Multifractal Analysis of Mass Distributions

John A. Rock

November 30, 2006

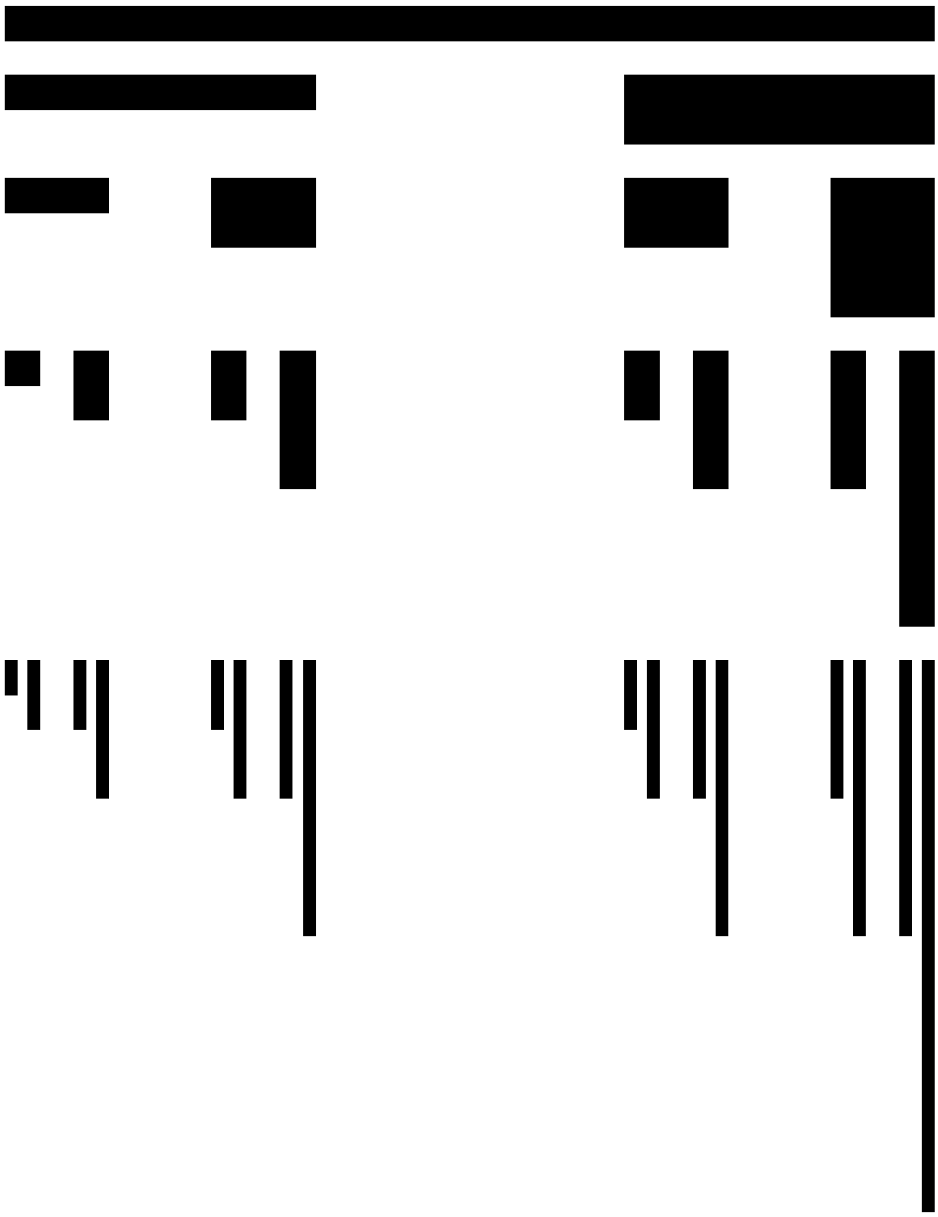

Figure 1: The first few stages in the construction of a mass distribution on the Cantor Set. At each stage there are 2^n segments of length 3^{-n}, each with mass split from the previous stage in ratios 1/3 on the left and 2/3 on the right.

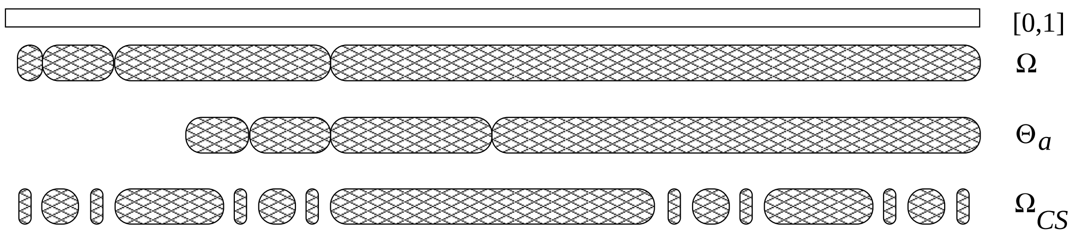

Figure 2: Three fractal strings.

Definition 1. *A* <u>*fractal string*</u> Ω *is a bounded open subset of the real line.*

A fractal string Ω has a sequence of lengths $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$ with the sequence of *distinct* lengths $\{l_n\}_{n=1}^{\infty}$ and multiplicities $\{m_n\}_{n=1}^{\infty}$.

Definition 2. *The* <u>*geometric zeta function*</u> *of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s,$$

where $\mathrm{Re} s > D_{\mathcal{L}}$.

$$\dim_M(\partial\Omega) = D_{\mathcal{L}} = \inf\left\{ r \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^r < \infty \right\}.$$

Let $\mathbf{X}([0,1])$ denote the space of closed subintervals of $[0,1]$.

Definition 3. *The regularity $A(U)$ of a Borel measure μ on $U \in \mathbf{X}([0,1])$ with range in $[0,\infty]$ is*

$$A(U) = \frac{\log \mu(U)}{\log |U|},$$

where $|\cdot| = \lambda(\cdot)$ is the Lebesgue measure on $\mathbb{R}$.

Equivalently, $A(U)$ is the exponent α that satisfies

$$|U|^{\alpha} = \mu(U).$$

We will consider regularity values α in the extended real numbers $[-\infty, \infty]$, where

$$\alpha = \infty = A(U) \Leftrightarrow \mu(U) = 0 \text{ and } |U| > 0,$$

and

$$\alpha = -\infty = A(U) \Leftrightarrow \mu(U) = \infty \text{ and } |U| > 0.$$

For each point $x \in [0,1]$, the regularity is defined as

$$\alpha_n(x) = A(U_n(x)),$$

where $U_n(x)$ is the 2^{-n}-dyadic subinterval of $[0,1]$ that contains x. Define

$$\begin{aligned} E_\alpha(\varepsilon, N) &:= \{x \mid n \geq N \Rightarrow |\alpha_n(x) - \alpha| \leq \varepsilon\}, \\ E_\alpha(\varepsilon) &:= \sup_{N \in \mathbb{N}} E_\alpha(\varepsilon, N) \\ &= \{x \mid \exists N \text{ such that } n \geq N \Rightarrow |\alpha_n(x) - \alpha| \leq \varepsilon\}. \\ E_\alpha &:= \lim_{\varepsilon \to 0} E_\alpha(\varepsilon) \\ &= \{x \mid \alpha_n(x) \to \alpha\}. \end{aligned}$$

Definition 4. *For any dimension* dim *and any real* α*, define the* *dimension spectra*

$$\begin{aligned} f_{\dim}(\alpha) &= \dim(E_\alpha) = \dim(\lim_{\varepsilon \to 0} \sup_{N \in \mathbb{N}} E_\alpha(\varepsilon, N)), \\ f_{\dim}^{\lim}(\alpha) &= \lim_{\varepsilon \to 0} \dim(E_\alpha(\varepsilon)) = \lim_{\varepsilon \to 0} \dim(\sup_{N \in \mathbb{N}} E_\alpha(\varepsilon, N)), \\ f_{\dim}^{\lim\sup}(\alpha) &= \lim_{\varepsilon \to 0} \sup_{N \in \mathbb{N}} \dim(E_\alpha(\varepsilon, N)). \end{aligned}$$

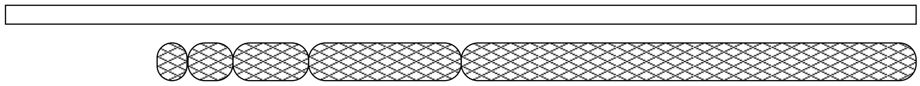

Figure 3: The a-String with $a = 1$.

Example 5 (**A Measure on the a-String**).

$$\nu = \lambda + \sum_{j=1}^{\infty} c_j \delta_{j-1},$$

where λ is the Lebesgue measure on $\mathbb{R}$ and $\sum_{j=1}^{\infty} c_j < \infty$, with $c_j > 0$ for all $j \geq 1$.

$$f_{\dim_M}^{\limsup}(\alpha) < f_{\dim_M}^{\lim}(\alpha),$$

where $\dim_M$ denotes the Minkowski dimension. The boundary of the a-String has Minkowski dimension $1/(1+a)$. For the spectra, the results are as follows:

$$\begin{aligned} f_{\dim_M}^{\limsup}(0) &= 0, \\ f_{\dim_M}^{\lim}(0) &= f_{\dim_M}(0) = \dim_M(\partial\Theta_a) = 1/2, \\ f_{\dim_M}^{\limsup}(1) &= f_{\dim_M}^{\lim}(1) = f_{\dim_M}(1) = 1. \end{aligned}$$

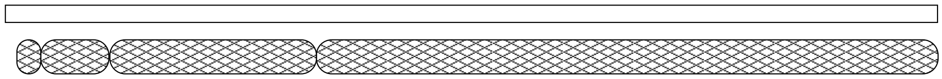

Figure 4: The complement of the support of σ.

Example 6 (**Another simple Measure**).

Consider the measure

$$\sigma = \lambda + \sum_{n=1}^{\infty} 3^{-n} \delta_{3^{-n}}.$$

The complement of the support of the singular part of σ, $\sum_{n=1}^{\infty} 3^{-n} \delta_{3^{-n}}$, is the fractal string Ω with sequence of lengths $\mathcal{L} = \{2 \cdot 3^{-n}\}_{n \in \mathbb{N}}$. Its geometric zeta function is

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \left(\frac{2}{3^n}\right)^s = \frac{2^s \cdot 3^{-s}}{1 - 3^s}.$$

The Minkowski dimension of $\partial\Omega$ is zero, but in the non-trivial manner discussed below.

$$\begin{aligned} f_{\dim_M}^{\limsup}(0) &= f_{\dim_M}^{\lim}(0) = f_{\dim_M}(0) = \dim_M(\partial\Omega_a) = 0, \\ f_{\dim_M}^{\limsup}(1) &= f_{\dim_M}^{\lim}(1) = f_{\dim_M}(1) = 1. \end{aligned}$$

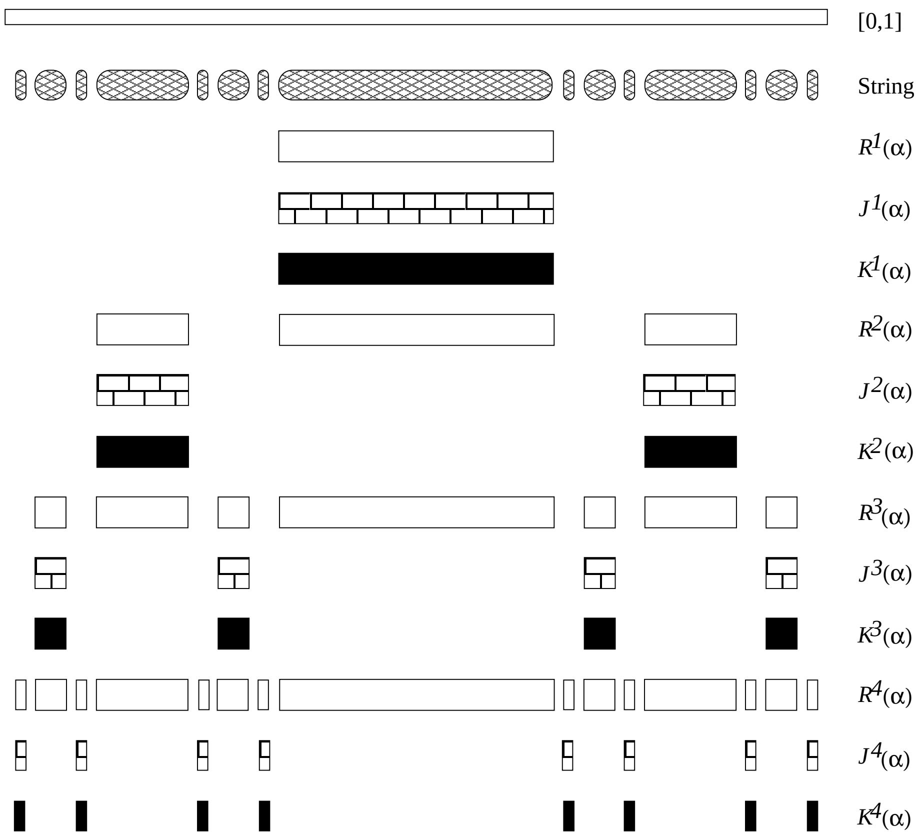

Figure 5: Construction of the multifractal zeta function $\zeta_{\mathcal{N}}^{\mu_{CS}}(\infty, s)$ where μ_{CS} is a measure supported on the Cantor Set and $\mathcal{N} = \{3^{-n-1}\}_{n=1}^{\infty}$.

Definition 7. *The* <u>*multifractal zeta function*</u> *of a measure μ, sequence $\mathcal{N} = \{\eta_n\}_{n=1}^{\infty}$ such that $\eta_n \searrow 0$ and associated regularity value $\alpha \in [-\infty, \infty]$ is*

$$\zeta_{\mathcal{N}}^{\mu}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{p=1}^{k_n(\alpha)} |K_p^n(\alpha)|^s,$$

for $\mathrm{Re}s$ *large enough.*

Definition 8. *For a measure* μ*, sequence* $\mathcal{N}$ *such that* $\eta_n \searrow 0$ *and regularity value* α*, the set of* <u>*complex dimensions with parameter* α</u> *is*

$$\mathcal{D}^{\mu}_{\mathcal{N}}(\alpha, W) = \{\omega \in W \mid \zeta^{\mu}_{\mathcal{N}}(\alpha, s) \text{ has a pole at } \omega\}.$$

Theorem 9. *The multifractal zeta function of the Borel measure* μ *with range in* $[0, \infty]$*, any sequence* $\mathcal{N}$ *such that* $\eta_n \searrow 0$ *and regularity* $\alpha = \infty$ *is the geometric zeta function of* $(supp(\mu))^c$*. That is,*

$$\zeta^{\mu}_{\mathcal{N}}(\infty, s) = \zeta_{\mathcal{L}_\mu}(s),$$

where $\mathcal{L}_\mu$ *is the sequence of lengths for the complement of the support of* μ*.*

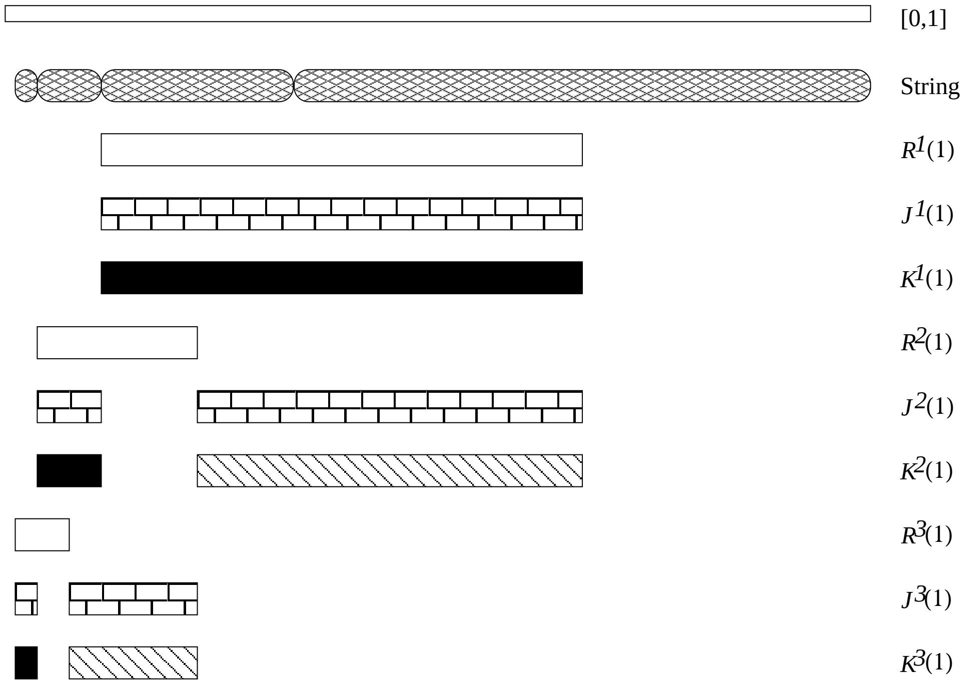

Figure 6: Construction of $\zeta_{\mathcal{N}}^{\sigma_1}(1,s)$.

Consider the measure

$$\sigma_1 = \sum_{n=1}^{\infty} 3^{-n}\delta_{3^{-n}}.$$

This measure has a nontrivial (not entire) multifractal zeta function for regularity $\alpha = 1$ and scales $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$.

$$\begin{aligned}\zeta_{\mathcal{N}}^{\sigma_1}(1,s) &= \left(\frac{1}{3}+\frac{2}{9}\right)^s + \sum_{n=2}^{\infty}\left(\frac{2}{3^{n+1}}\right)^s \\ &= \left(\frac{5}{9}\right)^s + \left(\frac{2}{27}\right)^s\left(\frac{1}{1-3^{-s}}\right).\end{aligned}$$

The sequence $\mathcal{L}_{\sigma_1}$ is $\{2\cdot 3^{-n}\}_{n\in\mathbb{N}}$ where each length has multiplicity one. Therefore

$$\begin{aligned} \zeta_{\mathcal{N}}^{\sigma_1}(\infty, s) &= \sum_{n=1}^{\infty}(2\cdot 3^{-n})^s \\ &= 2^s\sum_{n=1}^{\infty}3^{-ns} \\ &= \frac{2^s\cdot 3^{-s}}{1-3^{-s}}. \end{aligned}$$

The poles of $\zeta_{\mathcal{N}}^{\sigma_1}(\infty, s)$ are the same as $\zeta_{\mathcal{N}}^{\sigma_1}(1, s)$.

$$\mathcal{D}_{\mathcal{N}}^{\sigma_1}(1,\mathbb{C}) = \mathcal{D}_{\mathcal{N}}^{\sigma_1}(\infty,\mathbb{C}) = \left\{\frac{2\pi i m}{\log 3} \mid m\in\mathbb{Z}\right\}.$$