

FRactal Strings AND MULTIFRACTAL ZETA FUNCTIONS

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Fractal Strings

Definition 1. A fractal string Ω is a bounded open subset of the real line, thus $\Omega = \bigcup_{j=1}^{\infty} (a_j, b_j)$.

Let $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty} = \{|(a_j, b_j)|\}_{j=1}^{\infty}$ and let $\{l_n\}_{n=1}^{\infty}$ be the distinct lengths of \mathcal{L} with multiplicities m_n .

$$D_{\mathcal{L}} = \inf\{\sigma \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^{\sigma} < \infty\}.$$

Definition 2. The geometric zeta function of a fractal string with lengths \mathcal{L} is

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s$$

where $\operatorname{Re}(s) > D_{\mathcal{L}}$.

Definition 3. The set of complex dimensions of a fractal string with lengths \mathcal{L} is

$$\mathcal{D}_{\mathcal{L}}(W) = \{\omega \in W \mid \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}.$$

Multifractal Analysis

Let $\mathbf{X}([0, 1])$ be the space of closed subintervals of $[0, 1]$.

Definition 4. The regularity of a Borel measure μ on $U \in \mathbf{X}([0, 1])$ is

$$A(U) = \frac{\log \mu(U)}{\log |U|},$$

where $|\cdot| = m_L(\cdot)$ is the Lebesgue measure.

For $\alpha \in \mathbb{R} \cup \{\pm\infty\}$, let

$$\mathcal{R}_\eta(\alpha) = \{U \in \mathbf{X}([0, 1]) : |U| = \eta \text{ and } A(U) = \alpha\}.$$

Consider

$$\cup_{\mathcal{R}_\eta(\alpha)} U = \cup_{U \in \mathcal{R}_\eta(\alpha)} U.$$

Definition 5. The continuous large deviation spectrum is

$$\begin{aligned} \tilde{f}_g^c(\alpha) &= \limsup_{\eta \rightarrow 0} \frac{\log (|\cup_{\mathcal{R}_\eta(\alpha)} U|/\eta)}{|\log \eta|} \\ &= \limsup_{\eta \rightarrow 0} \left(1 - \frac{\log |\cup_{\mathcal{R}_\eta(\alpha)} U|}{|\log \eta|} \right). \end{aligned}$$

Definition of Multifractal Zeta Function

Given a sequence of positive real numbers $\mathcal{N} = \{\eta_n\}_{n=1}^{\infty}$ that tend to zero and a Borel measure μ , we wish to examine the way μ changes with respect to a fixed regularity α between stages $n - 1$ and n . Thus we consider the symmetric difference, denoted \ominus , between the unions of the closed intervals in $\mathcal{R}_{\eta_{n-1}}(\alpha)$ and $\mathcal{R}_{\eta_n}(\alpha)$. Let

$$J^1(\alpha) = \bigcup_{\mathcal{R}_{\eta_1}(\alpha)} U,$$

and for $n \geq 2$, let

$$J^n(\alpha) = \left(\bigcup_{\mathcal{R}_{\eta_{n-1}}(\alpha)} U \right) \ominus \left(\bigcup_{\mathcal{R}_{\eta_n}(\alpha)} U \right).$$

We have

$$J^n(\alpha) = \bigcup_{i=1}^{k_n(\alpha)} J_i^n(\alpha),$$

where $k_n(\alpha)$ is the number of connected components $J_i^n(\alpha)$ of $J^n(\alpha)$.

Definition 6. The multifractal zeta function of a measure μ and a sequence \mathcal{N} is

$$\zeta_{\mathcal{N}}^{\mu}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n(\alpha)} |J_i^n(\alpha)|^s.$$

Thus, for a fixed μ and \mathcal{N} , each α yields a sequence of lengths along with its geometric zeta function.

Two Results

$$\mu_\Omega = m_L + \sum_{j \in \mathbb{N}} (\delta_{a_j} + \delta_{b_j}),$$

where the (a_j, b_j) are the disjoint open intervals whose union is Ω .

Theorem 1. *Consider a fractal string Ω . For the measure μ_Ω as above and a sequence \mathcal{N} we have*

$$\zeta_{\mathcal{N}}^{\mu_\Omega}(1, s) = \zeta_{\mathcal{L}}(s).$$

Theorem 2. *Assume that a fractal string Ω has a perfect complement and associate to Ω its lengths \mathcal{L} and the measure μ_Ω . Suppose there exists a sequence \mathcal{N} such that $l_n > \eta_n \geq l_{n+1}$ and $l_n > 2\eta_n$. Then,*

$$\begin{aligned}\zeta_{\mathcal{N}}^{\mu_\Omega}(1, s) &= \zeta_{\mathcal{L}}(s) \\ \zeta_{\mathcal{N}}^{\mu_\Omega}(\alpha(n, 1), s) &= 4\left(\sum_{i=1}^n m_i\right)\eta_n^s\end{aligned}$$

$$\begin{aligned}\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s) &= (a_{j(1, m_1)} + \eta_1)^s + (1 - b_{j(1, 1)} + \eta_1)^s \\ &\quad + \sum_{i=1}^{m_1} (2\eta_1 + a_{j(1, i)} - b_{j(1, i+1)})^s \\ &\quad + \sum_{n=2}^{\infty} [m_n(l_n - 2\eta_n)^s \\ &\quad \quad + 2\left(\sum_{i=1}^{n-1} m_i\right)(\eta_{n-1} - \eta_n)^s],\end{aligned}$$

where $j(n, i)$ is the unique $j \in \mathbb{N}$ such that $l_j > \eta_n$ and I_j is the i th interval from the right. All other regularity values α yield identically zero multifractal zeta functions.

Two Examples

Let Ω_1 be the Cantor String and let Ω_2 be the open set with the intervals of lengths the same as the Cantor String arranged in nonincreasing order from right to left. Let

$$\mu_1 = m_L + \sum_{j \in \mathbb{N}} (\delta_{a_j} + \delta_{b_j})$$

and

$$\mu_2 = m_L + \sum_{j \in \mathbb{N}} \delta_{c_j},$$

where the (a_j, b_j) are the disjoint open intervals whose union is Ω_1 and the c_j are the endpoints of Ω_2 .

In [Lap-vF] it is shown that the *complex dimensions* of Ω_1 and Ω_2 are

$$\mathcal{D}_{CS} = \left\{ \log_3 2 + \frac{2i\pi k}{\log 3} : k \in \mathbb{Z} \right\}.$$

These are the poles of

$$\zeta_{\mathcal{N}}^{\mu_1}(1, s) = \zeta_{\mathcal{N}}^{\mu_2}(1, s) = \zeta_{CS}(s) = \frac{1}{1 - 2 \cdot 3^s}.$$

For simplicity, let \mathcal{N} be given by $\eta_n = l_{n+1} = 3^{-n-1}$. For Ω_2 and μ_2 we have

$$\begin{aligned}\zeta_{\mathcal{N}}^{\mu_2}(-\infty, s) &= \frac{1}{9^s} + \sum_{n=2}^{\infty} \left(\frac{1}{3^n} - \frac{1}{3^{n+1}} \right)^s \\ &= \frac{1}{9^s} + \left(\frac{2}{27} \right)^s \left(\frac{1}{1-3^s} \right)\end{aligned}$$

The poles of $\zeta_{\mathcal{N}}^{\mu_2}(-\infty, s)$ are

$$\mathcal{D}_{\mathcal{N}}^{\mu_2}(-\infty) = \left\{ \frac{2i\pi k}{\log 3} \right\}_{k \in \mathbb{Z}}.$$

For Ω_1 and μ_1 we have

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s) &= 2 \left(\frac{1}{3} + \frac{1}{9} \right)^s \\
&\quad + \sum_{n=2}^{\infty} 2^{n-1} \left(\frac{1}{3^n} + \frac{2}{3^{n+1}} \right)^s \\
&\quad + \sum_{n=2}^{\infty} (2^n - 2) \left(\frac{1}{3^n} - \frac{1}{3^{n+1}} \right)^s \\
&= 2 \left(\frac{4}{9} \right)^s \\
&\quad + \left(\frac{5^s}{4} + \frac{2^s}{2} \right) \left(\frac{8}{27^s} \right) \left(\frac{1}{1 - 2 \cdot 3^s} \right) \\
&\quad - 2 \left(\frac{2}{27} \right)^s \left(\frac{1}{1 - 3^s} \right)
\end{aligned}$$

So $\mathcal{D}_{\mathcal{N}}^{\mu_1}(-\infty) = \mathcal{D}_{\mathcal{N}}^{\mu_2}(-\infty) \cup \mathcal{D}_{CS}$.

Result for Singular Measures

Suppose $\sigma \perp m_L$. Since $(\text{supp}(\sigma))^c$ is open, there exists a *fractal string* \mathcal{L}_σ whose lengths are those of the intervals $I_j = (a_j, b_j)$ where $(\text{supp}(\sigma))^c = \bigcup_{j=1}^{\infty} I_j$. Let $\{\ell_j\}_{j=1}^{\infty}$ be the lengths of \mathcal{L}_σ and let $\{l_n\}_{n=1}^{\infty}$ be the distinct lengths of \mathcal{L}_σ with multiplicities m_n .

The following technical lemma is used in the proof of the theorem below.

Lemma 1. *Suppose $\{x\} = \text{supp}(\sigma) \cap U$ where $\sigma \perp m_L$ and $U \in \mathbf{X}([0, 1])$. For $\mu_\sigma = m_L + |\sigma|$,*

$$A(U) = 1 \Leftrightarrow \sigma(\{x\}) = 0$$

Proof:

$$\sigma(\{x\}) \neq 0 \Leftrightarrow \mu_\sigma(U) > |U| \Leftrightarrow A(U) \neq 1.$$

□

Theorem 3. *For $\sigma \perp m_L$, $\mu_\sigma = m_L + |\sigma|$ and \mathcal{N} such that $l_n > \eta_n > l_{n+1}$,*

$$\zeta_{\mathcal{N}}^{\mu_\sigma}(1, s) = \zeta_{\mathcal{L}_\sigma}(s).$$

Proof: $\forall n \in \mathbb{N}, U \in \mathcal{R}_{\eta_n}(1) \Leftrightarrow A(U) = 1$ and $|U| = \eta_n$.

$$\begin{aligned} 1 &= A(U) \\ &= \frac{\log(\mu_\sigma(U))}{\log |U|} \\ &= \frac{\log(|U| + |\sigma|(U))}{\log |U|} \end{aligned}$$

$\therefore \forall n \in \mathbb{N}, U \in \mathcal{R}_{\eta_n}(1)$ only if $|\sigma|(U) = 0$.
So

$$\bigcup_{\mathcal{R}_{\eta}(\alpha)} U = \bigcup_{\ell_j > \eta_n} \bar{I}_j \setminus T_\sigma = \bigcup_{\ell_j \geq l_n} \bar{I}_j \setminus T_\sigma,$$

where $T_\sigma = \{x \in \text{supp}(\sigma) : \sigma(\{x\}) \neq 0\}$. Thus

$$J^n(1) = \bigcup_{\ell_j = l_n} \bar{I}_j \setminus T_\sigma = \bigcup_{i=1}^{k_n(1)} J_i^n(1).$$

The lemma and the comments that follow it imply the existence of a length preserving bijection between the I_j such that $\ell_j = l_n$ and the $J_i^n(1)$. More specifically, $J_i^n(1) = l_n$ and $m_n = k_n(1) \forall i \in \{1, \dots, k_n(1)\}$ and $n \in \mathbb{N}$.

Therefore,

$$\begin{aligned}\zeta_{\mathcal{N}}^{\mu\sigma}(1, s) &= \sum_{n=1}^{\infty} \sum_{i=1}^{k_n(1)} |J_i^n(1)|^s \\ &= \sum_{n=1}^{\infty} m_n(l_n)^s \\ &= \zeta_{\mathcal{L}\sigma}(s)\end{aligned}$$

□

Proof of Theorem 2

Proof: The first formula is a consequence of Theorem 1. For the last statement, note that each point-mass is a limit point of other point-masses. Indeed, the endpoints of Ω are dense in $\partial\Omega$. Since $l_n > \eta_n \geq l_{n+1}$ and $l_n > 2\eta_n$, no closed interval in of size η_n can contain more than one but still less than infinitely many point-masses. Thus the only other α that yield multifractal zeta functions which are not identically zero are $\alpha(n, 1)$ and $-\infty$.

For the second formula, $J^n(\alpha(n, 1)) = J^{n+1}(\alpha(n, 1))$ and $J^m(\alpha(n, 1)) = \emptyset$ for all $n, m \in \mathbb{N}$ such that $m \neq n$ nor $n + 1$, as $\log(x + 1)/\log x$ is strictly decreasing. So there is a factor of 2 in the expansion of the multifractal zeta function with regularity $\alpha(n, 1)$. Since $l_n > \eta_n \geq l_{n+1}$ and $l_n > 2\eta_n$, there are two η_n^s terms corresponding to each of the open intervals of Ω with length $\ell_j > \eta_n$, resulting in another factor of 2. We have

$$\zeta_{\mathcal{N}}^{\mu_{\Omega}}(\alpha(n, 1), s) = 4 \left(\sum_{i=1}^n m_i \right) \eta_n^s.$$

Now we prove the third formula. Since $l_n > 2\eta_n$, we have

$$\bigcup_{\mathcal{R}_{\eta_n}(-\infty)} U = \left(\bigcup_{\ell_j > \eta_n} [a_j + \eta_n, b_j - \eta_n] \right)^C,$$

For $n \geq 2$, $J^n(-\infty)$ is made up of m_n intervals of length $l_n - 2\eta_n$ and $2 \sum_{i=1}^{n-1} m_i$ intervals of length $\eta_{n-1} - \eta_n$. That is, at each stage $n \geq 2$, we pick up two $\eta_{n-1} - \eta_n$ terms for each $\ell_j \geq l_{n-1}$ from previous the stage and one $l_n - 2\eta_n$ term for each $\ell_j = l_n$. Therefore,

$$\begin{aligned} \zeta_{\mathcal{N}}^{\mu\Omega}(-\infty, s) &= (a_{j(1,m_1)} + \eta_1)^s + (1 - b_{j(1,1)} + \eta_1)^s \\ &\quad + \sum_{i=1}^{m_1} (2\eta_1 + a_{j(1,i)} - b_{j(1,i+1)})^s \\ &\quad + \sum_{n=2}^{\infty} [m_n (l_n - 2\eta_n)^s \\ &\quad + 2 \left(\sum_{i=1}^{n-1} m_i \right) (\eta_{n-1} - \eta_n)^s], \end{aligned}$$

□