i'$, $f[L] = L'$, $f(Z_0) - Z_0'$, $f(A) = A'$, $f(D) = D'$, and $f(B) = B'$. If X is any point constructed from the unprimed point as in Section V.5, let X' be the corresponding point constructed from the primed points. Since f is a collineation, it is easy to verify that $f(X) = X'$ for all point X constructed in Section V.5. In particular, $f(C) = C'$ and $f(K) = K'$. But the latter equalities combined with Theorem V.28 and V.29 imply that

$$\Phi_{f,X_0,X_1,U}(a+b) \;=\; \Phi_{f,X_0,X_1,U}(a) \;+\; \Phi_{f,X_0,X_1,U}(b)\ .$$

Proof that the map $\Phi_{f,X_0,X_1,U}$ depends only on the line M containing X_0, X_1 and U. — It suffices to show that

$$\Phi_{f,X_0,X_1,U}\big(\mathsf{XR}(A,B,C,D)\big) \;=\; \mathsf{XR}\big(\,f(A),f(B),f(C),f(D)\,\big)$$

for every quadruple of distinct points on X_0X_1. There are several cases to be considered.

Case 1. None of the points $A,\ B,\ C,\ D$ is X_0. If we choose homogeneous coordinates ξ_i for X_i and ψ for U such that $\xi_0+\xi_1=\psi$, then homogeneous coordinates α, β, γ and δ for A, B, C and D are given as follows:

$$\begin{aligned}
\alpha &= \mathsf{XR}(X_0,X_1,U,A)\xi_0 \;+\; \xi_i\\
\beta &= \mathsf{XR}(X_0,X_1,U,B)\xi_0 \;+\; \xi_i\\
\gamma &= \mathsf{XR}(X_0,X_1,U,C)\xi_0 \;+\; \xi_i\\
\delta &= \mathsf{XR}(X_0,X_1,U,D)\xi_0 \;+\; \xi_i
\end{aligned}$$

and therefore we have

$$\Phi_{f,X_0,X_1,U}\big(\mathsf{XR}(A,B,C,D)\big) \;=\; \mathsf{XR}\big(\,f(A),f(B),f(C),f(D)\,\big)$$

by the formula established in Theorem V.13 and the fact that $\Phi_{f,X_0,X_1,U}$ is an isomorphism of fields.

Case 2. One of the points is X_0. We claim it suffices to consider the case $A=X_0$. For by Theorem V.12 there is a reordering $\big(\sigma(A),\sigma(B),\sigma(C),\sigma(D)\big)$ of (A,B,C,D) such that $\sigma(A)=X_0$ and

$$\mathsf{XR}\big(\sigma(A),\sigma(B),\sigma(C),\sigma(D)\big) \;=\; \mathsf{XR}(A,B,C,D)\;.$$

If the assertion is correct for quadruples whose first term is X_0, then

$$\Phi_{f,X_0,X_1,U}\Big(\big(\sigma(A),\sigma(B),\sigma(C),\sigma(D)\big)\Big) = \Big(\,\sigma\big(\,f(A)\big),\sigma\big(\,f(B)\big),\sigma\big(\,f(C)\big),\sigma\big(\,f(D)\big)\,\Big)\;.$$

Since the right hand side is equal to $\Phi_{f,X_0,X_1,U}\big(\mathsf{XR}(A,B,C,D)\big)$ and the right hand side is equal to $\mathsf{XR}\big(\,f(A),f(B),f(C),f(D)\big)$, the cases where X_0 is one of $B,\ C$ or D follow.

By the preceding discussion, we might as well assume that $X_0=A$ in Case 2. The remainder of the argument for Case 2 splits into subcases depending upon whether X_1 is equal to one of the remaining points.

Subcase 2.1. Suppose that $A=X_0$ and $B=X_1$. Then by Theorem V.11 we have

$$\mathsf{XR}(A,B,C,D) \;=\; \frac{\mathsf{XR}(A,B,U,D)}{\mathsf{XR}(A,B,U,C)} \;=\; \frac{\mathsf{XR}(X_0,X_1,U,D)}{\mathsf{XR}(X_0,X_1,U,C)}\;.$$

Note that the cross ratio $\mathsf{XR}(A,B,U,C)$ is nonzero because $B\neq C$. The assertion in this case follows from the formula above and the fact that Φ is an automorphism.

Subcase 2.2. Suppose that $A=X_0$ and $C=X_1$. Then $\mathsf{XR}(A,B,C,D) \;=\; 1-\mathsf{XR}(A,C,B,D)$, and hence the assertion in this subcase follows from Subcase 2.1 and the fact that Φ is an automorphism.

Subcase 2.3. Suppose that $A=X_0$ but neither B nor C is equal to X_1. Let

$$\begin{aligned}
b &= \mathsf{XR}(X_0,X_1,U,B)\\
c &= \mathsf{XR}(X_0,X_1,U,C)\\
d &= \mathsf{XR}(X_0,X_1,U,D)
\end{aligned}$$

so that homogeneous coordinates β, ξ_0 and ξ_1 for the points B, X_0 and X_1 can be chosen such that $\beta = b\xi_0 + \xi_1$, and hence the corresponding homogeneous coordinates $\gamma = c\xi_0 + \xi_1$ for C satisfy

$$\gamma \;=\; c\xi_0 \;+\; \xi_1 \;=\; (c-b)\xi_0 \;+\; (b\xi_0 + \xi_1)\ .$$

Since $B \neq C$, it follows that $c - b = 0$. Therefore homogeneous coordinates δ for D are given by

$$\delta \;=\; d\xi_0 \;+\; \xi_1 \;=\; \frac{d-b}{c-b}(c-b)\xi_0 \;+\; (b\xi_0 + \xi_1)\ .$$

Therefore we have the identity

$$\mathsf{XR}(A,B,C,D) \;=\; \frac{d-b}{c-b}\ .$$

The assertion in this subcase follows from the above formula and the fact that Φ is an isomorphism. This concludes the proof that Φ only depends upon the line $L = X_0X_1$.

Proof that the isomorphism $\Phi_f = \Phi_{f,M}$ does not depend upon the choice of the line M. — Once again, there are two cases.

CASE 1. Suppose we are given two lines M and M' which have a point in common; we claim that $\Phi_{f,M} = \Phi_{f,M'}$. Let V be a point in the plane of M and M' which is not on either line. If $X \in M$, let $X' \in M' \cap VX$; then

$$f(X') \;\in\; f[M'] \;\cap\; f(V)f(X)$$

because f is a collineation. Thus two applications of Theorem 15 imply

$$\mathsf{XR}(A,B,C,D) \;=\; \mathsf{XR}(A',B',C',D')$$
$$\mathsf{XR}\big(\,f(A),f(B),f(C),f(D)\,\big) \;=\; \mathsf{XR}\big(\,f(A'),f(B'),f(C'),f(D')\,\big)\ .$$

On the other hand, we also have

$$\Phi_{f,M}\big(\mathsf{XR}(A,B,C,D)\big) \;=\; \mathsf{XR}\big(\,f(A),f(B),f(C),f(D)\,\big) \text{ and}$$
$$\Phi_{f,M'}\big(\mathsf{XR}(A',B',C',D')\big) \;=\; \mathsf{XR}\big(\,f(A'),f(B'),f(C'),f(D')\,\big)\ .$$

Since every element of $\mathbb{F}$ has the form $\mathsf{XR}(A,B,C,D)$ for suitable points, the equations above imply that $\Phi_{f,M} = \Phi_{f,M'}$.

CASE 2. Suppose we are given two lines M and M' which have no points in common; we claim that $\Phi_{f,M} = \Phi_{f,M'}$. Let M'' be a line joining one point in M to one point in M'. Then two applications of the first case imply that $\Phi_{f,M} = \Phi_{f,M''} = \Phi_{f,M'}$.■

The characterization of Φ_f in terms of the cross ratio implies some useful properties of the construction Φ which sends a collineation $\mathbb{FP}^n \to \mathbb{EP}^n$ to the field isomorphism $\Phi_f : \mathbb{F} \to \mathbb{E}$.

THEOREM VI.11. *If $f : P \to P'$ and $f' : P' \to P''$ are collineations of coordinate projective n-spaces (where $n \geq 2$), then $\Phi_{gf} = \Phi_g \circ \Phi_f$. If $f : P \to P$ is the identity, then Φ_f is the identity on the underlying field. Finally, if $g : P' \to P$ is equal to f^{-1}, then $\Phi_g = (\Phi_f)^{-1}$.*

Proof. If f is the identity, then we have

$$\Phi_f\big(\mathsf{XR}(A,B,C,D)\big) \;=\; \mathsf{XR}\big(\,f(A),f(B),f(C),f(D)\,\big) \;=\; \mathsf{XR}(A,B,C,D)$$

because $f(X) = X$ for all X. If g and f are collineations then

$$\Phi_g \circ \Phi_f\big(\mathsf{XR}(A,B,C,D)\big) \;=\; \Phi_g\Big(\;\mathsf{XR}\big(\,f(A),f(B),f(C),f(D)\,\big)\;\Big) \;=$$

$$\mathsf{XR}\big(\, g\circ f(A), g\circ f(B), g\circ f(C), g\circ f(D)\,\big) \;=\; \Phi_{gf}\big(\,\mathsf{XR}(A,B,C,D)\,\big)\ .$$

To prove that $\Phi_{f^{-1}} = (\Phi_f)^{-1}$, note that $f\circ f^{-1}$ = identity and $f^{-1}\circ f$ = identity combine with the first two identities to show that the composites $\Phi_{f^{-1}}\circ\Phi_f$ and $\Phi_f\circ\Phi_{f^{-1}}$ are both identity maps, and these identities imply that $\Phi_{f^{-1}} = (\Phi_f)^{-1}$.■

Collineations of $\mathbb{FP}^1$

Of course, an incidence-theoretic definition of collineations for coordinate projective lines is meaningless. However, if $1+1 \neq 0$ in $\mathbb{F}$, then as in Section V.4 it is possible to defines collineations of $\mathbb{FP}^1$ as 1–1 correspondences which preserve harmonic quadruples. With this definition, an analog of Theorem 10 is valid. Details appear on pages 85–87 of the book by Bumcrot listed in the bibliography (this is related to the discussion of von Staudt's Theorem at the end of Section V.4).

Examples

We have already noted that every invertible $(n+1)\times(n+1)$ matrix A over $\mathbb{F}$ defines a geometric symmetry f_A of $\mathbb{FP}^n$, and by a straightforward extension of Exercise V.2.10 the mapping f_A preserves cross ratios; therefore, the automorphism Φ_{f_A} is the identity. On the other hand, if χ is an automorphism of $\mathbb{F}$ as above and g_χ is defined as at the beginning of this section, then for all distinct collinear points $A,\ B,\ C,\ D$ in $\mathbb{FP}^n$ we have

$$\chi\big(\,\mathsf{XR}(A,B,C,D)\,\big) \;=\; \mathsf{XR}\big(\, g_\chi(A), g_\chi(B), g_\chi(C), g_\chi(D)\,\big)$$

and therefore $\Phi_{g_\chi} = \chi$. In particular, the latter implies the following:

> *For every field $\mathbb{F}$, every automorphism χ of $\mathbb{F}$, and every $n > 0$, there is a collineation g from $\mathbb{FP}^n$ to itself such that $\Phi_g = \chi$.*■

Later in the section we shall prove a much stronger result of this type.

The Fundamental Theorem of Projective Geometry

Before stating and proving this result, we need to state and prove some variants of standard results from linear algebra. Let V and W be vector spaces over a field $\mathbb{F}$, and let α be an automorphism of $\mathbb{F}$. A mapping $T : V \to V$ is said to be an *α-semilinear transformation* if it satisfies the following conditions:

(1) $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x},\ \mathbf{y} \in V$.

(2) $T(c\mathbf{x}+\mathbf{y}) = \alpha(c)T(\mathbf{x})$ for all $\mathbf{x} \in V$ and $c \in \mathbb{F}$.

If α is the identity mapping, this reduces to the usual definition of a linear transformation.

THEOREM VI.12. *Let $V,\ W,\ \mathbb{F}, \alpha$ be as above. If $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is a basis for V and $\mathbf{w}_1, \cdots, \mathbf{w}_n \in W$, then there is a unique α-semilinear transformation $T : V \to W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for all i.*

Proof. Suppose that $v \in V$ and that T and S are α-semilinear transformations from V to W satisfying the conditions of the theorem. Express $\mathbf{v}$ as a linear combination of the form $\sum_i c_i\mathbf{v}_i$. Then we have

$$T(\mathbf{v}) = T\left(\sum_i c_i\mathbf{v}_i\right) = \sum_i \alpha(c_i)T(\mathbf{v}_i) = \sum_i \alpha(c_i)\mathbf{w}_i =$$

$$\sum_i \alpha(c_i)S(\mathbf{v}_i) = S\left(\sum_i c_i\mathbf{v}_i\right) = S(\mathbf{v})$$

and hence $S = T$. Conversely, if $\mathbf{v}$ is given as above, then $T(\mathbf{v}) = \sum_\mathbf{i} \alpha(\mathbf{c_i})\mathbf{w_i}$ is a well-defined α-semilinear transformation.■

This result has the following basic consequence:

THEOREM VI.13. *In the setting above, the mapping T is $1-1$ and onto if and only if the vectors $\mathbf{w}_1, \cdots, \mathbf{w}_n$ form a basis for W. In this case the inverse map T^{-1} is an α^{-1}-semilinear transformation.*

Proof. Since the image of T is contained in the subspace spanned by the vectors $\mathbf{w}_i$, it follows that T cannot be onto if these vectors do not span W. Conversely, suppose that these vectors do form a basis. Then by the previous result there is an α^{-1}-semilinear transformation $S : W \to V$ such that $S(\mathbf{w}_i) = \mathbf{v}_i$ for all i. It follows that $S \circ T$ is an $\alpha^{-1} \circ \alpha$-semilinear (hence linear) transformation from V to itself which sends $\mathbf{v}_i$ to $\mathbf{v}_i$ for all i, and hence $S \circ T$. Reversing the roles of V and W and also the roles of S and T in this argument, we conclude similarly that $T \circ S$ is the identity. Therefore the α-semilinear map S is an inverse to T and the latter is 1–1 and onto.■

If $\mathbb{F}$ and n are as in Theorem 12 and T is an invertible α-semilinear transformation from $\mathbb{F}^{n+1,1}$ to itself, then as in Section 4.3 there is a collineation f_T from $\mathbb{FP}^n$ to itself defined by

$$f_T(X) = T(\xi) \cdot \mathbb{F}$$

where ξ is an arbitrary set of homogeneous coordinates for X; this does not depend upon the choice of homogeneous coordinates, for if $\xi' = c\xi$ is another set of homogeneous coordinates for X we have

$$T(c\xi) \cdot \mathbb{F} = \alpha(c) \cdot T(\xi) \cdot \mathbb{F} = T(\xi) \cdot \mathbb{F}\ .$$

The proof that this map defines a collineation proceeds exactly as in the case of linear transformations, the only change being the need to substitute $T(c\mathbf{v}) = \alpha(c) \cdot T(\mathbf{v})$ in place of $T(c\mathbf{v}) = c \cdot T(\mathbf{v})$ when the latter appears.

The Fundamental Theorem of Projective Geometry is a converse to the preceding construction, and it shows that every collineation of $\mathbb{FP}^n$ to itself has the form f_T for a suitably chosen invertible α-linear mapping T from $\mathbb{F}^{n+1,1}$ to itself.

THEOREM VI.14. *(Fundamental Theorem of Projective Geometry) Let $\{X_0, \cdots, X_n, A\}$ and $\{Y_0, \cdots, Y_n, B\}$ be two sets of $(n+2)$ points in $\mathbb{FP}^n$ (where $n \geq 2$) such that no proper subset of either is dependent, and let χ be an automorphism of $\mathbb{F}$. Then there is a unique collineation f of $\mathbb{FP}^n$ to itself satisfying the following conditions:*

(i) $f(X_i) = Y_i$ *for* $0 \leq i \leq n$.

(ii) $f(A) = B$.

(*iii*) $\Phi_f \;=\; \chi$.

The theorem (with the proof given here) is also valid if $n = 1$ and $1 + 1 \neq 0$ in $\mathbb{F}$, provided collineations of $\mathbb{FP}^1$ are defined in the previously described manner (*i.e.*, preserving harmonic quadruples).

Proof. EXISTENCE. According to Theorem 8 we can choose homogeneous coordinates ξ for X_i, η_i for Y_i, α for A, and β for B so that $\alpha \;=\; \sum_i \xi_i$ and $\beta \;=\; \sum_i \eta_i$. The hypotheses imply that the vectors ξ_i and η_i form bases for $\mathbb{F}^{n+1,1}$, and therefore there is an invertible α-semilinear transformation of the latter such that $T(\xi_i) \;=\; \eta_i$ for all i. Then f_T is a collineation of $\mathbb{FP}^n$ sending X_i to Y_i and A to B. In order to compute the automorphism induced by f, let $Q_1,\ Q_2,\ Q_3,\ Q_4$ be distinct collinear points with homogeneous coordinates θ_i for Q_i chosen such that $\theta_3 \;=\; \theta_1 + \theta_2$ and $\theta_4 \;=\; q\theta_1 + \theta_2$, where $q \;=\; \mathsf{XR}(Q_1, Q_2, Q_3, Q_4)$. We then have $T(\theta_3) \;=\; {}_T(\theta_1) +_T (\theta_2)$ and $T(\theta_4) = \chi(q)\cdot T(\theta_1) + T(\theta_2)$, so that

$$\Big(\mathsf{XR}\big(Q_1, Q_2, Q_3, Q_4\big) \;=\; \chi\big(\mathsf{XR}\big(f_T(Q_1), f_T(Q_2), f_T(Q_3), f_T(Q_4)\big)\big)\ .$$

It follows that $\Phi_{f_T} \;=\; \chi$.$\square$

UNIQUENESS. Suppose that f and g are collineations of $\mathbb{FP}^n$ which satisfy $f(X_i) \;=\; g(X_i) \;=\; Y_i$ for $0 \leq i \leq n$, $f(A) \;=\; g(A) \;=\; B$, and $\Phi_f \;=\; \Phi_g \;=\; \chi$. Then $h = g^{-1}\circ f$ is a collineation which satisfies $h(X_i) = X_i$ for $0 \leq i \leq n$, $h(A) \;=\; A$, and Φ_h is the identity. If suffices to show that a collineation h satisfying these conditions must be the identity.

Let h be a collineation such that Φ_h is the identity, and suppose that h leaves three distinct points on a line fixed; we claim that h leaves every point on the line fixed. To see this, suppose that X_1, X_2 and X_3 are distinct collinear points such that $h(X_i) = X_i$ for all i, and let $Y \in X_1X_2$. Then we have

$$\mathsf{XR}\big(X_1, X_2, X_3, h(Y)\big) \;=\; \Phi_h\big(\mathsf{XR}(X_1, X_2, X_3, Y)\big) \;=\; \mathsf{XR}(X_1, X_2, X_3, Y)$$

and hence $h(Y) = Y$ by Theorem V.10.

Now assume that h satisfies the conditions in the first paragraph of this argument. We shall prove, by induction on r, that h fixes every point in the r-plane $X_0 \ \cdots\ X_r$. The statement is trivially true for $r = 0$, so assume that it is true for $r - 1$, where $r \geq 1$. By the dimension formula, the intersection of the subspaces $X_0 \ \cdots\ X_r$ and $AX_{r+1} \ \cdots\ X_n$ is a point which we shall call B_r. In fact, homogeneous coordinates β_r for B_r are given by

$$\beta_r \;=\; \xi_0 \;+\; \cdots \;+\; \xi_r$$

because the right hand side is set of homogeneous coordinates for a point in the intersection. Since $h(X_i) \;=\; X_i$ and $h(A) \;=\; A$, it follows that h maps the subspaces $X_0 \ \cdots\ X_r$ and $AX_{r+1} \ \cdots\ X_n$ into themselves. Thus the intersection of these subspaces (namely, the one point set $\{B_r\}$) must be mapped into itself and hence $h(B_r) \;=\; B_r$.

We claim that h fixes every point on the line X_rB_r fixed. By hypothesis $h(X_r) \;=\; X_r$, and by the preceding paragraph $h(B_r) = B_r$. Hence h maps X_rB_r into itself. Since X_rB_r and $X_0 \ \cdots\ X_{r-1}$ are both contained in $X_0 \ \cdots\ X_r$, the dimension formula implies that they intersect in a point W. Since $W \in X_0 \ \cdots\ X_{r-1}$, the induction hypothesis implies that $h(W) = W$. Homogeneous coordinates ω for W are given by

$$\omega \;=\; \beta_r \;-\; \xi_r \;=\; \xi_0 \;+\; \cdots \;+\; \xi_{r-1}$$

and hence the points X_r, V_r and W are distinct collinear points. Since each is left fixed by h, it follows that every other point in X_rB_r is also left fixed by h.

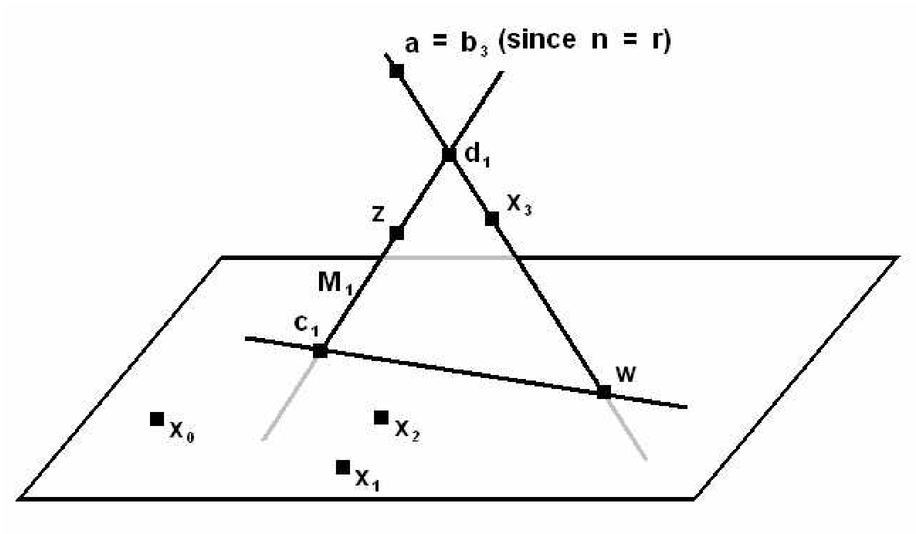

Figure VI.1

$(r = n = 3)$

Now let Z be any point of $X_0 \cdots X_r$ not on either $X_0 \cdots X_{r-1}$ or X_rB_r. We claim that $h(Z) = Z$. Since X_rB_r and $X_0 \cdots X_{r-1}$ are both contained in $X_0 \cdots X_r$, the dimension formula implies that $X_0 \cdots X_{r-1}$ and the plane ZX_rB_r intersect in a line we shall call L. The assumption that $Z \notin X_0 \cdots X_{r-1}$ implies that $Z \notin L$.

Let M_1 and M_2 be two distinct lines in ZX_rB_r containing Z; since there are at least three lines in the plane ZX_rB_r containing Z, we may choose M_1 and M_2 such that neither line contains the point B_r in which L meets X_rB_r; in particular, this means that the intersections of M_i with L and X_rB_r are distinct points.

Let $S_i \in M_i \cap L$, and let $D_i \in M_2 \cap X_rB_r$ (here $i = 1$ or 2). Then $h(C_i) = C_i$ and $h(D_i) = D_i$. Since the intersections of M_i with L and X_rB_r are distinct points, it follows that h leaves two distinct points of M_i fixed and hence h maps each line M_i into itself. Therefore it also follows that h maps $M_1 \cap M_2 = \{Z\}$ into itself, so that $h(Z) = Z$.

The preceding argument shows that h leaves every point of $X_0 \cdots X_r$ fixed, completing the inductive step of the argument. Therefore, by induction we conclude that h is the identity on $\mathbb{FP}^n = X_0 \cdots X_n$.■

One immediate consequence of the Fundamental Theorem is particularly worth stating at this point:

THEOREM VI.15. *Let f be a collineation of $\mathbb{FP}^n$, and let $\Phi_f = \alpha$. Then there is an invertible α-semilinear transformation T of $\mathbb{F}^{n+1,1}$ such that if $X \in \mathbb{FP}^n$ and ξ is a set of homogeneous coordinates for X then $f(X) = T(\xi) \cdot \mathbb{F}$.*

Proof. Let $\{X_0, \cdots, X_n, A\}$ be a set of of $(n+2)$ points in $\mathbb{FP}^n$ such that no proper subset is dependent. By the proof of existence in the Fundamental Theorem there is an invertible α-semilinear transformation T such that the associated collineation f_T satisfies the following conditions:

(i) $f(X_i) = f_T(X_i)$ for $0 \leq i \leq n$.

(ii) $f(A) = f_T(A)$.

Also, by construction the maps f and f_T determine the same automorphism of $\mathbb{F}$. We may now apply the uniqueness portion of the Fundamental Theorem to conclude that $f = f_T$.■

Definition. A collineation f of $\mathbb{FP}^n$ is *projective* if the associated automorphism Φ_f is the identity. Theorem 11 implies that the set of projective collineations is a subgroup — in fact, a normal subgroup — of the collineation group, and by the previous construction of examples we know that the quotient of the collineation group by the subgroup of projective collineations is equal to the automorphism group of $\mathbb{F}$. Further information along these lines is discussed in Exercise 9 below.■

Special cases

We conclude this section with some remarks on collineations if $\mathbb{F}$ is the real or complex numbers.

THEOREM VI.16. *For each $N \geq 2$, every collineation of real projective n-space $\mathbb{RP}^n$ is projective.*

By the previous results of this section, the proof of Theorem 15 reduces to showing the following:

THEOREM VI.17. *The only automorphism of the real numbers is the identity map.*

Proof(s). If χ is an isomorphism of $\mathbb{R}$, then $\chi(0) = 0$ and $\chi(0) = 1$. Suppose $\chi(r) = r$ for $r \geq 1$. Then $\chi(r+1) = \chi(r) + \chi(1) = r+1$, and hence χ agrees with the identity on all nonnegative integers. If k is a negative integer and $k = -m$, then

$$\chi(k) = \chi(-m) = -\chi(m) = -m = k$$

and hence χ is the identity on integers. If r is a rational number, write $r = m/n$ where m is an integer and n is a positive integer. Then $n = rm$ implies that

$$m = \chi(m) = \chi(nr) = \chi(n) \cdot \chi(r) = n \cdot \chi(r)$$

which implies that $\chi(r) = m/n = r$, and hence we see that χ fixes every rational number.

Suppose now that x is an arbitrary nonnegative real number. We claim that $\chi(x) \geq 0$. Recall that $x \geq 0$ if and only if $x = y^2$ for some y. Therefore $x \geq 0$ and $x = y^2$ imply that $\chi(x) = \chi(y)^2 \geq 0$. Similarly, if a and b are real numbers such that $a \geq b$, then

$$\chi(a) - \chi(b) = \chi(a-b) \geq 0$$

implies that $\chi(a) \geq \chi(b)$. Since χ is 1–1 it also follows that $a > b$ implies $\chi(a) > \chi(b)$.

Finally, suppose that we have an element $r \in \mathbb{R}$ such that $\chi(r) \neq r$. If $\chi(r) < r$, then there is a rational number q such that $\chi(r) < q < r$. But this implies $\chi(r) < \chi(q) = q$, and this contradicts the conclusion $\chi(r) > \chi(q)$ which follows from the previous paragraph. Therefore $\chi(r) < r$ is impossible, so that $\chi(r) \geq r$ for all real numbers r.

Now χ^{-1} is also an automorphism of $\mathbb{R}$, and if we apply the previous reasoning to this automorphism we conclude that $\chi^{-1}(r) \geq r$ for all r. Since we had previously shown that automorphisms are strictly increasing functions, if we apply χ to the previous inequality we obtain

$$r \;=\; \chi^{\circ}\chi^{-1}(r) \;\geq\; \chi(r)$$

and if we combine this with the final inequality of the preceding paragraph we conclude that $\chi(r) = r$ for all real numbers r.■

The analog of Theorem 16 does not hold for the complex numbers. In particular, at the beginning of this section we showed that the map g_χ of $\mathbb{CP}^n$ given by conjugating homogeneous coordinates is collineation that is not projective. Further information on automorphisms of the complex numbers and their applications to projective geometry appear in Appendix D.

EXERCISES

In the problems below, assume that $\mathbb{F}$ is a field and χ is an automorphism of $\mathbb{F}$.

1. Let A be an invertible $n \times n$ matrix over $\mathbb{F}$, and let f_A be the projective collineation of $\mathbb{FP}^n$ defined by A (in other words, if ξ are homogeneous coordinates for X, then $A\xi \cdot \mathbb{F}$ are homogeneous coordinates for $f_A(X)$). If H is a hyperplane in $\mathbb{FP}^n$ with homogeneous coordinates θ, prove that $T[H]$ has homogeneous coordinates $\theta \cdot A^{-1}$ (compare Exercise V.1.5).

2. In the notation of Exercise 1, suppose that a collineation T is defined such that if ξ are homogeneous coordinates for X, then $A\chi(\xi)$ are homogeneous coordinates for $T(X)$. Express homogeneous coordinates for $T[H]$ in terms of θ, A^{-1} and χ. You may use the product formula $\chi(A \cdot B) \;=\; \chi(A) \cdot \chi(B)$ for matrix multiplication. Also, recall that $\chi(0) \;=\; 0$.

3. Suppose that f is a collineation of $\mathbb{FP}^n$ with induced automorphism Φ_f, and suppose that $H_1,\ H_2,\ H_3,\ H_4$ are distinct hyperplanes containing a common $(n-2)$-plane. Prove that the cross ratio formula

$$\Phi_f\big(\mathsf{XR}(H_1, H_2, H_3, H_4)\big) \;=\; \mathsf{XR}\big(f[H_1], f[H_2], f[H_3], f[H_4]\big)$$

holds without using Exercise 2.

4. Suppose that T is an invertible χ-semilinear transformation of $\mathbb{F}^{n+1,1}$ where $n \geq 1$ such that the associated collineation f_T of $\mathbb{FP}^n$ is the identity. Prove that T is a scalar multiple of the identity. [*Hint:* By assumption, for each nonzero vector $\mathbf{x}$ there is a nonzero scalar $c_{\mathbf{x}}$ such that $T(\mathbf{x}) \;=\; c_{\mathbf{x}} \cdot \mathbf{x}$. If $c_{\mathbf{x}} \neq c_{\mathbf{y}}$, explain why $\mathbf{x}$ and $\mathbf{y}$ must be linearly independent. Consider $T(\mathbf{x}+\mathbf{y})$ in this case.]

5. (*a*) Let T be an invertible χ-semilinear transformation of $\mathbb{F}^n$ where $n \geq 1$, and let $\mathbf{z} \in \mathbb{F}^n$. Show that

$$G(\mathbf{x}) \;=\; T(\mathbf{x}) \;+\; \mathbf{z}$$

is a geometric symmetry of the affine incidence n-space $\mathbb{F}^n$. [*Hint:* Compare this statement to the examples following Theorem II.39.]

(*b*) Prove that G extends to a collineation g of $\mathbb{FP}^n$ for which $\Phi_g \;=\; \chi$; in other words, we have $g^{\circ}\mathrm{J} \;=\; \mathrm{J}^{\circ}G$ on $\mathbb{F}^n$. [*Hint:* Compare Exercise IV.4.14.]

(c) If $n \geq 2$, prove that every geometric symmetry f of $\mathbb{F}^n$ is given by a transformation of the type described in (a). [*Hint:* By Exercise 2 at the end of Chapter IV, the map f extends to a collineation g of $\mathbb{FP}^n$. Since the collineation leaves the hyperplane at infinity fixed, certain entries of an $(n+1) \times (n+1)$ matrix inducing g must vanish. But this implies the matrix has the form of one constructible by (b).]

(d) Determine whether $\mathrm{Aff}(\mathbb{F}^n)$ is the entire group of geometric symmetries of $\mathbb{F}^n$ when $\mathbb{F}$ is the real and complex numbers respectively.

6. Suppose that A is an invertible $m \times m$ matrix over a field $\mathbb{F}$ such that $1 + 1 \neq 0$ in $\mathbb{F}$. Prove that $\mathbb{F}^{m,1}$ contains two vector subspaces W_+ and W_- with the following properties:

(i) $A\mathbf{x} \ = \ \mathbf{x}$ if $\mathbf{x} \in W_+$ and $A\mathbf{x} \ = \ -\mathbf{x}$ if $\mathbf{x} \in W_-$.

(ii) $W_+ \ + \ W_- \ = \ \mathbb{F}^{m,1}$ and $W_+ \ \cap \ W_- \ = \ \{\mathbf{0}\}$.

[*Hint:* Let $W_\pm$ be the image of $A \pm I$. This yields the first part. To prove the rest, use the identity

$$I \ = \ \tfrac{1}{2}(A+I) \ - \ \tfrac{1}{2}(A-I) \ .]$$

> **Definition.** An *involution* of $\mathbb{FP}^n$ is a collineation f such that $f \circ f$ is the identity but f itself is not the identity. If $f(X) = X$, then X is called a **fixed point** of the involution.

7. (a) Let T be an invertible χ-semilinear transformation of $\mathbb{F}^{n+1,1}$ such that the induced collineation f_T of $\mathbb{FP}^n$ is an involution. Prove that T^2 is a scalar multiple of the identity. [*Hint:* Use Exercise 4.]

(b) Suppose that T is an involution of $\mathbb{RP}^n$. Prove that T is induced by an invertible $(n+1) \times (n+1)$ matrix A such that $A^2 = \pm 1$.

(c) In the previous part, prove that T has no fixed points if $A^2 = -I$. Using Exercise 6, prove that T has fixed points if $A^2 = 1$. [*Hint:* For the first part, suppose that X is a fixed point with homogeneous coordinates ξ such that $A \cdot \xi \ = \ c \cdot \xi$ for some real number c. However, $A^2 = -I$ implies that $c^2 = -1$.] — NOTATION. An involution is called *elliptic* if no fixed points exist and *hyperbolic* if fixed points exist.

(d) Using Exercise 6, prove that the fixed point set of a hyperbolic involution of $\mathbb{RP}^n$ has the form $Q_1 \cup Q_2$, where Q_1 and Q_2 are disjoint n_1- and n_2-planes and $n_1 + n_2 + 1 \ = \ n$.

8. Suppose that $A \neq B$, and that A and B are the only two points of the line AB left fixed by an involution f of $\mathbb{RP}^n$. Prove that $\mathsf{XR}\big(A, B, C, f(C)\big) \ = \ -1$ for all points C on $AB - \{A, B\}$. [*Hint:* Find an equation relating $\mathsf{XR}\big(A, B, C, f(C)\big)$ and $\mathsf{XR}\big(A, B, f(C), C\big)$.]

9. Let $\textsc{Coll}\,(\mathbb{FP}^n)$ denote the group of all collineations of $\mathbb{FP}^n$, let $\mathrm{Aut}(\mathbb{F})$ denote the group of (field) automorphisms of $\mathbb{F}$, and let $\Phi : \textsc{Coll}\,(\mathbb{FP}^n) \to \mathrm{Aut}(\mathbb{F})$ denote the homomorphism given by Theorem VI.10.

(a) Why is the kernel of Φ the group $\textsc{Proj}\,(\mathbb{FP}^n)$ denote the group of all projective collineations, and why does this imply that the latter is a normal subgroup of $\textsc{Coll}\,(\mathbb{FP}^n)$?

(b) Show that $\textsc{Coll}\,(\mathbb{FP}^n)$ contains a subgroup Γ such that the restricted homomorphism $\Phi|\Gamma$ is the identity, and using this prove that every element of $\textsc{Coll}\,(\mathbb{FP}^n)$ is expressible as a product of an element in $\textsc{Proj}\,(\mathbb{FP}^n)$ with an element in Γ. [*Hint:* Look at the set of all collineations of the form g_χ constructed at the top of the second page of this section, where $\chi \in \mathrm{Aut}(\mathbb{F})$, and show that the set of all such collineations forms a subgroup of COLL $(\mathbb{FP}^n)$ which is isomorphic to Aut$(\mathbb{F})$.]

(c) Suppose that A is an invertible $(n+1) \times (n+1)$ matrix over $\mathbb{F}$ and χ is an automorphism of $\mathbb{F}$, and let f_A and g_χ be the collineations of $\mathbb{FP}^n$ defined at the beginning of this section. By the previous parts of this exercise and Theorem 15, we know that $g_\chi \circ f_A \circ (g_\chi)^{-1}$ has the form f_B for some invertible $(n+1) \times (n+1)$ matrix B over $\mathbb{F}$. Prove that we can take B to be the matrix $\chi(A)$ obtained by applying χ to each entry of A. [*Note:* As usual, if two invertible matrices are nonzero scalar multiples of each other then they define the same projective collineation, and in particular we know that $f_B = f_{cB}$ for all nonzero scalars c; this is why we say that we take B to be equal to $\chi(A)$ and not that B is equal to $\chi(A)$.]

10. Let $\mathbb{F}$ be a field, let $0 < r \leq n$ where $n \geq 2$, let Q be an r-plane in $\mathbb{FP}^n$. Let $\{X_0, \cdots, X_r, A\}$ and $\{Y_0, \cdots, Y_r, B\}$ be two sets of $(r+2)$ points in Q such that no proper subset of either is dependent. Then there is a projective collineation f of $\mathbb{FP}^n$ to itself such that $f(X_i) = Y_i$ for $0 \leq i \leq r$ and $f(A) = B$. [*Hint:* Let W be the vector subspace of $\mathbb{F}^{n+1,1}$ such that $Q = \mathcal{S}_1(W)$, define an invertible linear transformation G on W which passes to a projective collineation of Q with the required properties as in the proof of the Fundamental Theorem, extend G to an invertible linear transformation $\overline{G}$ of $\mathbb{F}^{n+1,1}$, and consider the projective collineation associated to $\overline{G}$.]

4. Order and separation

All of the analytic projective geometry done up to this point is valid for an arbitrary $\mathbb{F}$ for which $1 + 1 \neq 0$. Certainly one would expect than real projective spaces have many properties not shared by other coordinate projective spaces just as the field of real numbers has many properties no shared by other fields. The distinguishing features of the real numbers are that it is an ordered field and is complete with respect to this ordering. In this section we shall discuss some properties of projective spaces over arbitrary ordered fields and mention properties that uniquely characterize real projective spaces.

Given points $\mathbf{u},\ \mathbf{v} \in \mathbb{R}^n$, let $d(\mathbf{u}, \mathbf{v})$ be the distance from $\mathbf{u}$ to $\mathbf{v}$. One characterization of the statement

$$\mathbf{y} \textit{ is between } \mathbf{x} \textit{ and } \mathbf{z}$$

is that it holds if and only if $d(\mathbf{x}, \mathbf{z}) \ = \ d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$. Another more algebraic characterization follows immediately from this.

THEOREM VI.18. *If* $\mathbf{x},\ \mathbf{y},\ \mathbf{z} \in \mathbb{R}^n$ *are distinct points, then* $d(\mathbf{x}, \mathbf{z}) \ = \ d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ *holds if and only if* $\mathbf{y} \ = \ t\mathbf{x} + (1 - t)\mathbf{z}$ *for some* t *satisfying* $0 < t < 1$.

Proof. Recall that $d(\mathbf{u}, \mathbf{v})$ is the square root of

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \ = \ |\mathbf{u} - \mathbf{v}|^2 \ .$$

The proof of the Triangle Inequality for inner (or dot) products is a consequence of the Cauchy-Schwarz inequality

$$(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{y} - \mathbf{z}) \ \leq \ \big|(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{y} - \mathbf{z})\big| \ \leq \ |\mathbf{x} - \mathbf{y}| \cdot |\mathbf{y} - \mathbf{z}|$$

and equality holds in the Triangle Inequality if and only if the end terms of the Cauchy-Schwarz inequality are equal.[2] However, the Cauchy-Schwarz inequality states that the middle term and right hand term are equal if and only if $\mathbf{x} - \mathbf{y}$ and $\mathbf{y} - \mathbf{z}$ are linearly dependent. Since both are nonzero, this means that $(\mathbf{y} - \mathbf{z}) \ = \ k(\mathbf{x} - \mathbf{y})$ for some $k \neq 0$. On the other hand, the left and right hand terms are equal if and only if both are nonnegative. Consequently, if the end terms are equal, then $(\mathbf{y} - \mathbf{z}) \ = \ k(\mathbf{x} - \mathbf{y})$ and also $k|\mathbf{x} - \mathbf{y}| \geq 0$. This implies that k must be positive. Conversely, if $k > 0$ then the end terms of the Cauchy-Schwarz inequality are equal.

Thus $d(\mathbf{x}, \mathbf{z}) \ = \ d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ if and only if $\mathbf{y} - \mathbf{z}$ is a positive multiple of $\mathbf{x} - \mathbf{y}$. But if $\mathbf{y} - \mathbf{z} \ = \ k(\mathbf{x} - \mathbf{y})$, then

$$\mathbf{y} \ = \ \frac{k}{k+1}\,\mathbf{x} \ + \ \frac{1}{k+1}\,\mathbf{z} \ .$$

Since $k > 0$ implies

$$0 \ < \ \frac{k}{k+1} \ < \ 1$$

it follows that if $d(\mathbf{x}, \mathbf{z}) \ = \ d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ then $\mathbf{y} \ = \ t\mathbf{x} + (1 - t)\mathbf{z}$ for some t satisfying $0 < t < 1$.

Conversely, if $\mathbf{y} \ = \ t\mathbf{x} + (1 - t)\mathbf{z}$ for some t satisfying $0 < t < 1$, then

$$\mathbf{y} - \mathbf{z} \ = \ \frac{t}{1-t}\,(\mathbf{x} - \mathbf{y}) \ .$$

[2]See pp. 177–178 of Birkhoff and MacLane or pp. 277–278 of Hoffman and Kunze for further details.

Since $t/(1-t)$ is positive if $0 < t < 1$, it follows that $d(\mathbf{x},\mathbf{z}) \;=\; d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z})$.■

With this motivation, we define betweenness for arbitrary vector spaces over arbitrary ordered fields.

Definition. Let $\mathbb{F}$ be an ordered field, let V be a vector space over $\mathbb{F}$, and let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be distinct points of V. We shall say that $\mathbf{y}$ is **between** $\mathbf{x}$ and $\mathbf{z}$, written $\boldsymbol{\mathcal{B}}(\mathbf{x},\mathbf{y},\mathbf{z})$, if there is some $t \in \mathbb{F}$ such that $0 < t < 1$ and $\mathbf{y} \;=\; t\mathbf{x} + (1-t)\mathbf{z}$. Frequently we shall also say that the points $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ satisfy the *ordering relation* $\boldsymbol{\mathcal{B}}(\mathbf{x},\mathbf{y},\mathbf{z})$. The *closed segment* $[\mathbf{x};\mathbf{z}]$ consists of $\mathbf{x}$, $\mathbf{z}$, and all $\mathbf{y}$ such that $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{z}$. In Exercise 1 below this is compared to the usual definition of closed interval in $\mathbb{R}$.

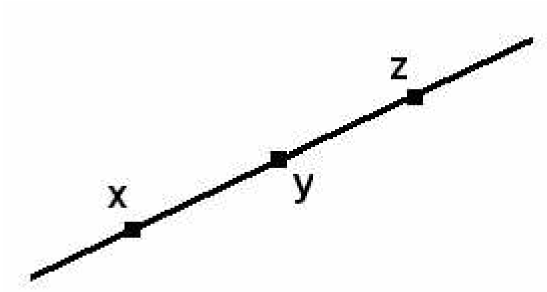

Figure VI.2

The *open segment* $(\mathbf{x};\mathbf{z})$ consists of all $\mathbf{y}$ such that $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{z}$.

The next results show that our definition of betweenness satisfies some properties that are probably very apparent. However, since we are dealing with a fairly abstract setting, it is necessary to give rigorous proofs.

THEOREM VI.19. *Let $\mathbb{F}$ be an ordered field, let V be a vector space over $\mathbb{F}$, and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be distinct vectors in V. If $\boldsymbol{\mathcal{B}}(\mathbf{a},\mathbf{b},\mathbf{c})$ is true, then so is $\boldsymbol{\mathcal{B}}(\mathbf{c},\mathbf{b},\mathbf{a})$. However, each of the four statements $\boldsymbol{\mathcal{B}}(\mathbf{b},\mathbf{a},\mathbf{c})$, $\boldsymbol{\mathcal{B}}(\mathbf{c},\mathbf{a},\mathbf{b})$, $\boldsymbol{\mathcal{B}}(\mathbf{a},\mathbf{c},\mathbf{b})$, $\boldsymbol{\mathcal{B}}(\mathbf{b},\mathbf{c},\mathbf{a})$ is false.*

Proof. By assumption $\mathbf{b} \;=\; t\mathbf{a}+(1-t)\mathbf{c}$ for some t satisfying $0 < t < 1$. The latter inequalities imply $0 < (1-t) < 1$, and since $t \;=\; 1-(1-t)$ it follows that $\boldsymbol{\mathcal{B}}(\mathbf{c},\mathbf{b},\mathbf{a})$ is true.

The equation $\mathbf{b} \;=\; t\mathbf{a} + (1-t)\mathbf{c}$ (where $0 < t < 1$) implies that $-t\mathbf{a} \;=\; -\mathbf{b} + (1-t)\mathbf{c}$, which in turn means that

$$t^{-1}\mathbf{b} \;-\; t^{-1}(1-t)\mathbf{c} \;=\; t^{-1}\mathbf{b} \;+\; (1-t^{-1})\mathbf{c}\ .$$

Therefore $\mathbf{a} \;=\; s\mathbf{b} + (1-s)\mathbf{c}$ then implies $s = t^{-1}$. Since $0 < t < 1$ implies $t^{-1} > 1$, it follows that $\boldsymbol{\mathcal{B}}(\mathbf{b},\mathbf{c},\mathbf{a})$ is false. Furthermore, it also follows that $\boldsymbol{\mathcal{B}}(\mathbf{a},\mathbf{c},\mathbf{b})$ is false, for if the latter were true then by the preceding paragraph the order relation $\boldsymbol{\mathcal{B}}(\mathbf{b},\mathbf{c},\mathbf{a})$ would also be true.

Finally, $\mathbf{b} \;=\; t\mathbf{a} + (1-t)\mathbf{c}$ (where $0 < t < 1$) implies that $(t-1)\mathbf{c} \;=\; t\mathbf{a} - \mathbf{b}$, which in turn implies that

$$\mathbf{c} \;=\; \frac{t}{t-1}\mathbf{a} \;+\; \frac{-1}{t-1}\mathbf{b}\ .$$

Now $0 < t < 1$ implies $t - 1 < 0$, so that

$$\frac{t}{t-1} \;<\; 0\ .$$

The latter means that $\boldsymbol{\mathcal{B}}(\mathbf{c},\mathbf{a},\mathbf{b})$ is false, and as in the previous paragraph it follows that $\boldsymbol{\mathcal{B}}(\mathbf{a},\mathbf{c},\mathbf{b})$ is also false.■

THEOREM VI.20. *Let* $\mathbb{F}$ *and* V *be as above, and let* $\mathbf{a}$ *and* $\mathbf{b}$ *be distinct vectors in* V. *Then* $\mathbf{c} \in V$ *lies on the line* $\mathbf{ab}$ *if and only if one of* $\mathbf{c} = \mathbf{a}$, $\mathbf{c} = \mathbf{b}$, $\boldsymbol{\mathcal{B}}(\mathbf{a},\mathbf{b},\mathbf{c})$, $\boldsymbol{\mathcal{B}}(\mathbf{c},\mathbf{a},\mathbf{b})$ *or* $\boldsymbol{\mathcal{B}}(\mathbf{b},\mathbf{c},\mathbf{a})$ *is true. Furthermore, these conditions are mutually exclusive.*

Proof. We know that $\mathbf{c} \in \mathbf{ab}$ if and only if $\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$ for some t. We claim the five conditions are equivalent to $t = 1$, $t = 0$, $t < 0$, $t > 1$ and $0 < t < 1$ respectively. Thus it will suffice to verify the following:

(1) $\boldsymbol{\mathcal{B}}(\mathbf{a},\mathbf{b},\mathbf{c})$ is true if and only if $\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$ for some $t < 0$.

(2) $\boldsymbol{\mathcal{B}}(\mathbf{c},\mathbf{a},\mathbf{b})$ is true if and only if $\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$ for some $t > 1$.

PROOF OF (1). The condition $\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$ with $t < 0$ is equivalent to

$$\mathbf{b} = \frac{t}{t-1}\,\mathbf{a} + \frac{-1}{t-1}\,\mathbf{c}\ .$$

The conclusion in this case follows because the map sending t to $t/(t-1)$ is a 1–1 correspondence from the unbounded set $\{\, u \in \mathbb{F} \mid u < 0 \,\}$ to the bounded open interval $\{\, v \in \mathbb{F} \mid 0 < v < 1 \,\}$.□

PROOF OF (2). The condition $\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$ with $t > 1$ is equivalent to

$$\mathbf{a} = \frac{1}{t}\,\mathbf{c} + \left(1 - \frac{1}{t}\right)\mathbf{b}\ .$$

The conclusion in this case follows because the map sending t to $1/t$ is a 1–1 correspondence from the unbounded set $\{\, u \in \mathbb{F} \mid u > 1 \,\}$ to the bounded open interval $\{\, v \in \mathbb{F} \mid 0 < v < 1 \,\}$.■

Betweenness and cross ratios

Not surprisingly, there are important relationships between the concept of betweenness and the notion of cross ratio. Here is the most basic result.

THEOREM VI.21. *Let* $\mathbb{F}$ *be an ordered field, and let* $\mathrm{J} : \mathbb{F}^n \to \mathbb{FP}^n$ *be the usual projective extension mapping. Then three collinear points* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *of* V *satisfy the order relation* $\boldsymbol{\mathcal{B}}(\mathbf{a},\mathbf{c},\mathbf{b})$ *if and only if*

$$\mathsf{XR}\big(\,\mathrm{J}(\mathbf{a}),\mathrm{J}(\mathbf{b}),\mathrm{J}(\mathbf{c}),L_\infty\,\big) \ < \ 0$$

where L_∞ *is the ideal point of the line* L *containing* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$.

Proof. By Theorem V.17, if $\mathbf{c} = t\mathbf{a} + (1-t)\mathbf{b}$ then

$$\mathsf{XR}\big(\,\mathrm{J}(\mathbf{a}),\mathrm{J}(\mathbf{b}),\mathrm{J}(\mathbf{c}),L_\infty\,\big) \ = \ \frac{t-1}{t}\ .$$

This is negative if $0 < t < 1$ because $t - 1 < 0 < t$. We claim that the cross ratio is positive if either $t < 0$ or $t > 1$. If $t > 1$, then $t - 1 > 0$ and therefore the cross ratio is positive. Similarly, if $t < 0$, then $t - 1 < t < 0$ implies that the cross ratio is positive.■

Affine transformations obviously preserve betweenness (see Exercise 10 below). However, if $\boldsymbol{\mathcal{B}}(\mathbf{a},\mathbf{b},\mathbf{c})$ in $\mathbb{F}^n$ and T is a projective collineation of $\mathbb{FP}^n$ such that the images $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ of

$\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ under T lie in (the image of) $\mathbb{F}^n$, then $\boldsymbol{\mathcal{B}}(\mathbf{a}', \mathbf{b}', \mathbf{c}')$ is not necessarily true. Specific examples are given by projective collineations which interchange $\mathbf{a}$ and $\mathbf{b}$, and send $\mathbf{c}$ to itself.

If one wants some aspect of order and betweenness which IS preserved by projective collineations, it is natural to try something involving the cross ratio, and the preceding result may be viewed as motivation for the following definition and theorem:

Definition. Let $\mathbb{F}$ be an ordered field, let V be a vector space over $\mathbb{F}$, and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be collinear vectors in V. Then $\mathbf{a}$ and $\mathbf{b}$ *separate* $\mathbf{c}$ and $\mathbf{d}$ if one of $\{\mathbf{c}, \mathbf{d}\}$ is between $\mathbf{a}$ and $\mathbf{b}$ but the other is not. We shall write this as $\mathsf{sep}(\mathbf{a}, \mathbf{b} : \mathbf{c}, \mathbf{d})$. It is trivial to see that $\mathsf{sep}(\mathbf{a}, \mathbf{b} : \mathbf{c}, \mathbf{d})$ is equivalent to $\mathsf{sep}(\mathbf{a}, \mathbf{b} : \mathbf{d}, \mathbf{c})$.

There is a very simple and important characterization of separation in terms of cross ratios.

THEOREM VI.22. *Let $\mathbb{F}$ be an ordered field, let V be a vector space over $\mathbb{F}$, and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be collinear vectors in V. Then $\mathsf{sep}(\mathbf{a}, \mathbf{b} : \mathbf{c}, \mathbf{d})$ is true if and only if*

$$\big(\, \mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}), \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d}) \,\big) \;<\; 0 \;.$$

Proof. *Suppose that $\mathsf{sep}(\mathbf{a}, \mathbf{b} : \mathbf{c}, \mathbf{d})$ is true.* Without loss of generality, we may assume that $\boldsymbol{\mathcal{B}}(\mathbf{a}, \mathbf{c}, \mathbf{b})$ is true but $\boldsymbol{\mathcal{B}}(\mathbf{a}, \mathbf{d}, \mathbf{b})$ is false (either this holds or else the corresponding statement with $\mathbf{c}$ and $\mathbf{d}$ interchanged is true – in the latter case, reverse the roles of the two points). Under these conditions we have $\mathbf{c} \;=\; t\mathbf{a} + (1-t)\mathbf{b}$ where $0 < t < 1$ and $\mathbf{d} \;=\; s\mathbf{a} + (1-s)\mathbf{b}$ where $s < 0$ or $s > 1$. By Theorem V.17 we have

$$\mathsf{XR}\big(\, \mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}), \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d}) \,\big) \;=\; \frac{s(1-t)}{t(1-s)} \;.$$

The sign of this cross ratio equals the sign of $s/(1-s)$, and the latter is negative if either $s < 0$ or $s > 1$.$\square$

Suppose that the cross ratio is negative. We need to show that one of s and t satisfies $0 < u < 1$ and the other does not. To do this, we eliminate all the other possibilities.

CASE 1. Suppose we have $0 < s, t < 1$. Then all the factors of the numerator and denominator are positive.

CASE 2. Suppose neither satisfies $0 < u < 1$. Then the previous argument shows that one of s and $1-s$ is positive and likewise for t and $1-t$. Therefore the formula above implies that the cross ratio must be positive.$\blacksquare$

We thus make the general definition in $\mathbb{FP}^n$ that two points A and B **separate** two points C and D on AB if and only if $\mathsf{XR}(A, B, C, D) \;<\; 0$. If all four of these points are ordinary, then Theorem 22 provides a geometrical description of separation. The cases where one point is ideal can be described using the following two special cases:

(1) $\mathsf{sep}\big(\, \mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}) : \infty, \mathrm{J}(\mathbf{c}) \,\big)$ and $\mathsf{sep}(\mathrm{J}(\mathbf{a}), \mathrm{J}(\mathbf{b}) : \mathrm{J}(\mathbf{c}), \infty)$ hold if and only if $\boldsymbol{\mathcal{B}}(\mathbf{a}, \mathbf{c}, \mathbf{b})$ is true (see Theorems V.12 and V.17).

(2) $\mathsf{sep}\big(\, \mathrm{J}(\mathbf{a}), \infty : \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d}) \,\big)$ and $\mathsf{sep}\big(\, \infty, \mathrm{J}(\mathbf{a}) : \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d}) \,\big)$ hold if and only if $\boldsymbol{\mathcal{B}}(\mathbf{c}, \mathbf{a}, \mathbf{d})$ is true because

$$\mathsf{XR}\big(\, \mathrm{J}(\mathbf{a}), \infty : \mathrm{J}(\mathbf{c}), \mathrm{J}(\mathbf{d}) \,\big) \;=\; \mathsf{XR}\big(\, \mathrm{J}(\mathbf{d}), \mathrm{J}(\mathbf{c}) : \infty, \mathrm{J}(\mathbf{a}) \,\big) \;.$$

The following observation is an immediate consequence of the definitions:

> *Let A, B, C, D be distinct collinear points in $\mathbb{FP}^n$, and let T be a projective collineation of $\mathbb{FP}^n$. Then* $\mathsf{sep}(A, B : C, D)$ *is true if and only if* $\mathsf{sep}\big(T(A), T(B) : T(C), T(D)\big)$ *is true.*■

A comprehensive visualization of separation for points on a real projective line may be given as follows:

As indicated in the picture below, there is a standard 1–1 correspondence (*stereographic projection*) between the points of $\mathbb{RP}^1$ and the points on the circle Γ in $\mathbb{R}^2$ which is tangent to the x-axis at the origin and whose center is $(0, \frac{1}{2})$. An ordinary point with standard affine coordinate u is sent to the intersection of Γ with the line joining $(u, 0)$ to $(0, 1)$, and the point at infinity is sent to $(0, 1)$. It is straightforward to check that this map σ defines a 1–1 correspondence from $\mathbb{RP}^1$ to Γ.[3]

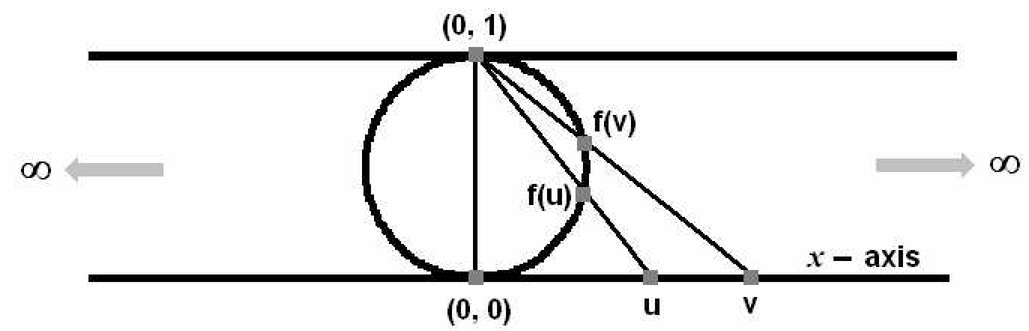

Figure VI.3

With respect to this correspondence, separation has the following interpretation. If $a,\ b \in \mathbb{RP}^1$, then $\Gamma - \{\sigma(a), \sigma(b)\}$ consists of two open arcs, and separation means that each arc contains exactly one of the points $\{c, d\}$.

We now summarize some basic properties of separation by means of the following theorem:

THEOREM VI.23. *If $\mathbb{F}$ is an ordered field and A, B, C, D are distinct collinear points of $\mathbb{FP}^n$, then the following hold:*

(a) $\mathsf{sep}(A, B : C, D)$ *implies* $\mathsf{sep}(A, B : D, C)$ *and* $\mathsf{sep}(C, D : A, B)$.

(b) One and only one of the relations $\mathsf{sep}(A, B : C, D)$, $\mathsf{sep}(B, C : D, A)$, *or and* $\mathsf{sep}(C, A : B, D)$ *is true.*

(c) If $\mathsf{sep}(A, B : C, D)$ *and* $\mathsf{sep}(B, C : D, E)$ *are true, then so is* $\mathsf{sep}(C, D : E, A)$.

(d) If L is a line meeting AB, Y is a coplanar point on neither line, and X' is the intersection point of PX and L for $X = A, B, C, D$, then $\mathsf{sep}(A, B : C, D)$ *implies* $\mathsf{sep}(A', B' : C', D')$.

[3]If we rotate the above picture about the y-axis in $\mathbb{R}^3$ we obtain a similar 1–1 correspondence between the complex projective line $\mathbb{CP}^1$ and the sphere of diameter 1 tangent to the xz-plane at the origin.

The proof is straightforward and is left as an exercise.■

One reason for listing the preceding four properties is that they come close to providing a complete characterization of separation.

THEOREM VI.24. *Let P be a Desarguian projective n-space, where $n \geq 2$, and suppose that P has an abstract notion of separation $\Sigma(\cdots,\cdots||\cdots,\cdots)$ which satisfies the four properties in the previous theorem. Assume that some (hence every) line contains at least four points. Then P is isomorphic to $\mathbb{FP}^n$, where $\mathbb{F}$ is an ordered skew-field, and the ordering of $\mathbb{F}$ has the property that* $\mathsf{sep}(A, B : C, D)$ *is true if and only if* $\Sigma(\cdots,\cdots||\cdots,\cdots)$ *is.*

In principle, this result is proved on pages 239–244 of Artzy, *Linear Geometry*. We say "in principle" because the result is only stated for projective planes in which Pappus' Theorem holds. However, the latter is not used explicitly in the argument on these pages,[4] and the restriction to planes is easily removed.

We would need only one more axiom to give a completely synthetic characterization of the real projective plane (and similarly for higher dimensional real projective spaces). Fairly readable formulations of the required *continuity condition* (as it is called) may be found in Coxeter, *The Real Projective Plane*, pages 161–162, and Artzy (*op. cit.*), page 244.

EXERCISES

Throughout these exercises $\mathbb{F}$ denotes and ordered field, and the ordering is given by the usual symbolism

1. In the real numbers $\mathbb{R}$, prove that the closed interval $[a, b]$, consisting of all x such that $a \leq x \leq b$, is equal to the closed segment $[a; b]$ joining a to b as defined here, and likewise for their open analogs (a, b) and $(a; b)$. [*Hint:* If $a \leq c \leq b$ and $t = (b - a)/(c - a)$, consider $ta + (1-t)b$. If $c = ta + (1-t)b$ for $0 \leq t \leq 1$, why does this and $a \leq b$ imply that $a \leq c \leq b$?]

Definition. A subset $K \subset \mathbb{F}^n$ is **convex** if $\mathbf{x}$ and $\mathbf{y}$ in K imply that the closed segment $[\mathbf{x}; \mathbf{y}]$ is contained in K. — In physical terms for, say, $\mathbb{R}^2$ or $\mathbb{R}^3$, this means that K has "no dents or holes."

2. Prove that the following subsets of $\mathbb{F}$ are convex for an arbitrary $b \in \mathbb{F}$:

(i) The set $\{ x \in \mathbb{F} \mid x > b \}$.

(ii) The set $\{ x \in \mathbb{F} \mid x < b \}$.

(iii) The set $\{ x \in \mathbb{F} \mid x \geq b \}$.

(iv) The set $\{ x \in \mathbb{F} \mid x \leq b \}$.

3. Prove that the intersection of an arbitrary family of convex subsets of $\mathbb{F}^n$ is also convex.

4. Let $f : \mathbb{F}^n \to \mathbb{F}$ be a linear function of the form $f(\mathbf{x}) = \sum_i a_i x_i - b$.

[4] An explicit recognition that Pappus' Theorem is unnecessary appears in Forder, *Foundations of Euclidean Geometry*, pp. 196–197 and 203–206.

(a) Prove that for all $t \in \mathbb{F}$ we have $f\big(t\mathbf{x} + (1-t)\mathbf{y}\big) = t \cdot f(\mathbf{x}) + (1-t)f(\mathbf{y})$.

(b) Prove that if $K \subset \mathbb{F}^n$ is convex, then so is its image $f[K]$.

(c) Prove that if $C \subset \mathbb{F}$ is convex, then so is its inverse image $f^{-1}[C]$.

5. Let f be as in Exercise 4. Then the subsets of $\mathbb{F}^n$ on which f is positive and negative are called the *(two)* sides *of the hyperplane* H_f defined by f or the *(two)* half-spaces *determined by the hyperplane* H_f. Prove that each half-space is (nonempty and) convex, and if we have points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{F}^n$ such that $\mathbf{x}$ lies on one of the half-spaces and $\mathbf{y}$ lies on the other, then the closed segment $[\mathbf{x};\mathbf{y}]$ contains a point of the hyperplane H_f defined by the equation $f(\mathbf{z}) = 0$. — This statement is called the *hyperplane separation property* for $\mathbb{F}^n$.

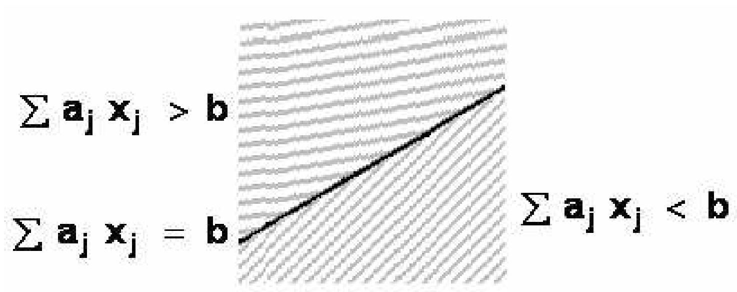

Figure VI.4

Also, explain why the hyperplane and its two sides are three pairwise disjoint subsets whose union is all of $\mathbb{F}^n$.

6. Formulate and prove a similar result to Exercise 5 for the set of all points in a k-plane $M \subset \mathbb{F}^n$ which are not in a $(k-1)$-plane $Q \subset M$.

7. Suppose that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are noncollinear points in $\mathbb{F}^2$. Define the **classical triangle** $\Delta^c\mathbf{xyz}$ to be the union of the closed segments $[\mathbf{x};\mathbf{y}]$, $[\mathbf{x};\mathbf{z}]$, and $[\mathbf{y};\mathbf{z}]$. Prove the **Theorem of Pasch:**[5] A line L containing a point $\mathbf{w}$ in an open side $(\mathbf{x};\mathbf{y})$ of $\Delta^c\,\mathbf{xyz}$ either passes through $\mathbf{z}$ or else meets one of the other open sides $[\mathbf{x};\mathbf{z}]$ or $[\mathbf{x};\mathbf{z}]$. [*Hint:* Explain why $\mathbf{x}$ and $\mathbf{y}$ are on opposite sides of the L through $\mathbf{w}$. What can be said about $\mathbf{z}$ if it does not lie on this line?]

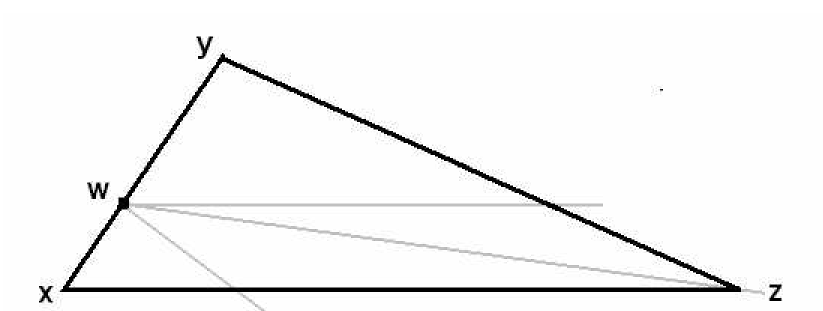

Figure VI.5

[5] MORITZ PASCH (1843–1930) is mainly known for his work on the foundations of geometry, and especially for recognizing the logical deficiencies in Euclid's *Elements* and developing logically rigorous methods for addressing such issues. The theorem in the exercise is one example of a geometrical result that is tacitly assumed – but not proved — in the *Elements*.

7. If $\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w}$ are distinct points in $\mathbb{F}^2$ such that no three are noncollinear, then the *classical quadrilateral* $\square^c\,\mathbf{xyzw}$ is the set

$$[\mathbf{x};\mathbf{y}]\ \cup\ [\mathbf{y};\mathbf{z}]\ \cup\ [\mathbf{z};\mathbf{w}]\ \cup\ [\mathbf{w};\mathbf{x}]\ .$$

It is called a *convex quadrilateral* if the following conditions hold:

$\mathbf{x}$ and $\mathbf{y}$ lie on the same side of $\mathbf{zw}$.
$\mathbf{y}$ and $\mathbf{z}$ lie on the same side of $\mathbf{wx}$.
$\mathbf{z}$ and $\mathbf{w}$ lie on the same side of $\mathbf{xy}$.
$\mathbf{w}$ and $\mathbf{x}$ lie on the same side of $\mathbf{yz}$.

The *diagonals* of a classical quadrilateral $\square^c\,\mathbf{xyzw}$ are the segments $[\mathbf{x};\mathbf{z}]$ and $[\mathbf{y};\mathbf{w}]$. Prove that the diagonals of a convex quadrilateral have a point in common. Why must this point lie on $(\mathbf{x};\mathbf{z})\ \cap\ (\mathbf{y};\mathbf{w})$?

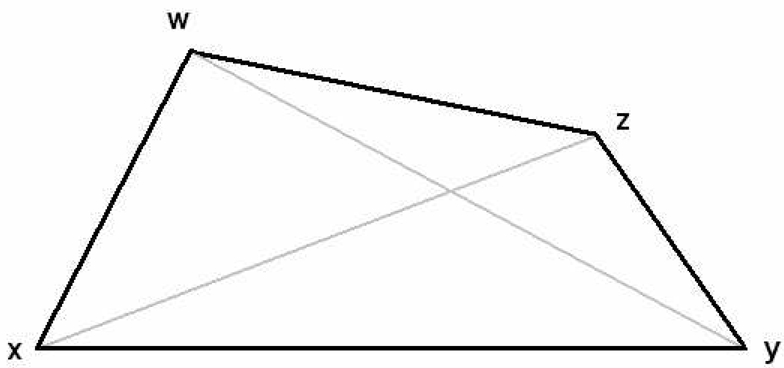

Figure VI.6

8. Give an explicit formula for the map defined by Figure VI.3 and the accompanying discussion.

9. Prove Theorem 24.

10. Suppose that $\mathbf{x},\ \mathbf{y},\ \mathbf{z}$ are points in $\mathbb{F}^n$ such that $\boldsymbol{\mathcal{B}}(\mathbf{x},\mathbf{y},\mathbf{z})$ is true and $T \in \mathrm{Aff}(\mathbb{F}^n)$. Prove that $\boldsymbol{\mathcal{B}}\big(T(\mathbf{x}),T(\mathbf{y}),T(\mathbf{z})\big)$ is also true.

11. In the notation of Exercise 5, let $\mathbf{y}_1,\ \cdots\ ,\mathbf{y}_n$ be an affine basis for the hyperplane H under consideration, and let $\mathbf{y}_0 \notin H$. Prove that $\mathbf{x} \in \mathbb{F}^n$ lies on the same side of H as $\mathbf{y}_0$ if the 0^{th} barycentric coordinate of $\mathbf{x}$ with respect to $\mathbf{y}_0,\ \mathbf{y}_1,\ \cdots\ ,\mathbf{y}_n$ (an affine basis for $\mathbb{F}^n$) is positive. What is the condition for $\mathbf{x}$ and $\mathbf{y}_0$ to lie on opposite sides?