Multifractal Analysis of Mass Distributions, Part II

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"A date which will live in infamy."
– *Franklin D. Roosevelt*

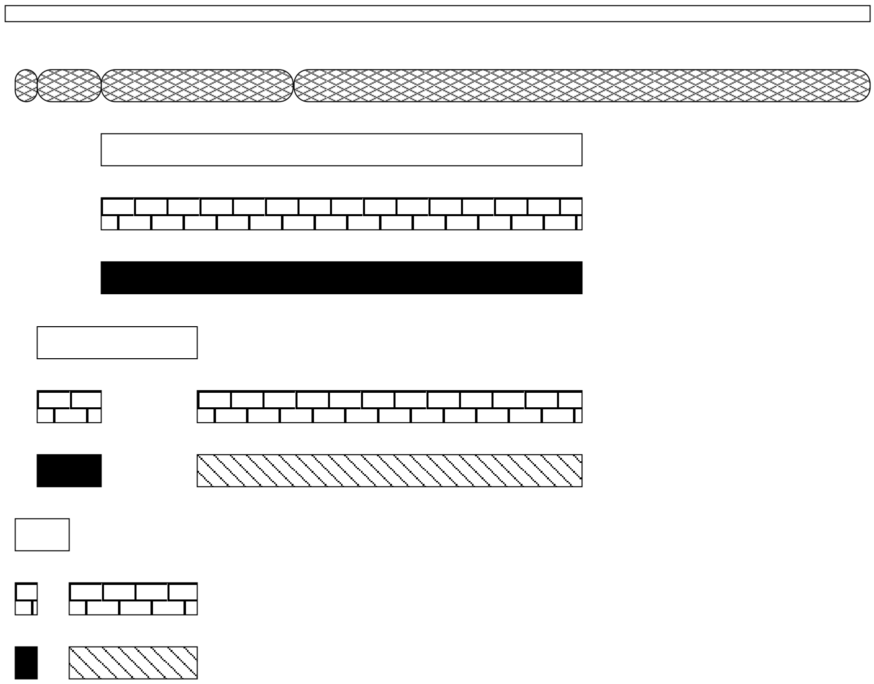

Figure 1: *Construction of* $\zeta_{\mathcal{N}}^{\sigma_1}(1, s)$.

Recall the measure

$$\sigma_1 = \sum_{n=1}^{\infty} 3^{-n} \delta_{3^{-n}}.$$

This measure has a nontrivial (not entire) multifractal zeta function for regularity $\alpha = 1$ and scales $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$.

$$\begin{aligned} \zeta_{\mathcal{N}}^{\sigma_1}(1, s) &= \left(\frac{1}{3} + \frac{2}{9}\right)^s + \sum_{n=2}^{\infty} \left(\frac{2}{3^{n+1}}\right)^s \\ &= \left(\frac{5}{9}\right)^s + \left(\frac{2}{27}\right)^s \left(\frac{1}{1 - 3^{-s}}\right). \end{aligned}$$

The sequence $\mathcal{L}_{\sigma_1}$ is $\{2 \cdot 3^{-n}\}_{n\in\mathbb{N}}$ where each length has multiplicity one. Therefore

$$\zeta_{\mathcal{N}}^{\sigma_1}(\infty, s) = \sum_{n=1}^{\infty}(2 \cdot 3^{-n})^s = \frac{2^s \cdot 3^{-s}}{1 - 3^{-s}}.$$

The poles of $\zeta_{\mathcal{N}}^{\sigma_1}(\infty, s)$ are the same as $\zeta_{\mathcal{N}}^{\sigma_1}(1, s)$. Moreover, they are the *complex dimensions* of the fractal string Ω with lengths $\mathcal{L}_{\sigma_1}$ is $\{2 \cdot 3^{-n}\}_{n\in\mathbb{N}}$.

$$\begin{aligned} \mathcal{D}_{\mathcal{N}}^{\sigma_1}(1, \mathbb{C}) = \mathcal{D}_{\mathcal{N}}^{\sigma_1}(\infty, \mathbb{C}) &= \mathcal{D}_{\mathcal{L}_{\sigma_1}} \\ &= \left\{ \frac{2\pi i m}{\log 3} \mid m \in \mathbb{Z} \right\}. \end{aligned}$$

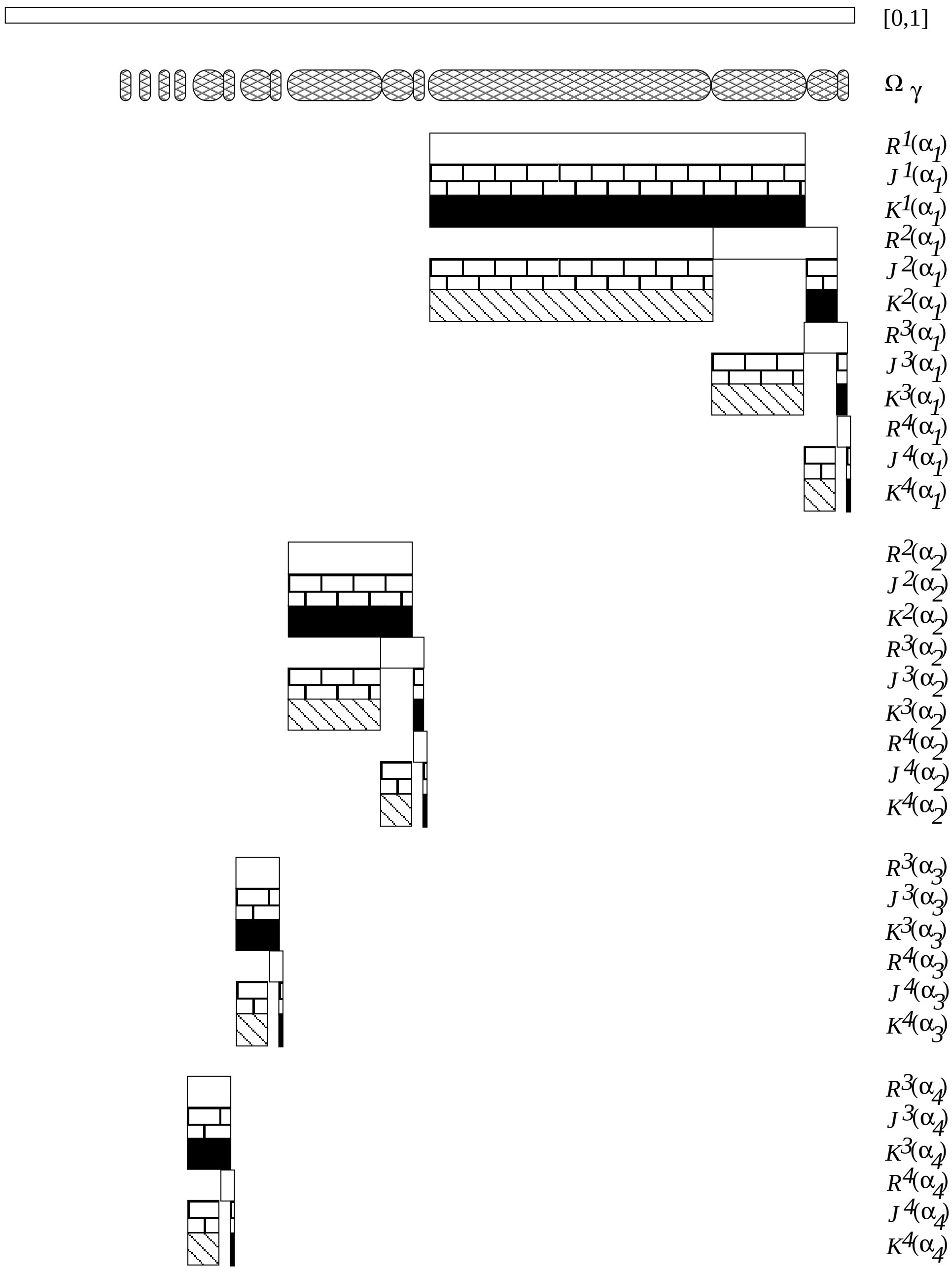

Figure 2: Constructing the multifractal zeta functions of the measure γ.

First, consider the regularity value $\alpha_1 = -\log_3 c_1$ and the multifractal zeta function $\zeta_{\mathcal{N}}^{\gamma}(\alpha_1, s)$. Each $K^n(\alpha_1)$ with $n > 1$ consists of one component of length 3^{-n-1}. Therefore,

$$\begin{aligned}\zeta_{\mathcal{N}}^{\gamma}(\alpha_1, s) &= \left(\frac{4}{9}\right)^s + \sum_{n=1}^{\infty}\left(\frac{1}{3^{n+2}}\right)^s \\ &= 4^s \cdot 9^{-s} + \frac{27^{-s}}{1-3^{-s}}.\end{aligned}$$

The multifractal zeta function of the measure γ, sequence $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$ and regularity value α_j is

$$\zeta_{\mathcal{N}}^{\gamma}(\alpha_j, s) = 4^s \cdot 3^{-(k+2)s} + \frac{3^{-(k+2)s}}{1-3^{-s}},$$

where $2^{k+1} \geq j > 2^k$.

The complex dimensions with parameters α_j all coincide. Namely, for all $j \in \mathbb{N}$,

$$\mathcal{D}_{\mathcal{N}}^{\gamma}(\alpha_j) = \left\{\frac{2m\pi i}{\log 3}\right\}_{m\in\mathbb{Z}.}$$

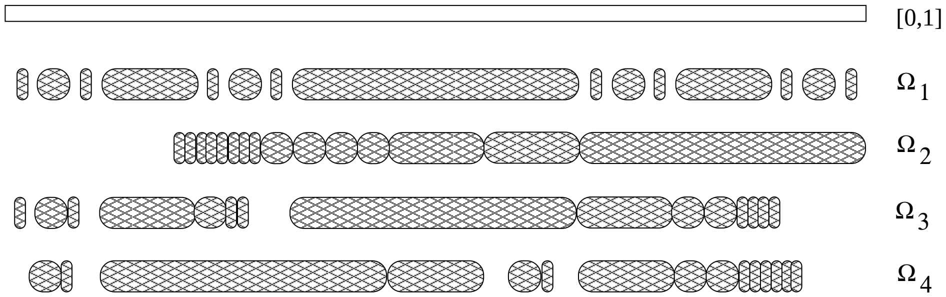

Figure 3: *The first four distinct lengths, with multiplicities, of four variants of the Cantor String.*

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} 2^{n-1} 3^{-ns} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}},$$

where $\mathcal{L}$ is the lengths of the Cantor String Ω_1 (also each Ω_q above), with the distinct lengths $l_n = 3^{-n}$ and multiplicities $m_n = 2^{n-1}$. The set of complex dimensions of each Ω_q above is

$$\mathcal{D}_{\mathcal{L}} = \left\{ \log_3 2 + \frac{2im\pi}{\log 3} \mid m \in \mathbb{Z} \right\}.$$

These fractal strings have identical geometric zeta functions and the same Minkowski dimension, namely $\log_3 2$. However,

$$\begin{aligned}
\dim_H(\partial\Omega_1) &= \dim_M(\partial\Omega_q) = \log_3 2, \\
\dim_H(\partial\Omega_2) &= \dim_H(\partial\Omega_3) = 0, \\
\dim_H(\partial\Omega_4) &= \log_9 2.
\end{aligned}$$

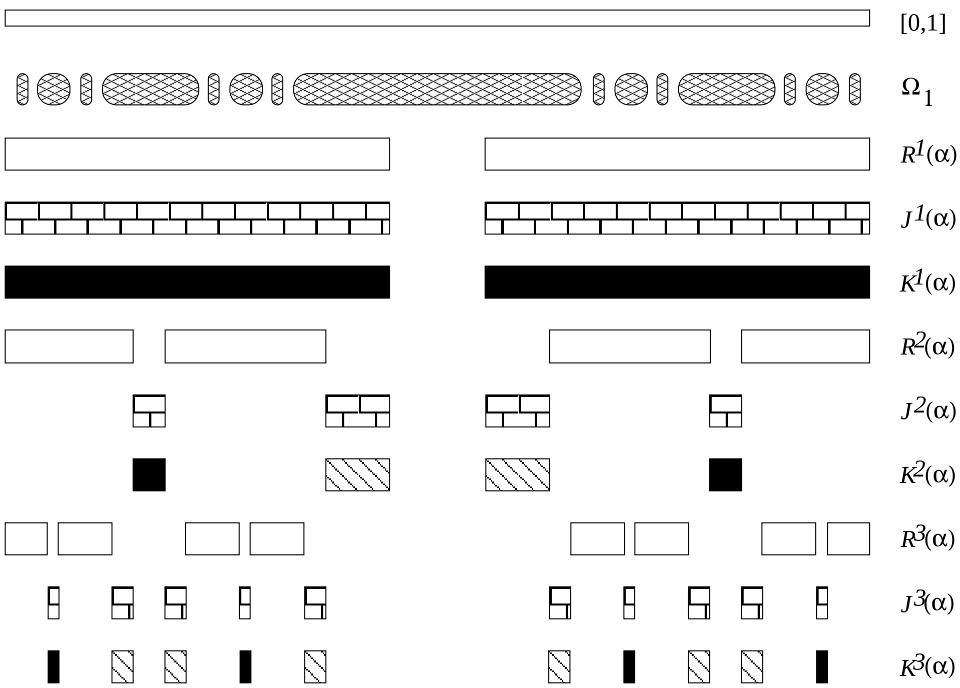

Figure 4: *The first three stages in the construction of* $\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s)$, *where* $\mathcal{N}$ *is the set of distinct lengths of the Cantor String beginning with* $1/9$. *The measure* μ_1 *is supported on the Cantor Set.*

$$\begin{aligned}\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s) &= 2\left(\frac{1}{3}+\frac{1}{9}\right)^s + \sum_{n=2}^{\infty} 2^{n-1}\left(\frac{1}{3^n}-\frac{2}{3^{n+1}}\right)^s \\ &= 2\left(\frac{4}{9}\right)^s + \frac{2}{27^s}\left(\frac{1}{1-2\cdot 3^{-s}}\right),\end{aligned}$$

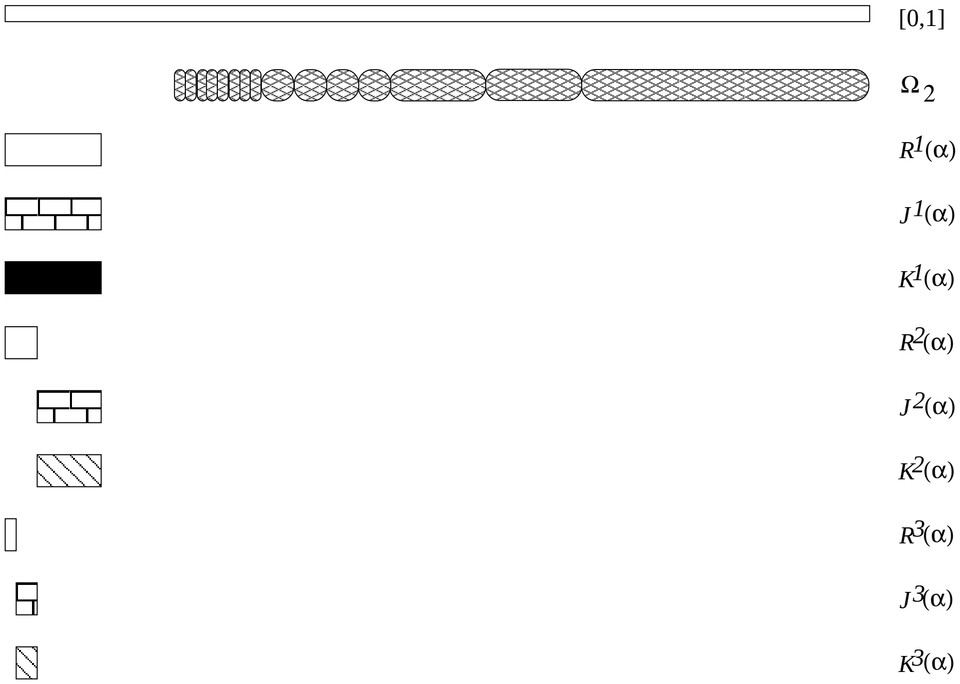

Figure 5: *The first three stages in the construction of* $\zeta_{\mathcal{N}}^{\mu_2}(-\infty, s)$, *where* $\mathcal{N}$ *is the set of distinct lengths of the Cantor String beginning with* $1/9$. *The measure* μ_2 *is supported on a set with a single accumulation point at* 0.

$$\zeta_{\mathcal{N}}^{\mu_2}(-\infty, s) = \eta_1^s = \frac{1}{9^s},$$

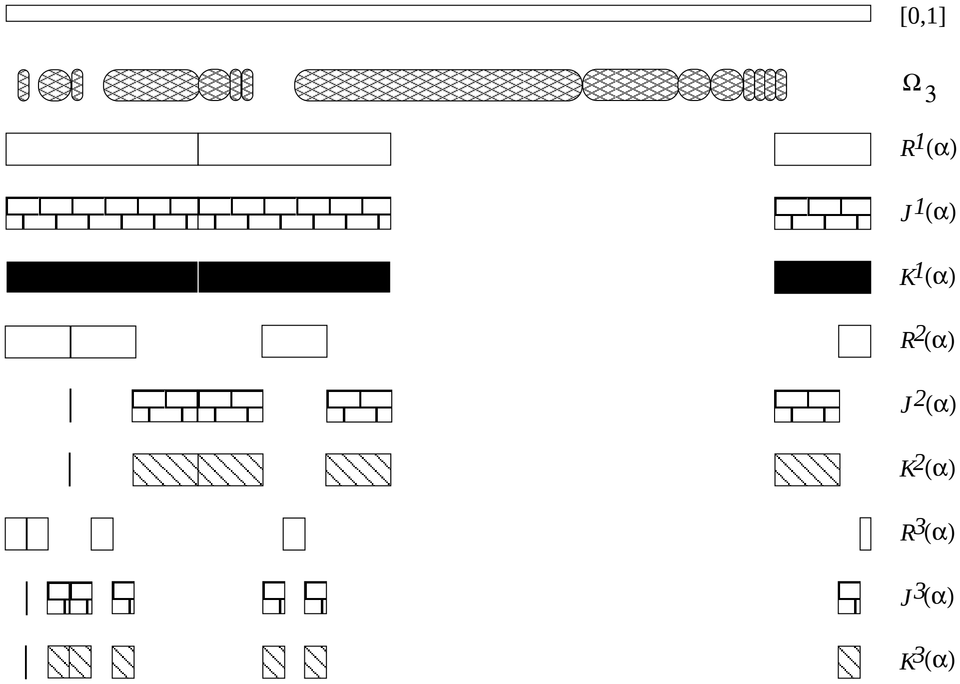

Figure 6: *The first three stages in the construction of* $\zeta_{\mathcal{N}}^{\mu_3}(-\infty, s)$, *where* $\mathcal{N}$ *is the set of distinct lengths of the Cantor String beginning with* $1/9$.

$$\zeta_{\mathcal{N}}^{\mu_3}(-\infty, s) \;=\; \left(\frac{1}{9}\right)^s + \left(\frac{4}{9}\right)^s,$$

This is misleading in that different choices of $\mathcal{N}$ can yield an infinite number of nonzero terms for the resulting multifractal zeta function, thus it may not be entire.

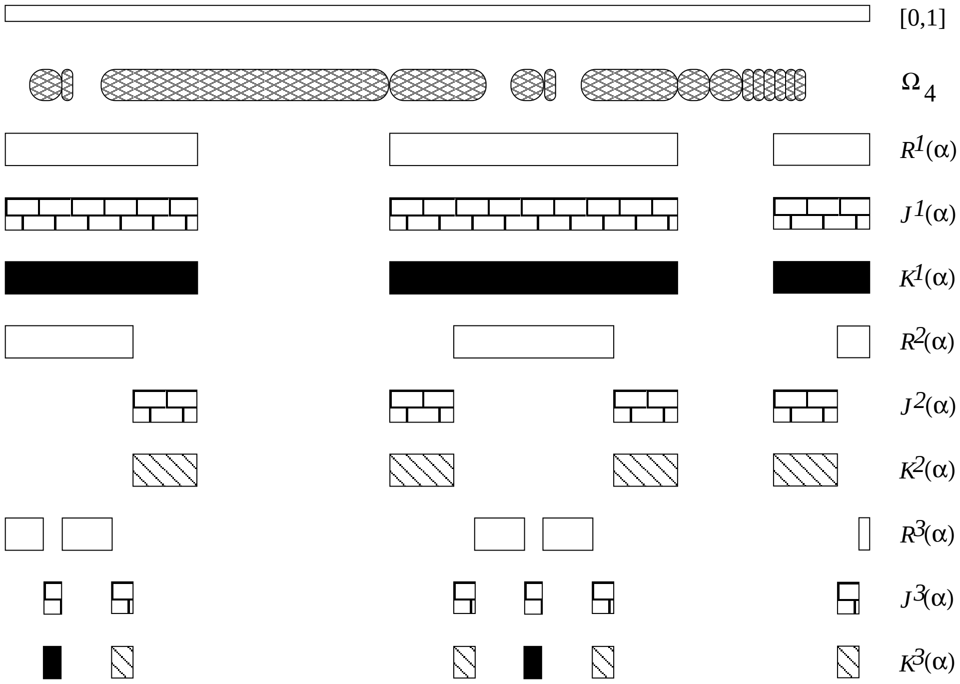

Figure 7: *The first three stages in the construction of* $\zeta_{\mathcal{N}}^{\mu_4}(-\infty, s)$, *where* $\mathcal{N}$ *is the set of distinct lengths of the Cantor String beginning with* $1/9$.

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\mu_4}(-\infty, s) &= h_4(s) + \sum_{n=2}^{\infty} 2^{n-1} \left(\frac{1}{3^{2n-1}} + \frac{1}{3^{2n}} - \frac{2}{3^{2n}} \right)^s \\
&= h_4(s) + \left(\frac{2^{s+1}}{81^s} \right) \left(\frac{1}{1 - 2 \cdot 9^{-s}} \right).
\end{aligned}$$