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MIRABOLIC FLAG VARIETIES: COMBINATORICS AND CONVOLUTION ALGEBRAS

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ABSTRACT

In this thesis we study some combinatorial and algebraic constructions arising from the action of a Lie group on the variety of triples of two flags and a vector. This is called the ‘mirabolic’ setting, compared to the classical setting of just the group action on pairs of flags. A theorem proved both by Spaltenstein and by Steinberg relates the relative position of two complete flags and the irreducible components of the flag variety in which they lie, using the Robinson-Schensted-Knuth correspondence. In Chapter 2, we generalize this result to the case of partial flags. Then we use this to generalize the mirabolic Robinson-Schensted-Knuth correspondence defined by Travkin, to the case of two partial flags and a vector. In chapter 3, we discuss algebras of functions on these varieties of flags that are defined by a convolution product. We start by recalling Solomon’s construction of the convolution algebra in the mirabolic setting for varieties defined over a finite field, which is closely related to the Hecke algebra of the symmetric group. Then we describe what can be done in the affine setting, working over a local field, defining what we call the ‘mirabolic affine Hecke algebra’ and proving some results about its generators and relations.
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Since we are on the subject, for what concerns the actual mathematical content of this thesis, in addition to people that have already been mentioned, I would like to thank Jonah Blasiak for useful discussions on the subject of the RSK correspondence, Joel Kamnitzer for pointing out the result in [H] and Anthony Henderson for pointing out the reference [Sp1]. I also thank Sergey Fomin for some useful feedback on an early version of what is now Chapter 2.

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CHAPTER 1
INTRODUCTION

When studying representations of Lie groups (or algebraic groups), we deal with several related algebraic structures, like universal enveloping algebras of Lie algebras, or group algebras of Weyl groups. It is often a very useful point of view, when studying such an abstract algebraic structure, to try to realize it geometrically as an algebra of functions on some space. It then becomes possible to use our knowledge of the geometry of the given space to answer questions about our algebra, for example to give a construction of its irreducible representations. Sometimes, rather than an algebra of functions we might want a cohomology algebra or $K$-theory. This approach of using geometric methods in representation theory has been used a lot, just to give a few examples, we can cite Kazhdan and Lusztig in [KL] and Nakajima in [N].

One important class of spaces that come up as an object of study in this way are Flag Varieties, which in particular are homogeneous spaces with an action by a Lie group. If $G$ is a semisimple Lie group and $B$ is a Borel subgroup (for example $G = \text{GL}_d$ and $B$ is the subgroup of upper triangular matrices), then $G$ acts by left multiplication on the space $G/B$, which is called the flag variety. For $G = \text{GL}_d$ we have

$$G/B \simeq \{ F : 0 = F_0 \subset F_1 \subset \ldots \subset F_{d-1} \subset F_d = V | \dim F_i = i \}.$$  \hfill (1.1)

More generally, if $P$ is a parabolic subgroup of $G$ (i.e. $P$ contains a Borel subgroup), then the space $G/P$ is called a partial flag variety. Similarly to the above, in the case of $\text{GL}_d$, if $P$ consists of block upper triangular matrices with blocks of sizes $\mu_1, \ldots, \mu_n$, we have

$$G/P \simeq \{ F : 0 = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = V | \dim(F_i/F_{i-1}) = \mu_i \}.$$  \hfill (1.2)

These spaces have a natural structure of projective varieties, hence we can use tools from algebraic geometry to study them. The flag variety $G/B$ can also be identified with the
variety $\mathcal{F}$ of Borel subalgebras of the Lie algebra $\mathfrak{g} = \text{Lie} G$. This is done in the following way: we fix the Borel subalgebra $\mathfrak{b} = \text{Lie} B$, and we consider the adjoint action of $G$ on $\mathfrak{g}$. For all $g \in G$, the image $\text{Ad}(g) \mathfrak{b} \subset \mathfrak{g}$ is also a Borel subalgebra, and moreover all such subalgebras are conjugate in this way. Since $B$ is the stabilizer of $\mathfrak{b}$ under this action, the surjective map

$$G \to \mathcal{F} \quad g \mapsto \text{Ad}(g) \mathfrak{b}$$

descends to the quotient and induces an isomorphism of algebraic varieties $G/B \simeq \mathcal{F}$.

There is an interesting interplay between subvarieties of the flag variety and adjoint orbits of elements in the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$. This comes from the Springer resolution, which is a resolution of singularities of the nilpotent cone. This can be defined by identifying the cotangent bundle of the flag variety with the set of pairs

$$T^*(G/B) = \{(b, x) \in \mathcal{B} \times \mathcal{N} | x \in [b, b] \} \quad (1.3)$$

and then taking the projection to the second factor

$$T^*(G/B) \to \mathcal{N} \quad (b, x) \mapsto x. \quad (1.4)$$

It is possible to associate discrete data to certain varieties in this setting by looking at ways to parametrize irreducible components or $G$-orbits. In Chapter 2 of this thesis we will study some interesting combinatorics that arise in this way.

### 1.1 Basic Combinatorics and Notation

Here we fix some notation that we will use throughout this thesis. If $d \in \mathbb{Z}_{\geq 0}$ is a nonnegative integer, we say that the sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a partition of $d$, if it satisfies $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$, and $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_m = d$. We will sometimes denote it by $\lambda \vdash d$. Partitions can be thought of as Young diagrams, consisting of $\lambda_i$ left justified boxes in the $i$-th row from the top.
Example 1.1.1. If $d = 9$ and $\lambda = (4, 2, 2, 1)$, then the Young diagram will be as follows.

Given a partition $\lambda$, we will denote its transpose partition by $\lambda^t$. A Young diagram filled with positive integers is called a *Standard tableau* if all the numbers $1, 2, \ldots, d$ are used and they are increasing along the rows and down the columns of the diagram. The partition giving the diagram will be called the *shape* of the tableau.

Example 1.1.2. A Standard tableau of shape $(4, 2, 2, 1)$ is the following.

A filling of a Young diagram such that the entries are strictly increasing along rows and weakly increasing down columns, will be called a *semistandard tableau* in this thesis. It should be noted that by the usual convention these would be transposes of semistandard tableaux (the strictness of the inequalities is switched from rows to columns and vice versa).

Definition 1.1.3. Given any tableau $T$ with entries in $\{1, \ldots, n\}$, we say that its *content* is the sequence $\mu = \mu(T) = (\mu_1, \ldots, \mu_n)$ where $\mu_i$ is the number of times the entry $i$ appears in $T$.

Example 1.1.4. The following is a semistandard tableau of shape $(4, 2, 2, 1)$ and content $(3, 2, 3, 1)$.

1.2 $G$-action on Flag Varieties and Irreducible Components of the Springer fiber

From now on we will restrict ourselves to the special case $G = \text{GL}(V)$, where $V$ is a $d$-dimensional space over a field $k$. Given two flags $F, F'$ in $V$ (either partial or complete) we...
define the **relative position** of \( F \) and \( F' \) to be the matrix of nonnegative integers \( M(F,F') \) with entries given by:

\[
M(F,F')_{ij} = \dim \left( \frac{F_i \cap F'_j}{F_i \cap F'_{j-1} + F_{i-1} \cap F'_j} \right).
\] (1.5)

Then, see [BLM, 1.1], the set of such matrices parametrizes the orbits of the diagonal action of \( \text{GL}_d \) on the variety of pairs of flags. If \( \mu = (\mu_1, \ldots, \mu_n) \) is a composition of \( d \), that is \( \mu_i \)'s are nonnegative integers and \( \sum_{i=1}^{n} \mu_i = d \), we denote by \( \mathcal{F}^\mu \) the partial flag variety defined in (1.2).

Note that if \( F \in \mathcal{F}^\mu \) and \( F' \in \mathcal{F}^\nu \), the row sums of \( M(F,F') \) will be \( \mu = (\mu_1, \ldots, \mu_n) \) and the column sums will be \( \nu = (\nu_1, \ldots, \nu_m) \). In particular, if we restrict ourselves to complete flags \( (\mu = \nu = (1,1,\ldots,1)) \), then orbits are parametrized by permutation matrices. This is in fact the same as the Bruhat decomposition \( G = \bigsqcup_{w \in W} BwB \) (where the Weyl group \( W \) is the symmetric group). Indeed, diagonal \( G \)-orbits on \( G/B \times G/B \) correspond to \( B \)-orbits on \( G/B \), and \( B \backslash G/B \simeq W \).

The description of the cotangent bundle of \( \mathcal{F}^\mu \) is analogous to the one given in (1.3) for the flag variety. In this case it can be re-interpreted as (see for example [BB]),

\[
T^*(\mathcal{F}^\mu) = \{(F,x) \in \mathcal{F}^\mu \times \text{End}(V)|x(F_i) \subset F_{i-1} \forall i\}.
\]

The \( G \)-action on \( \mathcal{F}^\mu \) induces an action on the cotangent bundle, which can be written as \( g \cdot (F,x) = (gF, gxg^{-1}) \). Then, just like in (1.4), we have a \( G \)-equivariant map from \( T^*(\mathcal{F}^\mu) \) to the nilpotent cone \( \mathcal{N} \subset \text{End}(V) \), given by projection on the second factor.

If \( x \in \mathcal{N} \), we will denote the fiber of this map over that point by \( \mathcal{F}^\mu_x = \{F \in \mathcal{F}^\mu|x(F_i) \subset (F_{i-1}) \} \) (in the case of complete flags for simplicity we will drop the superscript and we will let \( \mathcal{F}_x = \mathcal{F}_x^{(1,\ldots,1)} \)). These are called Spaltenstein varieties and they are an active area of research, for example their cohomology algebra has been described in [BO].

The group \( G \) acts on the nilpotent cone \( \mathcal{N} \) by conjugation, the orbits are parametrized by partitions of \( d \), using the Jordan canonical form. A conjugacy class is uniquely determined by the sizes of the Jordan blocks, which we can take in nonincreasing order. If \( x \in \mathcal{N} \), we let its Jordan type be \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \).
We consider the subvariety $F_x \subset F$ of flags preserved by $x$, that is
\[ F_x := \{ F \in F | x(F_i) \subset F_{i-1} \}. \]

**Definition 1.2.1.** Now let $T_\lambda$ be the set of standard Young tableaux of shape $\lambda$, we can define a map
\[ t : F_x \to T_\lambda \]
in the following way: given $F \in F_x$, consider the Jordan type of the restriction $x|_{F_i}$. This gives us an increasing sequence of Young diagrams each with one box more than the previous one. Filling the new box with the number $i$ at each step, we get a standard tableau.

Then (see [Sp2, II 5.21],[St]) for a tableau $T \in T_\lambda$, if we let $F_{x,T} = t^{-1}(T) \subset F_x$, we have that the closure $C_{x,T} = \overline{F_{x,T}}$ is an irreducible component of $F_x$. All the irreducible components are parametrized in this way by the set of standard tableaux of shape $\lambda$. In [Sp2], Spaltenstein actually uses a slightly different parametrization, to see how the two parametrizations are related, see [vL].

### 1.3 Mirabolic Setting

The action of $G$ on varieties of flags and pairs of flags is a classic topic of studies. Recently, in several settings, generalizations of these constructions have been appearing, including the extra data of a vector or a line in $V$. These are usually called the ‘mirabolic’ setting. The name comes from the ‘mirabolic’ subgroup $P \subset GL(V)$, which is the subgroup that fixes a nonzero vector in $V$. This is because in general, for a $G$-variety $X$, the $P$-orbits on $X$ are obviously in a 1-1 correspondence with $G$-orbits on $X \times (V \setminus \{0\})$.

The mirabolic subgroup $P$ has a lot of good properties (for example the fact that restrictions of irreducible representations of $G$ to $P$ are multiplicity-free [Z, 13.5a]) and has historically been used to study representations of $G$, for example in [GK] and [B].

One case in which such a generalization arises is the following. Let $F = G/B$ be the variety of complete flags, then it is interesting to study the action of $G$ on $F \times F \times V$. One reason why this is important is because $\mathcal{D}$-modules on $F \times F \times V$ are closely related to
mirabolic character $\mathcal{D}$-modules. These are certain $\mathcal{D}$-modules on $G \times V$, which arise when studying the spherical trigonometric Cherednik algebra, see [FG].

A second example has been extending the nilpotent cone $\mathcal{N} \subset \text{End}(V)$ to the ‘enhanced nilpotent cone’ $\mathcal{N} \times V$. The group $G$ acts on $V$ in the obvious way and on $\mathcal{N}$ by conjugation. The orbits $G \mathcal{N}$ are in 1-1 correspondence with the partitions of $d$, by the Jordan canonical form. Orbits on the enhanced cone, instead, are parametrized by bipartitions. This has been studied by Achar and Henderson: in [AH] they describe the closures of those orbits in the enhanced setting. Their motivation was the work of S.Kato ([K]), which had introduced the ‘exotic nilpotent cone’, of which $\mathcal{N} \times V$ is a simplification. Kato uses the exotic nilpotent cone to establish an ‘exotic Springer correspondence’ and to give a geometric construction of the affine Hecke algebra of type $C_n^{(1)}$.

A third reason of the importance of these constructions is the fact that in [AH] it is also proved that the intersection cohomology of the closures of orbits in the enhanced nilpotent cone can be described using type-$B$ Kostka polynomials, which were introduced by Shoji in [Sh]. This is an analogue of a classic result by Lusztig ([L]).

Finally, Magyar, Weyman and Zelevinsky in [MWZ] have classified all cases in which the $G$ action on triples of flags has finitely many orbits. In particular, the case of two flags and a vector is very interesting. The $G$-orbits on $\mathcal{F}^\mu \times \mathcal{F}^\nu \times V$ are in bijective correspondence with ‘decorated matrices’. These are pairs $(M, \Delta)$, where $M$ is a $n \times n'$ matrix as in (1.5) and $\Delta = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ is a (possibly empty) set that satisfies

$$1 \leq i_1 < \ldots < i_k \leq n, \quad 1 \leq j_k < \ldots < j_1 \leq n'$$  

and such that the entry $M_{ij} > 0$ for all $(i, j) \in \Delta$. Also, in [M] there is a combinatorial description of the closure relations among such orbits.

### 1.4 Robinson-Schensted-Knuth correspondence

The Robinson-Schensted-Knuth correspondence (RSK for short) was first discovered by Robinson (see [R]) as a bijection between permutations of $d$ letters and pairs of standard Young tableaux of the same shape on $d$ boxes, then independently rediscovered by Schensted (see [Sc]). It was eventually generalized by Knuth (see [Kn]) to the case of two rowed arrays
in lexicographic order (or equivalently matrices with nonnegative integer entries) and pairs of semistandard Young tableaux of the same shape.

This correspondence comes up when considering flag varieties. As we were discussing in Section 1.2, the orbits of the diagonal action of \(\text{GL}_d\) on the variety of pairs of complete flags correspond to the elements of the symmetric group \(S_d\), which follows from the Bruhat decomposition. Also, given a nilpotent \(x \in \text{End}(V)\), of Jordan type \(\lambda\), the irreducible components of the subvariety of complete flags that are preserved by \(x\) are parametrized by the standard tableaux of shape \(\lambda\).

We can now state the theorem, proved independently by Spaltenstein and by Steinberg, (see [Sp2, II 9.8], [St]), whose generalization was the starting point for the work of this thesis.

We use the notation of Section 1.2, and the following definition.

**Definition 1.4.1.** In this thesis, whenever we will refer to a *general* element in a variety or subvariety, we will mean any element in a suitable open dense subset.

**Theorem 1.4.2.** Let \(\mathcal{F}\) be the variety of complete flags on a vector space \(V\), and \(x \in \text{End}(V)\) a nilpotent transformation of Jordan type \(\lambda\). Let \(T, T'\) be standard Young tableaux of shape \(\lambda\) and \(C_{x,T}\) and \(C_{x,T'}\) the corresponding irreducible components of \(\mathcal{F}_x\). Then for general flags \(F \in C_{x,T}\) and \(F' \in C_{x,T'}\), the permutation \(w(F,F')\) that gives the relative position of the two flags is the same as the permutation \(w(T,T')\) given by the RSK correspondence.

Now, we look at the setting of Section 1.3, but we restrict ourselves to triples of two complete flags and a vector. Then the set parametrizing the orbits of \(G\) can be thought of as the set of colored permutations \(RB\), that is permutation words where every letter is assigned one of two colors (say red and blue).

In his paper [T], Travkin has introduced the *mirabolic Robinson-Schensted-Knuth correspondence*. It gives a bijection between \(RB\) and the set of all \(\{(\lambda, \theta, \lambda', T, T')\}\), where \(T, T'\) are standard Young tableaux of shape \(\lambda\) and \(\lambda'\) respectively, and \(\theta\) is another partition that satisfies \(\lambda_i \geq \theta_i \geq \lambda_{i-1}\) and \(\lambda'_i \geq \theta_i \geq \lambda'_{i-1}\) for all \(i\). The geometric meaning of this correspondence is: given a colored permutation indexing a \(\text{GL}(V)\)-orbit on the space of two complete flags and a line, it describes the type of a general conormal vector to the orbit.
In this thesis we will also see how to generalize Travkin’s work to the case of partial flags.

1.5 Convolution Algebras

As mentioned at the very beginning of this introduction, it is often useful to realize algebras of interest as algebras of functions over some space. A large source of examples are convolution algebras. The basic setting is the following: we have a finite set $X$ and we take $E = \mathbb{C}(X \times X)$ to be the vector space of all complex valued functions on $X \times X$. Then given two functions $f, g$ we define their convolution to be

$$f * g(x, y) = \sum_{z \in X} f(x, z)g(z, y). \quad (1.7)$$

This defines an associative product on $E$. In such a naive example, the algebra that we get is just the matrix algebra $\text{End}(\mathbb{C}(X))$, but there are several ways of making this more interesting, for example introducing a group action.

If $G$ is a group acting on $X$, then we have the diagonal action of $G$ on $X \times X$ which induces a $G$-action on $E$. We can then consider the algebra $E^G \simeq \mathbb{C}(G \setminus X \times X)$ of functions that are invariant under the group action, with the same convolution product of (1.7). Several algebras that are interesting for Representation theory arise in this way, for example if we take $G = \text{GL}_d(\mathbb{F}_q)$ and $X$ to be the space of all complete flags in $\mathbb{F}_q^d$, then the resulting convolution algebra $\mathcal{H}$ is the Iwahori-Hecke algebra of the symmetric group $S_d$. With the same $G$, but taking $X'$ to be the space of all $n$-step partial flags, that is

$$X' := \{F : 0 = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = \mathbb{F}_q^d\}$$

then the convolution algebra is the $q$-Schur Algebra, a finite dimensional quotient of the quantum group $U_q(\mathfrak{sl}_n(\mathbb{C}))$ associated to the universal enveloping algebra as is constructed in [BLM].

This geometric point of view also makes very explicit the Schur-Weyl duality between the two algebras that were just given as examples, see [GL]. In fact, taking $T = \mathbb{C}(X \times X')$, \[8\]
the product (1.7) defines an $(\mathcal{H}, U_q(\mathfrak{sl}_n(\mathbb{C})))$-bimodule structure on $T$. Then the images of the algebras $\mathcal{H}$ and $U_q(\mathfrak{sl}_n(\mathbb{C}))$ in $\text{End}(T)$ are the full centralizers of each other.

The finitess condition we assumed guarantees that the convolution is always well defined, but it is not absolutely necessary. Let $k = \mathbb{F}_q((t))$ be the field of Laurent series in one variable and let $G = \text{GL}_d(k)$. Let $I$ be the Iwahori subgroup of $G$, that is

$$I = \{(a_{ij}) \in \text{GL}_d(\mathbb{F}_q[[t]]) \mid a_{ii} \in \mathbb{F}_q[[t]]^\times \forall i, \ a_{ij} \in t\mathbb{F}_q[[t]] \text{ if } i > j\}. \quad (1.8)$$

The space $G/I$ is called the affine flag variety. Orbits for the diagonal action of $G$ on $G/I \times G/I$ are parametrized by periodic permutations of $\mathbb{Z}$ (they satisfy $w(i + d) = w(i) + d$ for all $i \in \mathbb{Z}$). This is not a finite set, but the convolution operation is still well defined on $\mathbb{C}_c(\mathbb{F}_q[[t]]$, the space of locally constant compactly supported functions on $G$ which are $I$ bi-invariant. The resulting convolution algebra is the Affine Hecke Algebra.

In all of these cases, we had to work over a finite field $\mathbb{F}_q$ to be able to count points in the definition of the product (1.7). Independently of the choice of $q$, we get the same structure constants for the algebra as polynomials in $q$. Therefore we will usually consider $q$ as the specialization of a formal parameter $q$ and extend the scalars to $\mathbb{C}[q, q^{-1}]$ (or to other similar rings, like $\mathbb{C}(q^{\frac{1}{2}})$).

In Chapter 3 of this thesis we will look at how these constructions can be adapted to the mirabolic setting of Section 1.3.
CHAPTER 2
CLASSIC AND MIRABOLIC
ROBINSON-SCHENSTED-KNUTH CORRESPONDENCE
FOR PARTIAL FLAGS

The content of this chapter is basically the same as the paper [Ro]. In the first part of the chapter we start by discussing different conventions and generalizations of the Robinson-Schensted-Knuth correspondence. We then prove a result (Theorem 2.3.1) which answers affirmatively to the very natural question of whether an analogue of Theorem 1.4.2 applies the more general case of partial flag varieties. This theorem is then used in the second part of the chapter to generalize Travkin’s construction from [T] to the case of triples of two partial flags and a vector. We will give a generalization of Travkin’s mirabolic RSK algorithm and then show that it agrees with the geometry of the varieties involved.

2.1 Combinatorics of Partial Flags

We will use notation that we introduced in Section 1.2. For \( \mu = (\mu_1, \ldots, \mu_n) \) a composition of \( d \), we consider the partial flag variety of type \( \mu \).

\[ \mathcal{F}^\mu := \{ F = (0 = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = V) \mid \dim(F_i/F_{i-1}) = \mu_i \}. \]

Then for \( x \in \mathcal{N} \) nilpotent of Jordan type \( \lambda \), we consider the Spaltenstein variety of partial flags that are preserved by \( x \):

\[ \mathcal{F}_x^\mu := \{ F \in \mathcal{F}^\mu \mid x(F_i) \subset F_{i-1} \}. \]

If \( F \in \mathcal{F}_x^\mu \), we can associate to \( F \) a tableau in an analogous way to definition 1.2.1, except this time at each step we are adding several boxes, none of which will be in the same row. The result will be a tableau which is strictly increasing along rows and weakly increasing.
down columns. As we did in Section 1.1, we will call this kind of tableaux semistandard, although by the usual definition this is the transpose of a semistandard tableau.

**Definition 2.1.1.** So, if we let $\mathcal{T}_\lambda^\mu$ be the set of semistandard tableaux of shape $\lambda$ and content $\mu$, we just defined a map

$$t : F_\mu \longrightarrow T_\lambda^\mu.$$ 

**Lemma 2.1.2.** The irreducible components of $F_\mu$ are the closures $C_{x,T} = \overline{F_{x,T}}$ where $T \in T_\lambda^\mu$ and $F_{x,T} = t^{-1}(T)$.

For a proof, see [Sp1] or [H]. Spaltenstein discusses this very briefly, and uses a slightly different convention, as was also mentioned earlier. In his result the indexing set is a subset of the standard tableaux. It can be seen that this subset consists of what we will define later in this paper to be the standardization of the semistandard tableaux.

On the other hand Haines, during the proof of Theorem 3.1 proves a more general result about irreducible components of fibers of convolution morphisms from convolution product of $G(O)$-orbits in the affine Grassmannian. In his result, the combinatorial data are sequences of dominant weights such that the difference of two consecutive weights is in the orbit of the Weyl group acting on a dominant minuscule weight. In our case these correspond to the semistandard tableaux.

### 2.1.1 Relative Position, Words and Arrays

Given two flags $F$, $F'$, we have defined in (1.5) their relative position $M(F, F')$. Notice that if $F \in F^\mu$ and $F' \in F'^\nu$, the row sums of this matrix will be $\mu = (\mu_1, \ldots, \mu_n)$ and the column sums will be $\nu = (\nu_1, \ldots, \nu_m)$. Then, see [BLM, 1.1], the set $M_\mu^\nu(Z \geq 0)$ of all such matrices parametrizes the orbits of the diagonal action of $GL_d$ on $F^\mu \times F'^\nu$.

In particular, if $F$ and $F'$ are both complete flags in $V$, $M(F, F')$ will be a permutation matrix. This data is equivalent to the word $w(F, F') = w(1) \ldots w(d)$ where $w(i) = j$ if 1 appears in the $(j, i)$-entry of the matrix.

**Definition 2.1.3.** If $F, F'$ are both partial flags, then $M(F, F')$ is just a matrix of non-negative integers. We can record the same data in a two-rowed array

$$\omega = \begin{pmatrix} u(1) & u(2) & \ldots & u(d) \\ w(1) & w(2) & \ldots & w(d) \end{pmatrix}.$$
which is defined as follows.

A pair \((j, i)\) appears in \(\omega\) a number of times equal to the \((j, i)\)-entry of \(M(F, F')\).

The array \(\omega\) is then ordered so that it satisfies the following relation:

\[
    u(1) \leq u(2) \leq \ldots \leq u(d) \quad \text{and} \quad w(k) \geq w(k + 1) \quad \text{if} \ u(k) = u(k + 1). \tag{2.1}
\]

**Example 2.1.4.** If \(M(F, F')\) is the matrix on the left, the corresponding array \(\omega\) is given on the right:

\[
    M(F, F') = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} \quad \omega = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ 2 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \end{pmatrix}.
\]

The set \(M^{\mu, \nu}(\mathbb{Z}_{\geq 0})\) is thus identified, with the convention just described, with the set of two rowed arrays such that the first row has content \(\nu\), the second row has content \(\mu\), and they satisfy the order (2.1).

Depending on what is more convenient at each time, we will use either description of this set.

**Remark 2.1.5.** Another way of looking at the set \(M^{\mu, \nu}(\mathbb{Z}_{\geq 0})\) is as the set of double cosets \(S_\mu \backslash S_d / S_\nu\). Here \(S_d\) is the symmetric group on \(d\) letters and \(S_\mu\) and \(S_\nu\) are the Young subgroups corresponding to the compositions \(\mu\) and \(\nu\).

**Remark 2.1.6.** Our convention is different from what is used in [F] and [S2], where the arrays are taken to be in **lexicographic order**, that is with

\[
    u(1) \leq u(2) \leq \ldots \leq u(d) \quad \text{and} \quad w(k) \leq w(k + 1) \quad \text{if} \ u(k) = u(k + 1).
\]

With the lexicographic convention, the matrix of example 2.1.4 would correspond to the array

\[
    \omega' = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 2 & 2 & 1 & 1 & 2 \end{pmatrix}.
\]
2.2 Robinson-Schensted-Knuth Correspondence and Standardization

In this section, we will review quickly some definitions and properties of the RSK correspondence, following mainly the conventions of [F, I] and [S2, 7.11]. Then we will see how to adapt the results to the conventions we are using.

2.2.1 Review of RSK

Just for this review, we will call a tableau semistandard if it is weakly increasing along rows and strictly increasing down columns. With this convention, the tableaux we defined in section 2.1 are transposes of semistandard tableaux. We will also identify matrices with arrays using the lexicographic order, as in Remark 2.1.6.

With increasing generality, the RSK correspondence gives a bijection between permutations and pairs of standard tableaux of same shape, or between two-rowed arrays in lexicographic order and pairs of semistandard tableaux of same shape.

Given a permutation word $w$ or a two rowed array $\omega$, where

$$w = w(1) \ldots w(d) \quad \omega = \begin{pmatrix} u(1) & u(2) & \ldots & u(d) \\ w(1) & w(2) & \ldots & w(d) \end{pmatrix},$$

the algorithm is given by inserting the entries of the word (or of the second row of the array) by row bumping in the first tableau. At the same time we record in the second tableau which box has been added at each step (in the more general case of the array, the added box at the $k$-th step will be recorded with $u(k)$ as opposed to $k$).

The row bumping algorithm can be described as follows:

- We have as input a semistandard tableau $T_k$ with $k$ boxes and an integer $z$;

- If $z$ is bigger or equal than all the integers in the first row of the tableau, we just add a new box at the end of the first row and we put $z$ into it, this gives as output a tableau $T_{k+1}$ with $k+1$ boxes;

- If there is an integer in the first row which is bigger than $z$, then $z$ replaces the left-most entry in the row which is strictly larger than $z$, call it $y$, and bumps $y$ to the next row;
• We repeat the above procedure using \( y \) instead of \( z \) and considering the following row;

• Possibly repeat it again with the entry bumped by \( y \) and so on;

• We continue until we have obtained a new tableau \( T_{k+1} \) with \( k + 1 \) boxes.

If \( T, T' \) are semistandard tableaux and \( \omega \) is an array in lexicographic order, we will denote the correspondence by

\[
M(T, T') = \omega; \quad \text{or} \quad (T, T') \xleftrightarrow{\text{RSK}} \omega.
\]

**Example 2.2.1.** This is an example of the row bumping and recording procedure. We take the array

\[
\omega = \begin{pmatrix}
1 & 2 & 2 & 3 & 3 \\
3 & 1 & 2 & 1 & 2
\end{pmatrix}
\]

• Step 1: We start with the empty tableau \( T_0 = \emptyset \), and we insert 3 into it. We obtain the tableau \( T_1 = [3] \) and we record its shape in \( T'_1 = [1] \).

• Step 2: Insert 1, bumping 3 to the second row. Record with 2.

\[
T_2 = \begin{array}{c}
1 \\
3
\end{array} \quad T'_2 = \begin{array}{c}
1 \\
2
\end{array}
\]

• Step 3: Insert 2 at the end of the first row. Record with 2.

\[
T_3 = \begin{array}{c}
1 \\
3
\end{array} \quad T'_3 = \begin{array}{c}
1 \\
2
\end{array}
\]

• Step 4: Insert 1, which bumps 2, which bumps 3 to the third row. Record with 3.

\[
T_4 = \begin{array}{c c c}
1 & 1 \\
2 & 3
\end{array} \quad T'_4 = \begin{array}{c c c}
1 & 2 \\
2 & 3
\end{array}
\]

• Step 5: Insert 2 at the end of the first row. Record with 3.

\[
T_5 = \begin{array}{c c c}
1 & 1 & 2 \\
2 & 3
\end{array} \quad T'_5 = \begin{array}{c c c}
1 & 2 & 3 \\
2 & 3
\end{array}
\]
In conclusion,

\[
\begin{pmatrix}
1 & 2 & 2 & 3 & 3 \\
3 & 1 & 2 & 1 & 2
\end{pmatrix}
\overset{RSK}{\leftrightarrow}
\begin{pmatrix}
\begin{bmatrix}
1 & 1 & 2 \\
2 & 2 & 3
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 3 \\
2 & 3
\end{bmatrix}
\end{pmatrix}.
\]

### 2.2.2 Standardization

As can be seen in [S2, 7.11], given a semistandard tableau \(T\) we can consider its standardization \(\tilde{T}\). It is a standard tableau of the same shape as \(T\). We construct it in this way: the \(\mu_1\) boxes that contain 1 in \(T\) will be replaced by the numbers 1, 2, \ldots, \(\mu_1\) increasingly from left to right. Then the boxes that originally contained 2's will be replaced by \(\mu_1 + 1, \ldots, \mu_1 + \mu_2\) also increasingly from left to right, and so on.

**Example 2.2.2.**

\[
\begin{array}{c|c|c}
\hline
1 & 1 & 2 \\
\hline
2 & \hline
3
\end{array}
\quad \begin{array}{c|c|c}
\hline
1 & 2 & 4 \\
\hline
3 & \hline
5
\end{array}.
\]

In a similar way, given an array in lexicographic order \(\omega = (u(1) u(2) \ldots u(d)) \omega(1) \omega(2) \ldots \omega(d))\) we can define the standardization \(\tilde{\omega}\). It is given by replacing \(u(i)\) with \(i\) in the first row, while in the second row we replace the 1’s with 1, 2, \ldots, \(\mu_1\) increasing from left to right, then the 2’s and so on. The standardization of an array will then be a permutation.

**Example 2.2.3.**

\[
\omega = \begin{pmatrix}
1 & 2 & 2 & 3 & 3 \\
3 & 1 & 2 & 1 & 2
\end{pmatrix} \quad \tilde{\omega} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
5 & 1 & 3 & 2 & 4
\end{pmatrix}.
\]

Standardization allows us to always reduce the RSK correspondence to the special case of permutations and standard tableaux, because standardization and RSK commute.

**Lemma 2.2.4.** The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{T}_\lambda^\mu \times \mathcal{T}_\lambda^\nu & \overset{RSK}{\rightarrow} & M_{\mu,\nu}(\mathbb{Z}_{\geq 0}) \\
\downarrow \text{std} \times \text{std} & & \downarrow \text{std} \\
\mathcal{T}_\lambda \times \mathcal{T}_\lambda & \overset{RSK}{\rightarrow} & S_d
\end{array}
\]
In the diagram, $T_\lambda$, $T_\lambda^\mu$, $T_\lambda^\nu$ are respectively the set of standard tableaux and the sets of semistandard tableaux with content $\mu$ and $\nu$, all of shape $\lambda$. Also, $M^{\mu,\nu}(\mathbb{Z}_{\geq 0})$ is the set of two rowed arrays in lexicographic order with row contents $\nu$ and $\mu$ and $\text{std}$ is the standardization map.

The lemma is proved in [S2, 7.11.6], but let us illustrate this with an example.

**Example 2.2.5.** Let $T$, $\omega$ be as in examples 2.2.2 and 2.2.3 and let

$$T' = \begin{array}{c} 1 \ 2 \ 3 \\ 2 \ 3 \end{array}$$

then we have $\text{std}(T') = \tilde{T}' = \begin{array}{c} 1 \ 3 \ 5 \\ 2 \ 4 \end{array}$

then $(T, T') \overset{RSK}{\longleftrightarrow} \omega$ and indeed $(\tilde{T}, \tilde{T}') \overset{RSK}{\longleftrightarrow} \tilde{\omega}$.

### 2.2.3 Variation on RSK

In this thesis we will need a slight variation on the RSK correspondence. This will agree with RSK on permutations, but will give different results in the case of general two rowed arrays. It will associate to an array satisfying (2.1), a pair of tableaux that are strictly increasing along rows and weakly increasing down columns. This is what we called *semistandard* in section 1.1 and we will keep using this terminology from now on. In the rest of this paper, we will also set the convention of identifying matrices and arrays using Definition 2.1.3.

The variation of the correspondence is defined modifying the row bumping algorithm to the following: a new entry $z$ will bump the left-most entry in the row which is greater or equal to $z$. The recording tableau will be constructed in the usual way.

This difference is clearly irrelevant in the case of standard tableaux, but our new choice of row bumping will produce tableaux that are strictly increasing along rows and weakly increasing down columns. This is similar to the dual RSK defined in [S2, 7.14], which however is only defined for matrices of 0’s and 1’s.

Since we will only use this variation on the correspondence, from now on we will call this one RSK and we will use the same notation as before, there should be no confusion.

**Lemma 2.2.6.** This procedure gives a bijection between matrices of non-negative integers and pairs of semistandard (strictly increasing along rows and weakly increasing down columns) tableaux of same shape.
Proof. This is completely analogous to the usual proofs of the RSK correspondence (see [F],[S2]).

If the array corresponding to the matrix is \( \omega = \begin{pmatrix} u(1) & \ldots & u(d) \\ w(1) & \ldots & w(d) \end{pmatrix} \) and by the correspondence it gives us the pair of tableaux \((P, Q)\), then it is clear that the insertion tableau \(P\) will be semistandard. To check that the recording tableau \(Q\) is also semistandard, it is enough to show that if \(u(i) = u(i + 1)\), then \(u(i + 1)\) will end up in a row of \(Q\) that is strictly below the row of \(u(i)\).

Since \(\omega\) satisfies (2.1), if \(u(i) = u(i + 1)\), then \(w(i) \geq w(i + 1)\). This means that if \(w(i)\) bumps an element \(y_i\) from the first row, then the element \(y_{i+1}\) bumped by \(w(i + 1)\) from the first row must be in the same box where \(y_i\) was or in a box to the left of it. In turn, this implies that \(y_i \geq y_{i+1}\) and we can iterate this argument for the following rows. Now, the bumping route \(R_i\) of \(w(i)\) must stop before the bumping route \(R_{i+1}\) of \(w(i + 1)\), which will then continue at least one row below that of \(R_i\), which shows what we want.

The fact that the correspondence is a bijection just follows from the fact that we can do the reverse row bumping algorithm by taking at each step the box that in the recording tableau contains the biggest number. In case of equal elements, we will take the one that is in the lowest row. \(\square\)

Remark 2.2.7. Basically in this version of RSK we are considering equal entries in a tableau to be ‘bigger’ if they are in a lower row and, while inserting, sequences of equal numbers are considered decreasing sequences.

This leads us to a new definition of standardization that will give us an analogous result to lemma 2.2.4. Given a semistandard tableau \(T\), we define its standardization \(\tilde{T}\) by replacing the 1’s with 1, 2, \ldots, \(\mu_1\) \textit{starting from the top row and going down}, and then the same for 2’s and so on. For an array \(\omega\) ordered as in (2.1), we define \(\tilde{\omega}\) by replacing the first row with 1, 2, \ldots, \(d\) and on the second row we replace the 1’s by 1, 2, \ldots, \(\mu_1\) \textit{decreasingly} from left to right and same for the rest, always decreasing from left to right.

Example 2.2.8.

\[
\begin{array}{c}
1 & 2 \\
1 & 2 \\
3
\end{array}
\quad
\begin{array}{c}
1 & 3 \\
2 & 4 \\
5
\end{array}
\]
\[
\omega = \begin{pmatrix}
1 & 2 & 2 & 3 & 3 \\
1 & 3 & 1 & 2 & 2
\end{pmatrix} \quad \tilde{\omega} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 5 & 1 & 4 & 3
\end{pmatrix}
\]

**Remark 2.2.9.** From the point of view of Remark 2.1.5, the standardization of an array corresponds to choosing the longest representative for the double coset.

With our new conventions for semistandard tableaux, order of arrays, RSK, standardization and the same notation of lemma 2.2.4 we have that

**Lemma 2.2.10.** Standardization and RSK commute, as in the following diagram:

\[
\begin{array}{ccc}
T_\lambda^\mu \times T_\lambda^\nu & \xrightarrow{RSK} & M_{\mu,\nu}(\mathbb{Z}_{\geq 0}) \\
\downarrow \text{std} \times \text{std} & & \downarrow \text{std} \\
T_\lambda \times T_\lambda & \xrightarrow{RSK} & S_d
\end{array}
\]

The proof, mutatis mutandis, is the same as the proof of lemma 2.2.4 in [S2, 7.11.6]. It is just the observation that the standardization we choose for the arrays is exactly the one that makes the insertion procedure work the way we want, turning sequences of equal numbers into decreasing sequences.

**Example 2.2.11.** Let \( T, \omega \) as in example 2.2.8 and let

\[
T' = \begin{pmatrix}
1 & 2 \\
2 & 3 \\
3
\end{pmatrix} \quad \text{then we have} \quad \tilde{T'} = \begin{pmatrix}
1 & 2 \\
3 & 4 \\
5
\end{pmatrix}
\]

then \( (T, T') \xleftarrow{RSK} \omega \) and \( (\tilde{T}, \tilde{T}') \xrightarrow{RSK} \tilde{\omega} \).

**Remark 2.2.12.** It is clear that if we fix the contents \( \mu \) and \( \nu \), two different arrays \( \omega_1 \neq \omega_2 \in M_{\mu,\nu}(\mathbb{Z}_{\geq 0}) \) when standardized will give two different permutations \( \tilde{\omega}_1 \neq \tilde{\omega}_2 \). That is we have an injective map

\[
\text{std} : M_{\mu,\nu}(\mathbb{Z}_{\geq 0}) \to S_d.
\]

We therefore have an inverse

\[
\text{std}^{-1} : \text{std}(M_{\mu,\nu}(\mathbb{Z}_{\geq 0})) \to M_{\mu,\nu}(\mathbb{Z}_{\geq 0})
\]
which is easily described as follows:

\[
\begin{pmatrix}
1 & 2 & \ldots & \nu_1 & \nu_1 + 1 & \ldots \\
(w(1) & w(2) & \ldots & w(\nu_1) & w(\nu_1 + 1) & \ldots)
\end{pmatrix}
\mapsto
\begin{pmatrix}
1 & 1 & \ldots & 1 & 2 & \ldots \\
w'(1) & w'(2) & \ldots & w'(\nu_1) & w'(\nu_1 + 1) & \ldots
\end{pmatrix}
\]

the first row is just replaced by \( \nu_1 \) 1’s, followed by \( \nu_2 \) 2’s and so on, while we have

\[
w'(k) = j \quad \text{if} \quad w(k) \in \{\mu_1 + \ldots + \mu_{j-1} + 1, \ldots, \mu_1 + \ldots + \mu_j\}.
\]

### 2.3 RSK and Partial Flag Varieties

In this section we will use all the conventions of section 2.2.3 and the notations of section 2.1.

We state and prove the main result of the first part of this chapter, which generalizes Theorem 1.4.2. The strategy for the proof is to use standardization and Lemma 2.2.10 to reduce the problem to the case of complete flags.

**Theorem 2.3.1.** Let \( x \in \text{End}(V) \) be a nilpotent transformation of Jordan type \( \lambda, T \in T^{\mu}_{\lambda}, \)
\( T' \in T^{\nu}_{\lambda} \) be semistandard tableaux, and let \( C_{x,T} \) and \( C_{x,T'} \) be respectively the irreducible components of \( F^{\mu}_{x} \) and \( F^{\nu}_{x} \) corresponding to the tableaux \( T \) and \( T' \).

Then, for generic \( F \in C_{x,T} \) and \( F' \in C_{x,T'} \), we have that the relative position matrix \( M(F,F') \) is the same as the matrix \( M(T,T') \) given by the RSK correspondence.

**Proof.** For a fixed \( \mu = (\mu_1, \ldots, \mu_n) \) with \( |\mu| = \mu_1 + \ldots + \mu_n = d \), consider the map

\[
p_{\mu} : \mathcal{F} \to \mathcal{F}^{\mu}
\]

that forgets some of the spaces, that is

\[
(0 = F_0, F_1, F_2, \ldots, F_{d-1}, F_d = V) \mapsto (0 = F_0, F_{\mu_1}, F_{\mu_1+\mu_2}, \ldots, F_{\mu_1+\ldots+\mu_{n-1}}, F_d = V).
\]
Clearly, if $F$ is any partial flag in $\mathcal{F}_x^\mu$ and $\tilde{F} \in p^{-1}_\mu(F)$, then $\tilde{F} \in \mathcal{F}_x$ because for all $j$ there is some $i$ such that

$$F_{\mu_1+...+\mu_i} \subset \tilde{F}_{j-1} \subset \tilde{F}_j \subset F_{\mu_1+...+\mu_i+1}$$

and

$$x(\tilde{F}_j) \subset x(F_{\mu_1+...+\mu_i+1}) = x(F_{i+1}) \subset \tilde{F}_i = F_{\mu_1+...+\mu_i} \subset \tilde{F}_{j-1}.$$

Now, let $t : \mathcal{F}_x^\mu \to \mathcal{T}_x^\mu$ be the map that associates a semistandard tableau to a partial flag, as in Definition 2.1.1.

We fix a semistandard tableau $T$ and we let $\mathcal{F}_{x,T} := t^{-1}(T)$, then $\mathcal{F}_{x,T}$ is a constructible dense subset of $C_{x,T}$.

Let $\tilde{T}$ be the standardization of $T$ and let $\mathcal{F}_{x,\tilde{T}} = t^{-1}(\tilde{T}) \subset \mathcal{F}_x$ be the dense subset of $C_{x,\tilde{T}}$. The set $C_{x,\tilde{T}}$ is the irreducible component of the complete flag variety associated to the standard tableau $\tilde{T}$.

It is clear that if $F \in \mathcal{F}_{x,\tilde{T}}$, then we have $F = p_\mu(\tilde{F}) \in \mathcal{F}_{x,T}$ because

$$x|_{F_i} = x|_{\tilde{F}_{\mu_1+...+\mu_i}}$$

also, the map

$$p_\mu : \mathcal{F}_{x,\tilde{T}} \to \mathcal{F}_{x,T}$$

is surjective. This is because we can always find appropriate subspaces to complete a partial flag $F$ to a flag $\tilde{F}$ such that the restriction of $x$ to those subspaces has the Jordan type we want.

What we have said so far applies in the same way if we fix another semistandard tableau $T'$ of content $\nu$ and we consider the sets $\mathcal{F}_{x,T'} \subset \mathcal{F}_x^\nu$ and $\mathcal{F}_{x,\tilde{T}'} \subset \mathcal{F}_x$.

Now, let us fix two semistandard tableaux $T$ and $T'$ as in the statement of the theorem, and consider their standardizations $\tilde{T}$ and $\tilde{T}'$. For general complete flags $\tilde{F} \in C_{x,\tilde{T}}$ and $\tilde{F}' \in C_{x,\tilde{T}'}$, Theorem 1.4.2 tells us that $M(\tilde{F}, \tilde{F}') = M(\tilde{T}, \tilde{T}')$. We let then $X_{\tilde{T}} \subset C_{x,\tilde{T}}$, $X_{\tilde{T}'} \subset C_{x,\tilde{T}'}$ be the open dense subsets such that this is true.
Then $X_{\tilde{T}} \cap F_{x,\tilde{T}}$ is constructible dense in $C_{x,\tilde{T}}$. Hence it contains an open dense subset and the image of 

$$p_\mu : X_{\tilde{T}} \cap F_{x,\tilde{T}} \to F_{x,T}$$

is constructible dense in $F_{x,T}$, therefore it is also dense in $C_{x,T}$. In the same way, $p_\nu(X_{\tilde{T}} \cap F_{x,\tilde{T}})$ is constructible dense in $F_{x,T'}$. Therefore, by Remark 2.2.10 we have that 

$$\tilde{\omega} = \text{std}(M(T, T')) = M(\tilde{T}, T').$$

Let $\tilde{F} \in p_\mu^{-1}(F)$ and $\tilde{F}' \in p_\nu^{-1}(F')$, then by Lemma 2.2.10 we have that 

$$\tilde{\omega}' = M(F, F') = M(\tilde{T}, \tilde{T}').$$

Now let $\omega' = M(F, F')$. By the definition of relative position of flags, the array $\tilde{\omega}' = M(\tilde{T}, \tilde{T}')$ is such that for all $i, j$

$$\text{card} \left\{ \begin{pmatrix} \tilde{u} \\ \tilde{w} \end{pmatrix} \in \tilde{\omega}' \bigg| \begin{pmatrix} \tilde{u} \\ \tilde{w} \end{pmatrix} \in \begin{pmatrix} \nu_1 + \ldots + \nu_{j-1} + 1, \ldots, \nu_1 + \ldots + \nu_j \\ \mu_1 + \ldots + \mu_{i-1} + 1, \ldots, \mu_1 + \ldots + \mu_i \end{pmatrix} \right\}$$

$$= \dim \left( \frac{\tilde{F}_{\mu_1 + \ldots + \mu_i} \cap \tilde{F}'_{\nu_1 + \ldots + \nu_j}}{(\tilde{F}_{\mu_1 + \ldots + \mu_i} \cap \tilde{F}'_{\nu_1 + \ldots + \nu_j}) + (\tilde{F}_{\mu_1 + \ldots + \mu_i} \cap \tilde{F}'_{\nu_1 + \ldots + \nu_j-1})} \right)$$

$$= \dim \left( \frac{F_i \cap F'_j}{F_i \cap F'_j + F_{i-1} \cap F'_j} \right)$$

$$= \text{card} \left\{ \begin{pmatrix} u \\ w \end{pmatrix} \in \omega' \bigg| \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} i \\ j \end{pmatrix} \right\}$$

Therefore, by Remark 2.2.12, $\omega' = \text{std}^{-1}(\tilde{\omega})$. It follows that $\text{std}(\omega') = \tilde{\omega}$, that is

$$\text{std}(M(F, F')) = \text{std}(M(T, T')).$$

Again by Remark 2.2.12, this implies that $M(F, F') = M(T, T')$. This concludes the proof of the claim.
Since \( p_\mu(X_T \cap F_{x,T}) \) and \( p_\nu(X_{T'} \cap F_{x,T'}) \) are constructible dense in \( C_{x,T} \) and \( C_{x,T'} \) respectively, they each contain an open dense subset of the respective irreducible component, which proves the theorem.

### 2.4 Mirabolic Flag Varieties

With this section, we start the second part of this chapter, where we generalize the construction of Travkin (see [T]). We keep the notation of Sections 1.1 and 2.1.

#### 2.4.1 GL(\(V\))-orbits in \( F_\mu \times F_{\mu'} \times V \)

Let \( \mu = (\mu_1, \ldots, \mu_n), \mu' = (\mu'_1, \ldots, \mu'_{n'}) \) be two compositions of \( d \). We consider the diagonal \( G \)-action on the set \( F_\mu \times F_{\mu'} \times V \). So, let \((F, F', v) \in F_\mu \times F_{\mu'} \times V \) and look at the orbit \( G \cdot (F, F', v) \).

If \( v = 0 \), this orbit lies in \( F_\mu \times F_{\mu'} \times \{0\} \approx F_\mu \times F_{\mu'} \). As in Section 2.1.1 we parametrize such orbits by the set \( M_{\mu,\mu'}(\mathbb{Z}_{\geq 0}) \) of matrices with row sums \( \mu \) and column sums \( \mu' \) which we can also identify with the set of two rowed arrays of positive integers with row contents \( \mu' \) and \( \mu \).

If \( v \neq 0 \), the orbit \( G \cdot (F, F', v) \) is the preimage of an orbit in \( F_\mu \times F_{\mu'} \times \mathbb{P}(V) \). This is because for all \( c \in k^\times \), \((F, F', cv) = c \text{Id} \cdot (F, F', v) \in G \cdot (F, F', v) \).

The \( G \)-orbits on \( F_\mu \times F_{\mu'} \times \mathbb{P}(V) \) have been parametrized in [MWZ, 2.11] (see also [M, 2.2]) by "decorated matrices". These are pairs \((M, \Delta)\), where \( M \) is a matrix in \( M_{\mu,\mu'}(\mathbb{Z}_{\geq 0}) \) and \( \Delta = \{(i_1, j_1), \ldots, (i_k, j_k)\} \) is a nonempty set that satisfies

\[
1 \leq i_1 < \ldots < i_k \leq n, \quad 1 \leq j_k < \ldots < j_1 \leq n'
\]

and such that the entry \( M_{ij} > 0 \) for all \((i, j) \in \Delta\).

We can concisely write down a pair \((M, \Delta)\), in a similar way to what is done in [M], by parenthesizing the entries of the matrix corresponding to \( \Delta \).

**Example 2.4.1.**

\[
M = \begin{pmatrix}
1 & 0 & 2 \\
1 & 1 & 0 \\
0 & 3 & 0
\end{pmatrix}; \quad \Delta = \{(1, 3), (2, 1)\}
\]
\[(M, \Delta) = \begin{pmatrix}
1 & 0 & (2) \\
(1) & 1 & 0 \\
0 & 3 & 0
\end{pmatrix}\]

**Lemma 2.4.2.** There is a 1-1 correspondence between the set of pairs \((M, \Delta)\) as above and the set of pairs \((\omega, \beta)\) where \(\omega\) is a two rowed array and \(\beta \subset \{1, \ldots, d\}\) is a nonempty subset such that if \(i \in \{1, \ldots, d\} \setminus \beta\) and \(j \in \beta\), either \(u(i) > u(j)\) or \(w(i) > w(j)\).

*Proof.* The correspondence between \(M\) and \(\omega\) is just the identification we discussed in Definition 2.1.3. Now, consider the map
\[\varphi : \{1, \ldots, d\} \rightarrow \{1, \ldots, n\} \times \{1, \ldots, n'\}\]
\[l \mapsto (w(l), u(l)).\]

Then \(\Delta\) will be the subset of \(\varphi(\beta)\) defined by
\[\Delta = \{(i, j) \in \varphi(\beta) | \forall k \geq 1 \ (i + k, j) \notin \varphi(\beta) \text{ and } (i, j + k) \notin \varphi(\beta)\}.

Given \(\Delta\) we can recover \(\beta\) in the following way: let
\[\Delta' = \{(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, n'\} | \exists (i_0, j_0) \in \Delta \text{ s.t. } i \leq i_0 \text{ or } j \leq j_0\}\]
then \(\beta = \varphi^{-1}(\Delta')\).

It is not difficult to see that these definitions give inverse correspondences.

Visually, \(\varphi(\beta)\) identifies a set of positions in the matrix that fits in a Young diagram, and such that no other nonzero positions are in the diagram. The set \(\Delta\) consists then of the outer corners of that diagram.

Vice versa, given \(\Delta\), \(\Delta'\) is the set of all positions of the matrix weakly northwest of \(\Delta\). Then \(\beta = \varphi^{-1}(\Delta')\) consists of all the columns of the array corresponding to the nonzero positions in \(\Delta'\). \(\Box\)

**Example 2.4.3.** If we take the decorated matrix \((M, \Delta)\) of Example 2.4.1, we have that
\[\Delta' = \{(1,1), (1,2), (1,3), (2,1)\}; \quad \omega = \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 \\
2 & 1 & 3 & 3 & 2 & 1
\end{pmatrix}\]
Then $\beta = \{1, 2, 7, 8\}$.

**Definition 2.4.4.** We define the set $\mathcal{D}^{\mu, \mu'}$ of "decorated arrays" to be the set of all pairs $(\omega, \beta)$, where $\omega \in M^{\mu, \mu'}(\mathbb{Z}_{\geq 0})$ and $\beta \subset \{1, \ldots, d\}$ is a (possibly empty) subset such that if $i \in \{1, \ldots, d\} \setminus \beta$ and $j \in \beta$, then either $u(i) > u(j)$ or $w(i) > w(j)$.

By Lemma 2.4.2, the set of decorated matrices (if we also allow $\Delta = \emptyset$) and decorated arrays are identified, so we might use either description of the set, depending on what is most convenient at each time.

By the result in [MWZ, 2.11] and Lemma 2.4.2, we can then parametrize the $G$-orbits on $\mathcal{F}^\mu \times \mathcal{F}^{\mu'} \times V$ with the set $\mathcal{D}^{\mu, \mu'}$. The pairs $(\omega, \beta) \in \mathcal{D}^{\mu, \mu'}$ with $\beta \neq \emptyset$ correspond to the $G$-orbits in $\mathcal{F}^\mu \times \mathcal{F}^{\mu'} \times \mathbb{P}(V)$, and the ones with $\beta = \emptyset$ correspond to the case of $v = 0$.

We are going to give a direct proof of this parametrization. In order to do that, we will use the following result of Travkin ([T, Lemma 1]).

**Lemma 2.4.5.** Let $A \subset \text{End}(V)$ be an associative algebra with identity and $A^\times$ the multiplicative group of $A$. Suppose that the $A$-module $V$ has finitely many submodules. Then the $A^\times$-orbits in $V$ are in 1-1 correspondence with these submodules. Namely, each $A^\times$-orbit has the form

$$\Omega_S := S \setminus \bigcup_{S' \subset S} S'$$

where $S$ is an $A^\times$-submodule of $V$ and the union is taken over all proper submodules of $S$.

**Proposition 2.4.6.** There is a 1-1 correspondence between $G$-orbits in $\mathcal{F}^\mu \times \mathcal{F}^{\mu'} \times V$ and the set $\mathcal{D}^{\mu, \mu'}$.

**Proof.** For each $\omega = \left( \begin{array}{cccc} u(1) & u(2) & \ldots & u(d) \\ w(1) & w(2) & \ldots & w(d) \end{array} \right)$ in $M^{\mu, \mu'}(\mathbb{Z}_{\geq 0})$, let $\Omega_{\omega}$ be the corresponding $G$-orbit in $\mathcal{F}^\mu \times \mathcal{F}^{\mu'}$.

In particular, $(F, F') \in \Omega_{\omega}$ if and only if there exists a basis $\{e_i | i = 1, \ldots, d\}$ of $V$ such that

$$F_i = \langle e_r | w(r) \leq i \rangle$$

$$F'_j = \langle e_s | u(s) \leq j \rangle.$$
For a fixed $\omega$, consider a point $(F, F') \in \Omega_\omega$ and let $H$ be its stabilizer in $G$. Then the $H$-orbits in $V$ are in 1-1 correspondence with the $G$-orbits of $\mathcal{F}^\mu \times \mathcal{F}^{\mu'} \times V$ consisting of points $(D, D', v)$ with $(D, D') \in \Omega_\omega$.

Let $A_F, A_{F'} \subset \text{End}(V)$ respectively be the subalgebras that leave the partial flags $F, F'$ invariant, i.e.  

$$A_F := \{ a \in \text{End}(V) \mid a(F_i) \subset (F_i) \quad \forall i \}$$

$$A_{F'} := \{ a \in \text{End}(V) \mid a(F'_i) \subset (F'_i) \quad \forall i \}$$

Let $A = A_F \cap A_{F'}$, then $H = A^G$. So pick a basis $\{e_i\}$ of $V$ satisfying (2.2), and let $E_{ij}$ be the linear operator such that  

$$E_{ij}e_r = \delta_{jr}e_i.$$  

Then  

$$A = \bigoplus_{u(i) \leq u(j); w(i) \leq w(j)} kE_{ij},$$  

From this it follows that all the $A$-submodules of $V$ have the form $S(\beta) := \oplus_{i \in \beta} ke_i$, where $\beta$ is like in Definition 2.4.4. In particular, they are finite, so we can apply lemma 2.4.5 to conclude the proof.  

**Definition 2.4.7.** We will denote by $\Omega_{\omega, \beta}$ the $G$-orbit in $\mathcal{F}^\mu \times \mathcal{F}^{\mu'} \times V$ corresponding to $(\omega, \beta)$.

**Remark 2.4.8.** The orbit $\Omega_{\omega, \beta}$ consists exactly of the triples $(F, F', v)$ such that there exists a basis $\{e_i \mid i = 1, \ldots, d\}$ of $V$ that satisfies (2.2) and with  

$$v = \sum_{i \in \beta} e_i.$$  

### 2.4.2 Conormal Bundles and Mirabolic RSK

We consider the variety $X^{\mu, \mu'} := \mathcal{F}^\mu \times \mathcal{F}^{\mu'} \times V$, and its cotangent bundle $T^*(X^{\mu, \mu'})$. We know that, see [CG, 4.1.2],  

$$T^*(\mathcal{F}^\mu) = \{(F, x) \in \mathcal{F}^\mu \times \mathcal{N} \mid x(F_i) \subset F_{i-1} \quad \forall i \}.$$
Therefore

\[ T^*(X^\mu,\mu') = \{ (F, F', v, x, x', v^*) \in X^\mu,\mu' \times \mathcal{N} \times \mathcal{N} \times V^* | F \in F^\mu_x; \quad F' \in F^\mu'_x \}. \]

We have the moment map

\[ T^*(X^\mu,\mu') \to \mathfrak{gl}(V)^* \simeq \mathfrak{gl}(V) \]

\[ (F, F', v, x, x', v^*) \mapsto x + x' + v^* \otimes v. \]

We let \( Y^\mu,\mu' \) be the preimage of 0 under the moment map, then \( Y^\mu,\mu' \) is the union of the conormal bundles of the \( G \)-orbits in \( X^\mu,\mu' \):

\[ Y^\mu,\mu' := \{ (F, F', v, x, x', v^*) \in T^*(X^\mu,\mu') | x + x' + v^* \otimes v = 0 \} = \bigsqcup_{\omega,\beta} N^\ast \Omega_{\omega,\beta}. \]

Hence, all the irreducible components of \( Y^\mu,\mu' \) are the closures \( N^\ast \Omega_{\omega,\beta} \).

Now, consider the variety \( Z \) of quadruples

\[ Z := \{ (x, x', v, v^*) \in N \times N \times V \times V^* | x + x' + v^* \otimes v = 0 \}. \]

We then have a projection

\[ \pi : Y^\mu,\mu' \to Z \]

\[ (F, F', v, x, x', v^*) \mapsto (x, x', v, v^*). \]

We let \( \mathfrak{P} \) be the set of pairs of partitions \((\lambda, \theta)\) such that \(|\lambda| = d\) and \(\lambda_i \geq \theta_i \geq \lambda_{i+1}\) for all \(i\).

**Remark 2.4.9.** The set \( \mathfrak{P} \) parametrizes \( G \)-orbits on \( \mathcal{N} \times V \), as is proved independently in both [T, Theorem 1] and [AH, Proposition 2.3]. In particular, \((x, v)\) is in the orbit corresponding to \((\lambda, \theta)\) if the Jordan type of \(x\) is \(\lambda\) and the Jordan type of \(x|_{V/k[x]}\) is \(\theta\).

Define the set of triples \( \mathbf{T} := \{ (\lambda, \theta, \lambda') | (\lambda, \theta) \in \mathfrak{P}; (\lambda', \theta) \in \mathfrak{P} \} \).

For any \( \mathbf{t} = (\lambda, \theta, \lambda') \in \mathbf{T} \), we write \( Z^\mathbf{t} \) for the subset of quadruples \((x, x', v, v^*) \in Z\) such that the Jordan types of \(x, x'\) and \(x|_{V/k[x]}\) are respectively \(\lambda, \lambda'\) and \(\theta\).
Remark 2.4.10. Notice that in the previous statement we did not break any symmetry by choosing \( x \) instead of \( x' \), because if \( x + x' + v^* \otimes v = 0 \), then \( k[x]v = k[x']v \) and \( x|_{V/k[x]}v = -x'|_{V/k[x']}v \).

Now if \( \tilde{\omega} = (\omega, \beta) \in \mathcal{D}^\mu,\mu' \), we can consider a point \( y = (F, F', v, x, x', v^*) \) in the variety \( Y_{\tilde{\omega}} := N^*(\Omega_{\tilde{\omega}}) \). In particular we can take \( y \in N^*(\Omega_{\tilde{\omega}}) \).

Then \( \pi(y) \in Z^t \) for some \( t = (\lambda, \theta, \lambda') \in T \). Now, \( Z^t \) is irreducible, as is shown in the proof of Proposition 1 in [T]. Hence this \( t = t(y) \) will be the same for all \( y \) in an open dense subset of \( Y_{\tilde{\omega}} \). With this choice of \( y \), we can then denote \( t = t(\tilde{\omega}) \) to emphasize that it depends only on \( \tilde{\omega} \). Let \( T = T(\tilde{\omega}) \in \mathcal{T}_\lambda^\mu \) and \( T' = T'(\tilde{\omega}) \in \mathcal{T}_{\lambda'}^\mu' \) such that \( F = F(y) \in \mathcal{F}_{x,T} \) and \( F' = F'(y) \in \mathcal{F}_{x',T'} \).

Proposition 2.4.11. The assignment \( \tilde{\omega} \mapsto (t(\tilde{\omega}), T(\tilde{\omega}), T'(\tilde{\omega})) \), that we just described, gives a 1-1 correspondence

\[
\mathcal{D}^\mu,\mu' \longleftrightarrow \{(\lambda, \theta, \lambda') \mid (\lambda, \theta, \lambda') \in T, T \in \mathcal{T}_\lambda^\mu, T' \in \mathcal{T}_{\lambda'}^\mu'\}.
\]

Proof. Consider the set \( Y^{t,T,T'} \subset Y^\mu,\mu' \) defined by

\[
Y^{t,T,T'} = \{y \in Y^\mu,\mu' \mid \pi(y) \in Z^t, F(y) \in \mathcal{F}_{x,T}, F'(y) \in \mathcal{F}_{x',T'}\}.
\]

Then for all \( (t, T, T') \), \( Y^{t,T,T'} \) is a locally closed subset of \( Y^\mu,\mu' \) and

\[
Y^\mu,\mu' = \bigsqcup_{t,T,T'} Y^{t,T,T'}.
\]

Claim 2.4.12. These locally closed subsets are irreducible and \( \dim Y^{t,T,T'} = \dim Y^\mu,\mu' \) for all \( t, T, T' \).

We look at the projection \( \pi|_{Y^{t,T,T'}} : Y^{t,T,T'} \to Z^t \). All the fibers of this map are of the form

\[
\pi^{-1}(x, x', v, v^*) = \{y \in Y^{t,T,T'} \mid F(y) \in \mathcal{F}_{x,T}, F'(y) \in \mathcal{F}_{x',T'}\} \simeq \mathcal{F}_{x,T} \times \mathcal{F}_{x',T'}.
\]

It follows that they are irreducible and they have the same dimension. The set \( Z^t \) is also irreducible, hence the sets \( Y^{t,T,T'} \) are irreducible.
From now on in the paper we will use the notation \((a^{d_1}, b^{d_2}, \ldots)\) for the sequence \((a, \ldots, a, b, \ldots, b, \ldots)\) where \(a\) appears \(d_1\) times, \(b\) appears \(d_2\) times and so on.

From Travkin's proof of Proposition 1 in [T], it follows that the statement about dimensions is true when we consider the case of complete flags. That is, when \(\mu = \mu' = (1^d)\), we let \(Y := Y(1^d, 1^d)\) and we have \(\dim Y = \dim Y^{t, T, T'}\) where \(T, T'\) are standard tableaux. That implies that if \(t = (\lambda, \theta, \lambda')\), then

\[
\dim Z^t = \dim Y^{t, T, T'} - \dim(F_{x, T} \times F_{x', T'}).
\]

\[
= \dim Y - (\dim F_x + \dim F_{x'})
\]

\[
= d^2 - n_\lambda - n_\lambda'
\]

where \(n_\lambda = \sum_i (i - 1)\lambda_i\).

In the case of partial flags, we know that

\[
\dim X^{\mu, \mu'} = \dim Y^{\mu, \mu'} = d^2 + d - \frac{1}{2} \sum_i \mu_i^2 - \frac{1}{2} \sum_j \mu_j^2.
\]

Further, for \(t = (\lambda, \theta, \lambda')\),

\[
\dim Y^{t, T, T'} = \dim Z^t + \dim(F_{x, T} \times F_{x', T'})
\]

\[
= \dim Z^t + \dim F_x^\mu + \dim F_{x'}^{\mu'}
\]

\[
= d^2 - n_\lambda - n_\lambda' + \left(n_\lambda - \frac{1}{2} \left(-d + \sum_i \mu_i^2\right)\right) + \left(n_\lambda' - \frac{1}{2} \left(-d + \sum_j \mu_j^2\right)\right)
\]

\[
= d^2 + d - \frac{1}{2} \sum_i \mu_i^2 - \frac{1}{2} \sum_j \mu_j^2
\]

This concludes the proof of the claim. Now, the claim implies that the irreducible components of \(Y^{\mu, \mu'}\) are exactly the closures of the sets \(Y^{t, T, T'}\). This is enough to prove the proposition because the set \(D^{\mu, \mu'}\) also parametrizes the same irreducible components. □

**Definition 2.4.13.** The map \(\tilde{\omega} \mapsto (t(\tilde{\omega}), T(\tilde{\omega}), T'(\tilde{\omega}))\) of Proposition 2.4.11 is called the **mirabolic Robinson-Schensted-Knuth correspondence**.
2.5 Combinatorial description of the mirabolic RSK correspondence

In this section we will describe an algorithm that takes as input a decorated array \( \tilde{\omega} = (\omega, \beta) \in D_{\mu, \mu'} \) and gives as output a triple \( (t, T, T') \), with \( t = (\lambda, \theta, \lambda') \in T, T \in T^\mu_{\lambda}, T' \in T^\mu_{\lambda'} \). We will then prove that this is the same as the mirabolic RSK correspondence defined geometrically in the previous section.

2.5.1 The Algorithm

In the algorithm we describe, the row bumping convention is that a new entry \( z \) will bump the left-most entry in the row which is greater or equal to \( z \), as in Section 2.2.3.

**Definition 2.5.1.** As an input, we have \( \tilde{\omega} = (\omega, \beta) \) where

\[
\omega = \begin{pmatrix} u(1) & u(2) & \ldots & u(d) \\ w(1) & w(2) & \ldots & w(d) \end{pmatrix}, \quad \beta \subset \{1, \ldots, d\}.
\]

- At the beginning, set \( T_0 = T'_0 = \emptyset \) and let \( R \) be a single row consisting of the numbers \( d + 1, \ldots, 2d \)

\[
R = \begin{array}{cccc}
d + 1 & d + 2 & \ldots & 2d \\
\end{array}
\]

- For \( i = 1, 2, \ldots, d \)
  - If \( i \in \beta \), let \( T_i \) be the tableau obtained by inserting \( w(i) \) into the tableau \( T_{i-1} \) via row bumping.
  - If \( i \notin \beta \), insert \( w(i) \) into \( R \) by replacing the least element \( z \in R \) that is greater or equal to \( w(i) \). Then let \( T_i \) be the tableau obtained by inserting \( z \) into \( T_{i-1} \) via row bumping.
  - Construct \( T'_i \) by the usual recording procedure. That is add a new box to \( T'_{i-1} \) in the same place where the row bumping for \( T_i \) terminated, and put \( u(i) \) in the new box.

- At this point we have \( T_d, T'_d \) two semistandard tableaux with \( d \) boxes, and the single row \( R \). We let \( T' := T'_d \) and \( \lambda' \) will be its shape (which is also the same shape of \( T_d \)).
• Insert, via row bumping, $R$ into $T_d$, starting from the left. Call $T_{2d}$ the resulting tableau.

• Let $\nu = (\nu_1, \nu_2, \ldots)$ be the shape of $T_{2d}$, then we have $\theta = (\theta_1, \theta_2, \ldots) := (\nu_2, \nu_3, \ldots)$. That is we define $\theta$ to be the partition obtained from $\nu$ by removing the first part.

• We let $T := T_{2d}^{(d)}$, that is $T$ is the tableau obtained from $T_{2d}$ by removing all the boxes with numbers strictly bigger than $d$. We then have $\lambda$ be the shape of $T$.

• The output is $((\lambda, \theta, \lambda'), (T, T'))$.

**Theorem 2.5.2.** For all $\tilde{\omega} \in D^{\mu, \mu'}$, the triple $(t(\tilde{\omega}), T(\tilde{\omega}), T'(\tilde{\omega}))$ of Definition 2.4.13 is the same as the triple obtained by applying the algorithm 2.5.1 to $\tilde{\omega}$.

The next section of this chapter is a proof of this theorem. In Section 2.6 we give an example that illustrates the result and the algorithm.

For simplicity of notation, throughout the following proof we will drop the zeros in the compositions. For example we will write $(1^d, \mu, 1^d)$ instead of $(1^d, \mu, 0^{d-n}, 1^d)$. What this means is that we are implicitly using the appropriate canonical isomorphisms of partial flag varieties and of sets of arrays; e.g. $F(1^d, \mu, 0^{d-n}, 1^d) \simeq F(1^d, \mu, 1^d)$ and

$$M(\mu, 0^{d-n}, 1^d, \mu', 0^{d-n'}, 1^d)(\mathbb{Z}_{\geq 0}) \simeq M(\mu, 1^d, \mu', 1^d)(\mathbb{Z}_{\geq 0}).$$

### 2.5.2 Proof of Theorem 2.5.2

Let $\tilde{\omega} = (\omega, \beta) \in D^{\mu, \mu'}$ with $\omega = \begin{pmatrix} u(1) & \ldots & u(d) \\ w(1) & \ldots & w(d) \end{pmatrix}$, $\beta \subset \{1, \ldots, d\}$. We want to show that the triple $(t(\tilde{\omega}), T(\tilde{\omega}), T'(\tilde{\omega}))$ of Definition 2.4.13 is the same as what we get applying the algorithm of Definition 2.5.1.

Consider $\tilde{\omega}_+ = (\omega_+, \beta_+) \in D^{(1^d, \mu, 1^d), (1^d, \mu', 1^d)}$ defined by

$$\omega_+ = \begin{pmatrix} u_+(1) & u_+(2) & \ldots & u_+(3d) \\ w_+(1) & w_+(2) & \ldots & w_+(3d) \end{pmatrix}$$
where

\[
\begin{align*}
  u_+(i) & = \begin{cases} 
  i & \text{if } i \leq d \text{ or if } 2d + 1 \leq i \leq 3d \\
  u(i - d) + d & \text{if } d + 1 \leq i \leq 2d \\
  i + 2d & \text{if } i \leq d \\
  w(i - d) + d & \text{if } d + 1 \leq i \leq 2d \\
  i - 2d & \text{if } 2d + 1 \leq i \leq 3d
  \end{cases} \\
  w_+(i) & = \begin{cases} 
  i + 2d & \text{if } i \leq d \\
  w(i - d) + d & \text{if } d + 1 \leq i \leq 2d \\
  i - 2d & \text{if } 2d + 1 \leq i \leq 3d
  \end{cases}
\end{align*}
\]

\[
\beta_+ = \{i + d | i \in \beta\}
\]

If we look at \( \omega_+ \) as a matrix, we can visualize it as a block matrix:

\[
\omega_+ = \begin{pmatrix} 0 & 0 & I_d \\ 0 & \omega & 0 \\ I_d & 0 & 0 \end{pmatrix}
\]

where \( \mathbf{0} \) is a block of zeros and \( I_d \) is the identity \( d \times d \) matrix.

Or, as an array,

\[
\omega_+ = \begin{pmatrix} 1 & \ldots & d & u(1) + d & \ldots & u(d) + d & 2d + 1 & \ldots & 3d \\ 2d + 1 & \ldots & 3d & w(1) + d & \ldots & w(d) + d & 1 & \ldots & 3d \end{pmatrix}
\]

Then we have a corresponding variety \( Y_{\omega_+} \) which is the closure of \( N^* \Omega_{\tilde{\omega}_+} \). Since \( Y_{\omega_+} \) is irreducible, all the discrete combinatorial data associated to a point \( y \in Y_{\tilde{\omega}_+} \) will agree on an open dense subset. So we let \( y = (F, F', v, x, x', v^*) \) be such a general point, where \( F \) and \( F' \) are partial flags in a \( 3d \)-dimensional vector space \( V_+ \), \( v \in V_+ \) and \( x + x' + v^* \otimes v = 0 \).

Choose a basis \( \{e_1, e_2, \ldots, e_{3d}\} \) of \( V_+ \) that satisfies

\[
\begin{align*}
F_i &= \langle e_r | w_+(r) \leq i \rangle \\
F'_j &= \langle e_s | u_+(s) \leq j \rangle \\
v &= \sum_{i \in \beta_+} e_i
\end{align*}
\]

and let \( \{e_i^*\} \) be the dual basis of \( V_+^* \).
Definition 2.5.3. For \( m \geq 1 \), we define inductively two sequences \( \{ \gamma_m \}, \{ \delta_m \} \) of subsets of \( \{1, \ldots, 3d\} \).

\[
\gamma_1 := \{1, \ldots, 3d\} \setminus \beta_+
\]

\[
\delta_m := \{ i \in \gamma_m \mid \forall j \in \gamma_m, \quad u_+(j) \geq u_+(i) \text{ or } w_+(j) \geq w_+(i) \}
\]

\[
\gamma_{m+1} := \gamma_m \setminus \delta_m
\]

It is easy to see that for all \( m = 1, \ldots, d \), the set \( \delta_m \) consists of the elements \( m, 2d + m \) plus some subset of \( \{d+1, \ldots, 2d\} \). Also, \( \delta_m = \gamma_m = \emptyset \) for all \( m > d \).

Lemma 2.5.4. For a general conormal vector \((x, x', v^*)\) at the point \((F, F', v)\), we have for \( 1 \leq m \leq d - 1 \)

\[
(x^*)^m v^* = \sum_{i \in \gamma_{m+1}} \alpha_{m,i} e_i^*
\]

with \( \alpha_{m,m+1}, \alpha_{m,m+2d+1} \) both nonzero.

Proof. Since \( x \) preserves the flag \( F \), we have that \( e_i^*(x(e_j)) = 0 \) if \( w(i) \geq w(j) \). Analogously \( e_i^*(x'(e_j)) = 0 \) if \( u(i) \geq u(j) \). Also, we have that \( \text{Im}(v^* \otimes v) \subset \langle e_{d+1}, \ldots, e_{2d} \rangle \). Therefore the condition that \( x + x' + v^* \otimes v = 0 \) implies that in the basis \( \{e_1, \ldots, e_{3d}\} \) the three operators have the following block matrix form:

\[
v^* \otimes v = \begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix} \quad x = \begin{pmatrix} A & 0 & 0 \\ 0 & B & * \\ 0 & 0 & C \end{pmatrix} \quad x' = \begin{pmatrix} A' & 0 & 0 \\ * & B' & 0 \\ 0 & 0 & C' \end{pmatrix}
\]

where \( A, B, C, A', B', C' \) are strictly upper triangular \( d \times d \) matrices and the *'s are some possibly nonzero matrices depending on \( \beta_+ \). They satisfy \( A' = -A \) and \( C' = -C \). Now, let

\[
A = \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,d} \\ 0 & \ddots & \cdots & \vdots \\ 0 & \cdots & a_{d-1,d} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & c_{1,2} & \cdots & c_{1,d} \\ 0 & \ddots & \cdots & \vdots \\ \vdots & \cdots & c_{d-1,d} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}
\]

Then, since \( F \in F(1^d, \mu, 1^d) \), the set of conormal vectors such that \( \text{rank } A = d - 1 = \text{rank } C \)
(or equivalently such that \(a_{1,2}, \ldots, a_{d-1,d}, c_{1,2}, \ldots, c_{d-1,d}\) are all nonzero) is open dense in \(N^*\Omega_{\tilde{\omega}+}(F,F',v)\).

Also, the set of conormal vectors \((x, x', v^*)\) such that \(v^*(e_1), v^*(e_{2d+1})\) are both nonzero is open dense.

Let \(J\) be the intersection of these two sets, then \(J \subset N^*\Omega_{\tilde{\omega}+}(F,F',v)\) is open dense.

From now on we assume that \((x, x', v^*)\) \(\in J\) and we are going to prove that the conclusion of the lemma is true.

For \(i \in \beta_+\), we have
\[
e_i^*(x(e_i)) = e_i^*(x'(e_i)) = 0
\]

therefore
\[
0 = e_i^*(x(e_i)) + e_i^*(x'(e_i))
\]
\[
= e_i^*((-v^* \otimes v)(e_i))
\]
\[
= -e_i^*(v(e_i) \sum_{k \in \beta_+} e_k)
\]
\[
= -v^*(e_i).
\]

So the elements of the basis \(\{e_i| i \in \beta_+\}\) are such that \(v^*\) vanishes on them, hence
\[
v^* = \sum_{i \in \beta_+} \alpha_{0,i} e_i^* = \sum_{i \in \gamma_1} \alpha_{0,i} e_i^*
\]

for some coefficients \(\{\alpha_{0,i}\}\) with \(\alpha_{0,1} = v^*(e_1) \neq 0\) and \(\alpha_{0,2d+1} = v^*(e_{2d+1}) \neq 0\) because \((x, x', v^*) \in J\).

Now, inductively, let us assume that
\[
(x^*)^{m-1}v^* = \sum_{i \in \gamma_m} \alpha_{m-1,i} e_i^*
\]
with \(\alpha_{m-1,m} \neq 0 \neq \alpha_{m-1,m+2d}\)
then

$$(x^*)^m v^*(e_j) = (x^*)^{m-1} v^*(xe_j)$$

$$= \sum_{i \in \gamma_m} \alpha_{m-1,i} e_i^*(xe_j)$$

now if $i \in \gamma_m$, in particular $i \notin \beta_+$, hence

$$0 = e_i^* (x + x' + v^* \otimes v) e_j$$

$$= e_i^* (xe_j + x'(e_j) + v^*(e_j)v)$$

$$= e_i^* (xe_j) + e_i^* (x'(e_j))$$

therefore

$$e_i^* (xe_j) = -e_i^* (x'(e_j)). \quad (2.6)$$

The LHS of (2.6) is nonzero if and only if $w^+(i) < w^+(j)$, while the RHS is nonzero if and only if $u^+(i) < u^+(j)$.

This shows that $(x^*)^m v^*(e_j) = 0$ if for all $i \in \gamma_{m-1}$, we have $u^+(i) \geq u^+(j)$ or $w^+(i) \geq w^+(j)$. This is equivalent to

$$(x^*)^m v^* = \sum_{i \in \gamma_{m+1}} \alpha_{m,i} e_i^*.$$
for some $\alpha_{m,i}$. Moreover,

\[
\alpha_{m,m+1} = (x^*)^m v^* (e_{m+1}) \\
= (x^*)^{m-1} v^* (xe_{m+1}) \\
= (x^*)^{m-1} v^* \left( \sum_{j=1}^{m} a_{j,m+1} e_j \right) \\
= \sum_{i \in \gamma_m} \alpha_{m-1,i} e_i^* \left( \sum_{j=1}^{m} a_{j,m+1} e_j \right) \\
= \alpha_{m-1,m} a_{m,m+1} \neq 0
\]

because $j \notin \gamma_m$ for $j < m$ and $(x,x',v^*) \in J$.

Analogously,

\[
\alpha_{m,m+2d+1} = (x^*)^m v^* (e_{m+2d+1}) \\
= (x^*)^{m-1} v^* (xe_{m+2d+1}) \\
= (x^*)^{m-1} v^* ((-x' - v^* \otimes v)e_{m+2d+1}) \\
= -(x^*)^{m-1} v^* \left( \sum_{j=1}^{m} (-c_{j,m+1}) e_{j+2d} + v^* (e_{m+2d+1}) v \right) \\
= \sum_{i \in \gamma_m} \alpha_{m-1,i} e_i^* \left( \sum_{j=1}^{m} c_{j,m+1} e_{j+2d} \right) + 0 \\
= \alpha_{m-1,m} c_{m,m+1} \neq 0
\]

because $j \notin \gamma_m$ for $2d < j < m + 2d$ and $(x,x',v^*) \in J$.

We let $S := (k[x^*]v^*)^\perp \subset V_+$, that is $S$ is the annihilator of the span of

\[
\{ v^*, x^* v^*, (x^*)^2 v^*, \ldots \} \subset V_+^*.
\]

We want to describe the relative position of the partial flags $F \cap S$ and $F' \cap S$. 

Definition 2.5.5. We define a new array \( \omega' \in M(\mu,1^d), (\mu',1^d)(\mathbb{Z}_{\geq 0}) \) by

\[
\omega' = \begin{pmatrix}
u'(d+1) & \ldots & \nu'(3d) \\
w'(d+1) & \ldots & w'(3d)
\end{pmatrix}
\]

where

\[
u'(i) = u_+(i)
\]

\[
w'(i) = \begin{cases}
w_+(i) & \text{if } i \in \beta_+ \\
w_+(j), \text{ where } j = \max\{l \in \delta_m | l < i\} & \text{if } i \in \delta_m \text{ for some } m.
\end{cases}
\]

Notice that this is well defined because \( \beta_+ \sqcup \bigsqcup_m \delta_m = \{1,2,\ldots,3d\} \).

Lemma 2.5.6. The relative position of the flags \( F \cap S \) and \( F' \cap S \) is \( \omega' \).

Proof. Remark that \( S = \bigcap_{m=0}^{d-1} \ker((x)^mv^*) \), therefore \( F_d \cap S = 0 = F'_d \cap S \). This follows from Lemma 2.5.4, since \((x)^m v^*(e_{m+1})\) and \((x)^m v^*(e_{m+2d+1})\) are both nonzero.

This implies that the types of the partial flags \( F \cap S \) and \( F' \cap S \) are respectively \((\mu,1^d)\) and \((\mu',1^d)\). In particular we have \( (F \cap S)_{\mu_1+\ldots+\mu_i} = F_{d+\mu_1+\ldots+\mu_i} \cap S \) and analogously for \( F' \cap S \).

If we let

\[
R_{ij}(\omega') := \{l \in \{d+1,\ldots,3d\} | w'(l) \leq i \text{ and } u'(l) \leq j\}
\]

then if we let \( r_{ij}(\omega_+) = \text{card} R_{ij}(\omega_+) \), we have that \( r_{ij}(\omega_+) = \dim(F_i \cap F'_j) \). More precisely, the vectors \( \{e_l | l \in R_{ij}(\omega_+)\} \) form a basis for \( F_i \cap F'_j \).
Remark that if \( m' \leq m \) and \( \delta_m \cap R_{ij}(\omega_+) \neq \emptyset \), then \( \delta_{m'} \cap R_{ij}(\omega_+) \neq \emptyset \). Hence there exist integers \( k_{ij} \geq 0 \) defined by the property that \( \delta_m \cap R_{ij}(\omega_+) \neq \emptyset \) if and only if \( m \leq k_{ij} \). Furthermore, since \( \gamma_m = \delta_m \sqcup \delta_{m+1} \sqcup \ldots \sqcup \delta_d \), we have that \( \gamma_m \cap R_{ij}(\omega_+) \neq \emptyset \) if and only if \( m \leq k_{ij} \).

This implies that \((x^*)^{m-1}v^*|_{F_i \cap F_j'} \neq 0\) if and only if \( m \leq k_{ij} \). Actually, by Lemma 2.5.4 these linear functionals on \( F_i \cap F_j' \) are linearly independent for \( m = 1, \ldots, k_{ij} \). Therefore

\[
\dim(F_i \cap F_j' \cap S) = \dim\left(\bigcap_{m=1}^{k_{ij}} \ker(x^*)^{m-1}v^*|_{F_i \cap F_j'}\right)
= \dim(F_i \cap F_j') - k_{ij}
= r_{ij}(\omega_+) - k_{ij}
\]

To conclude the proof of the lemma now we need to show that \( r_{ij}(\omega_+) - k_{ij} = r_{ij}(\omega') \). If we now define, in analogy to \( R_{ij}(\omega_+) \), \( R_{ij}(\omega') \) to be the set such that \( r_{ij}(\omega') = \text{card} \ R_{ij}(\omega') \), we have that

\[
R_{ij}(\omega_+) = (R_{ij}(\omega_+) \cap \beta_+) \sqcup \left( \bigcup_{m=1}^{d} R_{ij}(\omega_+) \cap \delta_m \right)
\]

\[
R_{ij}(\omega') = (R_{ij}(\omega') \cap \beta_+) \sqcup \left( \bigcup_{m=1}^{d} R_{ij}(\omega') \cap \delta_m \right).
\]

By definition of \( \omega' \), we have \( R_{ij}(\omega') \cap \beta_+ = R_{ij}(\omega_+) \cap \beta_+ \).

If \( m > k_{ij} \), then \( R_{ij}(\omega') \cap \delta_m = R_{ij}(\omega_+) = \emptyset \).

If \( m \leq k_{ij} \), then \( R_{ij}(\omega') \cap \delta_m = R_{ij}(\omega_+) \cap \delta_m \setminus \{s_m\} \), where \( s_m \) is the minimal element of \( R_{ij}(\omega_+) \cap \delta_m \).

Therefore \( R_{ij}(\omega') = R_{ij}(\omega_+) \setminus \{s_1, \ldots, s_{k_{ij}}\} \) which implies

\[
r_{ij}(\omega') = r_{ij}(\omega_+) - k_{ij}.
\]

\[\square\]

**Lemma 2.5.7.** The subspace \( S \) of \( V_+ \) has dimension \( 2d \) and it is invariant under both \( x \) and \( x' \).
Proof. The dimension claim just follows from the fact that $S = \bigcap_{m=0}^{d-1} \ker ((x^*)^m v^*)$ and that those functionals are linearly independent by Lemma 2.5.4. Therefore $\dim S = 3d - d = 2d$. If $z \in S$, then $(x^*)^m v^*(z) = 0$ for all $m$, therefore

$$(x^*)^m v^*(xz) = (x^*)^m v^*(z) = 0, \quad \text{for all } m$$

and $xz \in S$. In a similar way, for all $m$,

$$(x^*)^m v^*(x'z) = (x^*)^m v^* ((-x - v^* \otimes v)z) = -(x^*)^{m+1} v^*(z) - v^*(z)(x^*)^m v^*(v) = 0.$$

Definition 2.5.8. Let $\bar{x} = x|_S = -x'|_S$. We then have a map

$$g : Y_{\bar{\omega}+} \to \mathcal{F}_{\bar{x}} \times \mathcal{F}_{\bar{x}}$$

$$(F, F', v, x, x', v^*) \mapsto (F \cap S, F' \cap S).$$

Since $Y_{\bar{\omega}+}$ is irreducible, $\Im g$ lies in an irreducible component of $\mathcal{F}_{\bar{x}} \times \mathcal{F}_{\bar{x}}$.

So there exist two semistandard tableaux $\bar{T}, \bar{T}'$ such that, for all $y \in Y_{\bar{\omega}+}$, $(F \cap S, F' \cap S) \in C_{\bar{x}, \bar{T}} \times C_{\bar{x}, \bar{T}'}$.

In particular, by what we remarked at the beginning of the proof of Lemma 2.5.6, we have that $\bar{T}$ has content $(\mu, 1^d)$ and $\bar{T}'$ has content $(\mu', 1^d)$.

Lemma 2.5.9. The map

$$g : Y_{\bar{\omega}+} \to C_{\bar{x}, \bar{T}} \times C_{\bar{x}, \bar{T}'}$$

is surjective.

Proof. Let $y = (F, F', v, x, x', v^*) \in Y_{\bar{\omega}+}$, so that $g(y) = (F \cap S, F' \cap S)$. Given $(\bar{F}, \bar{F}') \in C_{\bar{x}, \bar{T}} \times C_{\bar{x}, \bar{T}'}$, define two flags $\bar{F}, \bar{F}'$ in $V_+$ as follows

$$\bar{F}'_i = \begin{cases} F_i & \text{if } i \leq d \\ \bar{F}'_{i-d} + F_d & \text{if } i > d \end{cases}$$
and \( \hat{F}' \) is defined in the same way, replacing \( F' \) and \( \hat{F}' \) where necessary.

Clearly, \( x \) preserves the flag \( \hat{F} \) and the same is true for \( x' \) and \( \hat{F}' \). We can then consider the point \( \hat{y} = (\hat{F}, \hat{F}', v, x, x', v^*) \in Y(1^d, \mu, 1^d, 1^d, \mu', 1^d) \). By construction, \( \hat{y} \) is such that \( \hat{F} \cap S = \hat{F} \) and \( \hat{F}' \cap S = \hat{F}' \). Consider the maps

\[
(\hat{F}, \hat{F}') \mapsto (\hat{F}, \hat{F}') \mapsto \hat{y}.
\]

Let \( f \) be the composition of those, then

\[
f : C_{\tilde{\omega}, \tilde{T}} \times C_{\tilde{\omega}, \tilde{T}'} \to Y(1^d, \mu, 1^d, 1^d, \mu', 1^d).
\]

Since \( C_{\tilde{\omega}, \tilde{T}} \times C_{\tilde{\omega}, \tilde{T}'} \) is irreducible, the image of \( f \) lies in an irreducible component of \( Y(1^d, \mu, 1^d, 1^d, \mu', 1^d) \). Notice that \( f(F \cap S, F' \cap S) = y \in Y_{\tilde{\omega}+} \), hence \( \text{Im} f \subset Y_{\tilde{\omega}+} \). Therefore \( \hat{y} \in Y_{\tilde{\omega}+} \) and \( g(\hat{y}) = (\hat{F}, \hat{F}') \), thus the lemma is proved. \( \square \)

**Remark 2.5.10.** For \((F, F', v, x, x', v^*)\) in an open dense subset of \( Y_{\tilde{\omega}+} \), we know by Lemma 2.5.6 that \( \omega' \) is the relative position of the partial flags \( F \cap S \) and \( F' \cap S \). Also, by Lemma 2.5.9, the preimage of the open dense subset of \( C_{\tilde{\omega}, \tilde{T}} \times C_{\tilde{\omega}, \tilde{T}'} \) for which Theorem 2.3.1 applies, contains an open dense subset of \( Y_{\tilde{\omega}+} \). Therefore we have

\[
(\tilde{T}, \tilde{T}') \xrightarrow{\text{RSK}} \omega'.
\]

Now, consider the spaces \( F_{d+n}, F'_{d+n'} \) where \( n, n' \) are the number of parts of \( \mu \) and \( \mu' \) respectively, i.e. \( \mu = (\mu_1, \ldots, \mu_n), \mu' = (\mu'_1, \ldots, \mu'_{n'}) \). By (2.4), they are invariant under both operators \( x \) and \( x' \), therefore the same is true for \( V := F_{d+n} \cap F'_{d+n'} \). Notice that in the basis of (2.3), we have that \( V = (e_{d+1}, \ldots, e_{2d}) \).

Consider the flags \( F \cap V \) and \( F' \cap V \). It is clear that the relative position \( M(F \cap V, F' \cap V) = \omega \), and that \( \tilde{y} = (F \cap V, F' \cap V, v, x|_V, x'|_V, v^*|_V) \in Y_{\tilde{\omega}} \).

Applying the mirabolic RSK correspondence of Definition 2.4.13 to \( Y_{\tilde{\omega}} \) we get

\[
(t(\tilde{\omega}), T(\tilde{\omega}), T'(\tilde{\omega})), \quad \text{with} \quad t(\tilde{\omega}) = (\lambda(\tilde{\omega}), \theta(\tilde{\omega}), \lambda'(\tilde{\omega})). \tag{2.7}
\]

Thus we have \( F \cap V \in C_{x|_V, T(\tilde{\omega})}, F' \cap V \in C_{x'|_V, T'(\tilde{\omega})} \) and \( \theta(\tilde{\omega}) \) is the Jordan type of \( x|_V/k[x]|_V \).
Lemma 2.5.11. The semistandard tableau $T(\tilde{\omega})$ (resp. $T'(\tilde{\omega})$) is obtained from the tableau $\bar{T}$ (resp. $\bar{T}'$) of Definition 2.5.8 by removing all boxes with numbers $n+1, \ldots, n+d$ (resp. $n'+1, \ldots, n'+d$).

Proof. By symmetry, it is enough to prove the case of $T(\tilde{\omega})$. The tableau $\bar{T}$ is defined by the condition that $F \cap S \in C_{x|V;\bar{T}}$. If we let $T^{(n)}$ be the tableau obtained by removing from $\bar{T}$ all numbers greater than $n$, we have

$$F \cap S \cap F_{d+n} \in C_{x|S \cap F_{d+n};T^{(n)}}.$$ 

By the remark at the beginning of the proof of Lemma 2.5.6 and by the definition of $V$, the spaces $V$ and $S \cap F_{d+n}$ are both complementary to $F_d$ inside $F_{d+n}$. So they can both be identified with the image of the map

$$F_{d+n} \rightarrow F_{d+n}/F_d.$$ 

(2.8)

Notice that under this map

$$F \cap V \mapsto (F \cap F_{d+n})/F_d$$

and both operators $x|V$ and $x|S \cap F_{d+n}$ get identified via (2.8) with $x|F_{d+n}/F_d$. Therefore it follows that

$$(F \cap F_{d+n})/F_d \in C_{x|F_{d+n}/F_d;T(\tilde{\omega})}$$

$$(F \cap F_{d+n})/F_d \in C_{x|F_{d+n}/F_d;T^{(n)}}.$$ 

By Lemma 2.5.9 the set of all $F \cap F_{d+n}$, for varying $y \in Y_{\tilde{\omega}+}$, covers all points in these irreducible components. Therefore they must be equal, i.e.

$$C_{x|F_{d+n}/F_d;T(\tilde{\omega})} = C_{x|F_{d+n}/F_d;T^{(n)}}$$

which implies that $T(\tilde{\omega}) = T^{(n)}$. \qed
Suppose that applying the algorithm 2.5.1 to $\tilde{\omega}$ we obtain $(\lambda^c, \theta^c, (\lambda')^c, T^c, (T')^c)$. We want to show that this coincides with the quintuple $(\lambda(\tilde{\omega}), \theta(\tilde{\omega}), \lambda'(\tilde{\omega}), T(\tilde{\omega}), T'(\tilde{\omega}))$ of (2.7).

Remark that $i + d \in \delta_m$ if and only if at the $i$-th step of the algorithm, the number $w(i)$ is being inserted in the $m$-th position of the row $R$. In this case, $w'(i)$ is the number bumped from $R$ and inserted in $T_i$.

Therefore, if we apply the RSK correspondence from Section 2.2.3 to $\omega'$ we get a pair of tableaux $(T(\omega'), T'(\omega'))$ that satisfy the following:

- the tableau $T_{2d}$ from the algorithm is the same as $T(\omega')$;
- the tableau $T'_d$ is obtained from $T'(\omega')$ by removing all numbers strictly greater than $n'$.

We also know that $\omega' \xrightarrow{RSK} (\bar{T}, \bar{T}')$, therefore $\bar{T} = T(\omega')$ and $\bar{T}' = T'(\omega')$.

By Lemma 2.5.11, this implies that both $T^c$ and $T(\tilde{\omega})$ are obtained from $\bar{T} = T(\omega') = T_{2d}$ by removing the last $d$ numbers, so $T^c = T(\tilde{\omega})$.

Again by Lemma 2.5.11, $(T')^c = T'_d$ and $T'(\tilde{\omega})$ are both obtained from $\bar{T}' = T'(\omega')$ by removing all numbers greater than $n'$, so $(T')^c = T'(\tilde{\omega})$.

It also follows immediately that $\lambda^c = \lambda(\tilde{\omega})$ and $(\lambda')^c = \lambda'(\tilde{\omega})$ since those are respectively the shape of $T(\tilde{\omega})$ and of $T'(\tilde{\omega})$.

The only thing left to prove is that $\theta^c = \theta(\tilde{\omega})$, which will follow from the next Lemma.

**Lemma 2.5.12.** If we let $\nu$ be the shape of the tableau $\bar{T}$, then $\theta := \theta(\tilde{\omega})$ is obtained from $\nu$ by removing the first part of the partition. That is $\theta = (\theta_1, \theta_2, \ldots) = (\nu_2, \nu_3, \ldots)$.

**Proof of Lemma.** From Definition 2.5.8, the shape of $\bar{T}$ is the Jordan type of $\bar{x} = x|_S$. On the other hand, $\theta$ is the type of $x|_{V/k[x]v}$.

Consider the space $D := (F_d + F'_d) \cap S + k[x]v$. Since $v \in S$ by Lemma 2.5.4 and since $S$ is $x$-invariant, we have $k[x]v \subset S$. Therefore $D = (F_d + F'_d + k[x]v) \cap S$. Now, $F_d$ is $x$-invariant by (2.4), and if $z \in F'_d$, then $x(z) = -x'(z) - v^*(z)v \in F'_d + k[x]v$. So $F'_d + k[x]v$ is also $x$-invariant. It follows that $D$ is invariant under $x$.

**Claim 2.5.13.** In Jordan normal form, the nilpotent operator $x|_D$ has a single block.
Proof of Claim. We can assume that the matrices $A$ and $C$ of (2.5) have rank $n - 1$. Therefore $x|_{F_d'}$, which is represented by the matrix $A$, has a single Jordan block. In the same way, $-C$ represents $x'|_{F'_d}$ which also has a single Jordan block.

Now, given the basis $\{e_i|i = 1, \ldots, 3d\}$ of $V_+$ defined in (2.3), we have that

$$F_d + F'_d = \langle e_1, \ldots, e_d, e_{2d+1}, \ldots, e_{3d}\rangle$$

hence, by Lemma 2.5.4 for $m = 0, \ldots, d - 1$, the linear functionals

$$(x^*)^m v^* = \sum_{i \in \gamma_{m+1}} \alpha_{m,i} e_i^*$$

are linearly independent on $F_d + F'_d$. It follows that

$$\dim(F_d + F'_d) \cap S = 2d - d$$

$$= d$$

$$\dim D = \dim(F_d + F'_d) \cap S + k[x]v$$

$$= d + \dim k[x]v$$

since $(F_d + F'_d) \cap S \cap k[x]v = 0$. (This is because $k[x]v \subset \langle e_{d+1}, \ldots, e_{2d}\rangle$ by (2.4)).

Now, let $z = \sum_{i=1}^d z_i e_i \in F_d$, with $z_d \neq 0$, and let $z' = \sum_{i=2d+1}^{3d} z_i e_i \in F'_d$, with $z_{3d} \neq 0$. We want to show that we can choose the $z'_i$s in such a way that $z + z' \in S$. Consider the equation

$$0 = (x^*)^{d-1} v^*(z + z')$$

$$= \left( \sum_{i \in \gamma_d} \alpha_{d-1,i} e_i^* \right) \left( \sum_{j=1, \ldots, d} z_j e_j \right)_{2d+1, \ldots, 3d}$$

$$= \alpha_{d-1,d} z_d + \alpha_{d-1,3d} z_{3d}$$

Since, by Lemma 2.5.4, $\alpha_{d-1,d}$ and $\alpha_{d-1,3d}$ are both nonzero, we can find nonzero $z_d, z_{3d}$
such that the equation holds. We find

\[ 0 = (x^*)^{d-2}v^*(z + z') \]

\[ = \left( \sum_{i \in \gamma_{d-1}} \alpha_{d-2,i}e_i^* \right) \left( \sum_{j=1,\ldots,d} z_j e_j \right) \]

\[ = \alpha_{d-2,d-1}z_{d-1} + \alpha_{d-2,d}z_d + \alpha_{d-2,3d-1}z_{3d-1} + \alpha_{d-2,3d}z_{3d}. \]

Since \( \alpha_{d-2,d-1} \) and \( \alpha_{d-2,3d-1} \) are both nonzero, we can choose \( z_{d-1}, z_{3d-1} \) so that the equation holds. Iterating this procedure, we find \( z, z' \) such that \( (x^*)^m v^*(z + z') = 0 \) for all \( 0 \leq m \leq d - 1 \), hence \( z + z' \in (F_d + F'_d) \cap S \).

Remark that, since \( z_d \neq 0 \), and since \( x \) acts as the matrix \( A \) from (2.5) on \( F_d \), \( F_d = \langle z, \ldots, x^{d-1}z \rangle \). In the same way, we have \( F'_d = \langle z', \ldots, (x')^{d-1}z' \rangle \).

We are now going to prove that \( z + z' \) is a cyclic vector for \( x \) on \( D \).

We have

\[ x(z + z') = x(z) + x(z') \]

\[ = x(z) - x'(z') - v^*(z')v \]

\[ x^2(z + z') = x(x(z) - x'(z') - v^*(z')v) \]

\[ = x^2(z) - v^*(z')x(v) + (-x' - v^* \otimes v)(-x'(z')) \]

\[ = x^2(z) + (x')^2(z') + v^*(x'z')v - v^*(z')x(v) \]

\[ \ldots \]

\[ x^d(z + z') = x^d(z) + (x')^d(z') + (-1)^d v^*((x')^{d-1}z')v + \ldots - v^*(z')x^{d-1}v. \]

Remark that, since \( z_d \neq 0 \), and since \( x \) acts as the matrix \( A \) from (2.5) on \( F_d \), \( F_d = \langle z, \ldots, x^{d-1}z \rangle \). In the same way, we have \( F'_d = \langle z', \ldots, (x')^{d-1}z' \rangle \). Also, notice that

\[ v^*((x')^{d-1}z') = v^*((-1)^{d-1}c_1, \ldots, c_{d-1}, d e_{2d+1}) \]

\[ = (-1)^{d-1}c_1, \ldots, c_{d-1}, d \alpha_{0,2d+1} \neq 0 \]

Since \( x^d(z) = (x')^d(z') = 0 \), we have \( x^d(z + z') \in k[x]v \) and has a nonzero coefficient in \( v \).
Therefore, it follows from the computation (2.9) that \( x^m(z + z') \) are linearly independent for \( m = 0, \ldots, d \).

Moreover, the elements

\[
x^m(z + z') \quad \text{with} \quad d \leq m \leq d + (\dim k[x]v - 1),
\]

span \( k[x]v \). In conclusion, the set \( \{x^m(z + z') \mid m = 0, \ldots, d + (\dim k[x]v - 1)\} \) spans \( D = (F_d + F'_d) \cap S + k[x]v \). This means that \( z + z' \) is a cyclic vector, hence \( x|_D \) has a single block in Jordan normal form.

Now, the identification of (2.8) gives us an isomorphism of \( x \)-modules

\[
\alpha : V \xrightarrow{\sim} S \cap F_{d+n}.
\]

Remark that \( D \cap V = k[x]v \), and that \( D + (S \cap F_{d+n}) = S \). Also

\[
D \cap \alpha(V) = D \cap (F_{d+n} \cap S)
\]

\[
= (F_d + F'_d + k[x]v) \cap F_{d+n} \cap S
\]

\[
= (F_d + k[x]v) \cap S
\]

\[
= F_d \cap S + k[x]v
\]

\[
= k[x]v
\]

We have then isomorphisms of \( x \)-modules

\[
V/k[x]v = V/(V \cap D)
\]

\[
\simeq (D + V)/D
\]

\[
\simeq (D + \alpha(V))/D
\]

\[
= S/D.
\]

Hence, \( x|_{V/k[x]v} = x|_{S/D} \). So \( \theta \) is also the Jordan type of \( x|_{S/D} \).

We know that \( \dim D \geq d \), \( \dim S = 2d \) and \( x|_D \) is a single Jordan block. It follows that \( \theta \), which is the Jordan type of \( x|_{S/D} \), is obtained from the one of \( x|_S \) by removing the
maximal part of the partition. This concludes the proof of the Lemma and consequently of the Theorem. □

2.6 Example of the mirabolic RSK correspondence

Let $V \simeq k^7$, and let a basis of $V$ be $\{u_1, u_2, \ldots, u_7\}$. We consider the nilpotents $x$, $x'$, expressed as matrices in the basis $\{u_i\}$.

$$
x = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} ;
\quad x' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Then we have that the Jordan type of $x$ is $\lambda = (4, 2, 1)$ and the type of $x'$ is $\lambda' = (3, 2, 1, 1)$. If we let $v = u_2$, $v^* = -u_3^*$, we have indeed

$$
x + x' + v^* \otimes v = 0.
$$

Now, $x(v) = 0$, therefore $k[x]v = \langle v \rangle$ and $V/k[x]v \simeq \langle u_i | i \neq 2 \rangle$. We then have that the type of $x|_{V/k[x]v}$ is $\theta = (3, 2, 1)$.

Let us define the flag $F$ by

$$
F_1 = \langle u_2, u_6 \rangle \\
F_2 = \langle u_2, u_6, u_3, u_1 + u_7 \rangle \\
F_3 = \langle u_2, u_6, u_3, u_1 + u_7, u_4 \rangle \\
F_4 = V.
$$
Then $F \in \mathcal{F}_x^\mu$ for $\mu = (2,2,1,2)$. We also define $F' \in \mathcal{F}_{x'}^{\mu'}$, with $\mu' = (2,2,3)$, by

$$F'_1 = \langle u_1, u_3 \rangle$$
$$F'_2 = \langle u_1, u_3, u_4, u_6 \rangle$$
$$F'_3 = V.$$  

The semistandard tableaux associated to $F$ and $F'$ are respectively

$$T = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & & \\
4 & & & \\
\end{array}$$

$$T' = \begin{array}{cccc}
1 & 2 & 3 & \\
1 & 3 & & \\
2 & & & \\
3 & & & \\
\end{array}$$

In which $\text{GL}_7$-orbit does the point $(F, F', v)$ lie? The relative position of $F$ and $F'$ is $\omega \in M(2,2,1,2),(2,2,3)(\mathbb{Z}_{\geq 0})$ which we can see as a matrix or as an array

$$\omega = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 & 3 \\
4 & 2 & 3 & 1 & 4 & 2 & 1 \\
\end{pmatrix}$$

and since $v \in F_1 \setminus (F_1 \cap F'_2)$, we have that $\beta = \{4, 7\}$. So for $\tilde{\omega} = (\omega, \beta)$, we have $y = (F, F', v, x, x', v^*) \in N^*\Omega_{\omega, \beta}$. Now, by the mirabolic RSK correspondence of Definition 2.4.13, we have

$$(\omega, \beta) \rightarrow (\lambda, \theta, \lambda', T, T').$$

Let us verify that this is indeed the result we obtain when we apply the algorithm 2.5.1. Our input is

$$(\omega, \beta) = \left( \begin{pmatrix}
1 & 1 & 2 & 2 & 3 & 3 & 3 \\
4 & 2 & 3 & 1 & 4 & 2 & 1 \\
\end{pmatrix}, \{4,7\} \right).$$

To start, we set $T_0 = T'_0 = \emptyset$, $R = \underline{8 \, 9 \, 10 \, 11 \, 12 \, 13 \, 14}$

- $1 \notin \beta$.

$$R = \underline{4 \, 9 \, 10 \, 11 \, 12 \, 13 \, 14} \quad T_1 = \underline{8} \quad T'_1 = \underline{1}$$
\[ R = 2 \overline{9} 10 11 12 13 14 \quad T_2 = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \quad T'_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ R = 2 3 \overline{10} 11 12 13 14 \quad T_3 = \begin{bmatrix} 4 \\ 9 \\ 8 \end{bmatrix} \quad T'_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \]

\[ R = 2 3 \overline{10} 11 12 13 14 \quad T_4 = \begin{bmatrix} 1 \\ 9 \\ 4 \\ 8 \end{bmatrix} \quad T'_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \]

\[ R = 2 3 4 \overline{11} 12 13 14 \quad T_5 = \begin{bmatrix} 1 \\ 9 \\ 10 \\ 4 \\ 8 \end{bmatrix} \quad T'_5 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} \]

\[ R = 2 3 4 \overline{11} 12 13 14 \quad T_6 = \begin{bmatrix} 1 \\ 2 \\ 10 \\ 4 \\ 9 \\ 8 \end{bmatrix} \quad T'_6 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} \]

\[ R = 2 3 4 \overline{11} 12 13 14 \quad T_7 = \begin{bmatrix} 1 \\ 2 \\ 10 \\ 1 \\ 9 \\ 4 \\ 8 \end{bmatrix} \quad T'_7 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 3 \\ 2 \\ 3 \end{bmatrix} \]

\[ T' = T'_7 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 3 \\ 2 \\ 3 \end{bmatrix} \]

which agrees with what we had before.

- Insert \( R \) into \( T_7 \), get

\[ T_{14} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 11 \\ 12 \\ 13 \\ 14 \\ 1 \\ 2 \\ 10 \\ 4 \\ 9 \\ 8 \end{bmatrix} \]

- The shape of \( T_{14} \) is \( \nu = (8, 3, 2, 1) \), so \( \theta = (3, 2, 1) \) as we wanted.

- Removing all numbers greater than 7 from \( T_{14} \), we get

\[ T = T_{14}^{(7)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 1 \\ 2 \\ 4 \end{bmatrix} \]
CHAPTER 3
MIRABOLIC CONVOLUTION ALGEBRAS

In this chapter we discuss convolution algebras over flag varieties both in the finite and affine setting. We will then see how Solomon ([So]) and Travkin ([T]) have generalized some of these constructions to the mirabolic case. Finally, we will present some results on what can be done in the mirabolic affine situation.

3.1 Iwahori-Hecke Algebra of $S_d$

The symmetric group $S_d$ is a Coxeter group, with simple reflections being the adjacent transpositions $s_i = (i \ i + 1)$ for $i = 1, \ldots, d - 1$. We can define the corresponding Iwahori-Hecke Algebra $\mathcal{H}$ as follows.

Definition 3.1.1. $\mathcal{H}$ is a free $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module, with basis $\{T_w \mid w \in S_d\}$ satisfying the following relations:

\[
\begin{align*}
T_{s_i}^2 &= (q - 1)T_{s_i} + qT_e \\
T_{s_i}T_{s_{i+1}} &= T_{s_{i+1}}T_{s_i} \\
T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} \quad |i - j| \geq 2
\end{align*}
\]

(3.1)

As we mentioned in Section 1.5 of the Introduction, the specialization $\mathcal{H}_q$ of the Hecke algebra at $q \mapsto q$ is realized as a convolution algebra. This is done by taking $G = \text{GL}_d(\mathbb{F}_q)$, $B$ its Borel subgroup, and $\mathcal{F} = G/B$ the variety of complete flags in $V = \mathbb{F}_q^d$. We let $E = \mathbb{C}(\mathcal{F} \times \mathcal{F})$ be the vector space of all complex valued functions on $\mathcal{F} \times \mathcal{F}$. Then given two functions $f, g \in E$ we define their convolution to be

\[
f * g(F, F') = \sum_{H \in \mathcal{F}} f(F, H)g(H, F').
\]

(3.2)

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This is an associative product with identity element the characteristic function of the diagonal, and it descends to a well defined product on $E^G \simeq \mathbb{C}(G \setminus \mathcal{F} \times \mathcal{F})$. This is the space of functions that are invariant under the diagonal group action or equivalently the space of functions that are constant on the $G$-orbits. As we know, the orbits in this case are given by elements of the symmetric group. We define then $T_w$ to be the characteristic function of the $G$-orbit of all pairs of flags in relative position $w$. It is then pretty easy to see that, with this identification, there is an algebra isomorphism

$$E^G \simeq \mathcal{H}/(q - q) = \mathcal{H}_q.$$ 

**Example 3.1.2.** Let us verify that the first relation in (3.1) is satisfied by computing $T_{s_i} * T_{s_i}$. First of all, notice that two flags $F, F' \in \mathcal{F}$ are in relative position $s_i$ if and only if $F_i \neq F'_i$ and $F_j = F'_j$ for $j \neq i$.

$$T_{s_i} * T_{s_i}(F, F') = \sum_{H \in \mathcal{F}} T_{s_i}(F, H)T_{s_i}(H, F').$$

For the RHS to be nonzero we need $(F, H)$ to be in relative position $s_i$ and same for $(H, F')$, which implies that $F_j = H_j = F'_j$ for $j \neq i$. Hence two possibilities arise for the LHS to be nonzero: either $F_i = F'_i$, or $F_i \neq F'_i$. So $T_{s_i} * T_{s_i}$ is supported on the orbits corresponding to the identity and to $s_i$. Now the only thing left is to compute the coefficients of $T_{s_i}$ and $T_e$ in the product. In order to do that, we just need to count how many terms are in the sum on the RHS, i.e. how many flags $H$ satisfy the condition of being in relative position $s_i$ with both $F$ and $F'$ in each of the two cases. Since $F_j = H_j = F'_j$ for $j \neq i$, the only thing we can choose is $H_i \in \mathbb{P} (H_{i+1}/H_{i-1}) = \mathbb{P} (F_{i+1}/F'_{i-1})$. There are $q + 1$ lines in a 2-dimensional space over $\mathbb{F}_q$, and we need $F_i \neq H_i \neq F'_i$. In conclusion, if $F_i = F'_i$, we have $q + 1 - 1 = q$ choices for $H_i$, while if $F_i \neq F'_i$, we have $q + 1 - 2 = q - 1$ choices for $H_i$.

### 3.2 Mirabolic Hecke Algebra

For this section we will use the same notation as in [T], which is similar to what we used in Section 2.4.1. The only difference is that, since we are only concerned with complete flags, we will index orbits by pairs $(w, \beta)$ where $w \in S_d$, and $\beta \subset \{1, \ldots, d\}$ satisfying the
condition of 2.4.4 (with $u(i) = i$). As in Section 3.1, we let $G = \text{GL}_d(\mathbb{F}_q)$ and we consider the space of $G$-invariant functions $R = \mathbb{C}(\mathcal{F} \times \mathcal{F} \times V)^G = \mathbb{C}(\mathcal{G} \setminus \mathcal{F} \times \mathcal{F} \times V)$. Travkin in [T] has studied this space as an $\mathcal{H}$-bimodule. The action is given by convolution: if we think of $\mathcal{H}_q$ as the convolution algebra in Section 3.1, for $\alpha \in \mathcal{H}_q$ and $\beta \in R$ we can define

$$
\alpha \ast \beta(F, F', v) = \sum_{H \in \mathcal{F}} \alpha(F, H) \beta(H, F', v);
\beta \ast \alpha(F, F', v) = \sum_{H \in \mathcal{F}} \beta(F, H, v) \alpha(H, F').
$$

Again, it is immediate that this gives well defined $G$-invariant functions. More interestingly, Solomon in [So] had defined an associative algebra structure on $R$, which can be stated in terms of convolution as follows. If, $\alpha, \beta \in R$,

$$
\alpha \ast \beta(F, F', v) = \sum_{H \in \mathcal{F}, u \in V} \alpha(F, H, u) \beta(H, F', v - u).
$$

Denoting the characteristic functions of orbits by $T_{w, \beta}$ as in [T], the identity element for this product is $T_{e, \emptyset}$ which is the characteristic function of the orbit $\{(F, F', v) | F = F', v = 0\}$.

**Definition 3.2.1.** Since all the structure constants appearing are polynomials in $q$, we can consider $R$ to be the specialization at $q \mapsto q$ of a $\mathbb{C}[q^{1/2}, q^{-1/2}]$-algebra $\mathcal{R}$. We call $\mathcal{R}$, with the product (3.3), the *Mirabolic Hecke Algebra*.

**Remark 3.2.2.** The Mirabolic Hecke Algebra contains $\mathcal{H}$ as a subalgebra, with the inclusion being given by $T_w \mapsto T_{w, \emptyset}$. It is clear from the definition of the product that this inclusion agrees with the bimodule structure defined by Travkin.

**Remark 3.2.3.** The involution on $\mathcal{F} \times \mathcal{F} \times V$ defined by $(F, F', v) \mapsto (F', F, v)$ induces an algebra anti-automorphism of $\mathcal{R}$. In the natural basis for $\mathcal{R}$, this can be written as $T_{w, \beta} \mapsto T_{w^{-1}, w(\beta)}$.

**3.2.1 Comparison with Solomon’s Conventions**

In this section we explain how to translate the work from Solomon’s paper [So] into the notation that we use. We consider the group $P$ of affine transformations in $V$, which is isomorphic to the semidirect product $\text{GL}(V) \ltimes V$. We can think of $P$ as the group of
$d + 1 \times d + 1$ matrices that fixes a nonzero vector, i.e. the mirabolic subgroup of $\text{GL}_{d+1}$.

We can write it as block matrices as follows:

$$P = \left\{ \begin{pmatrix} 1 & 0 \\ v & g \end{pmatrix} \bigg| v \in V, g \in \text{GL}(V) \right\}.$$ 

Solomon then considers the Borel subgroup $B \subset \text{GL}(V)$ as a subgroup of $P$ and defines the algebra of double cosets $B \backslash P / B$. The multiplication is given by the following formula, if $p_1, \ldots, p_m$ are a set of representatives for the double cosets,

$$(Bp_iB) \cdot (Bp_jB) = \sum_{k=1}^{m} c_{ijk} (Bp_kB)$$

with

$$c_{ijk} = \frac{\text{card} \{(Bp_iB)^{-1}p_k \cap Bp_jB\}}{\text{card}(B)}$$

which is the same as saying that the coefficient $c_{ijk}$ equals $\text{card}(B)$ times the number of pairs $(x, y)$, $x \in Bp_iB$, $y \in Bp_jB$ such that $xy = p_k$.

To see how this is the same algebra as the one defined by (3.3), first of all we need to observe that we have the following isomorphisms of sets of orbits

$$B \backslash P / B \simeq G \backslash (G/B \times P / B) \simeq G \backslash (G/B \times G/B \times P / G) \simeq G \backslash (G/B \times G/B \times V)$$

that are given by

$$B \begin{pmatrix} 1 & 0 \\ v & g \end{pmatrix} B \mapsto G \cdot \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} B, \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} B \mapsto G \cdot (gB, 1B, v).$$

Now, given $p_k = \begin{pmatrix} 1 & 0 \\ v_k & g_k \end{pmatrix}$, counting the pairs of elements $(x, y)$ such that $xy = p_k$ is the same as counting the set $\{(g, g', v, v') | gg' = g_k, gv + v' = v_k\}$. And it is now clear from this description that the structure constants in the algebra defined in (3.4) are exactly the same as the ones obtained by convolution in (3.3).
3.2.2 Generators and Relations

Solomon has given a description of the Mirabolic Hecke Algebra in terms of generators and relations in [So], proving the following theorem.

**Theorem 3.2.4.** The algebra $\mathcal{R}$ is isomorphic to the associative algebra with generators $\{x_i | i = 0, 1, \ldots, d-1\}$ and relations

\[
x_0^2 = (q-2)x_0 + (q-1) \cdot 1 \quad (3.5)
\]

\[
x_i^2 = (q-1)x_i + q \cdot 1 \quad i \geq 1 \quad (3.6)
\]

\[
x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1} \quad i \geq 1 \quad (3.7)
\]

\[
x_0x_1x_0x_1 = (q-1)(x_1x_0x_1 + x_1x_0) - x_0x_1x_0 \quad (3.8)
\]

\[
x_1x_0x_1x_0 = (q-1)(x_1x_0x_1 + x_0x_1) - x_0x_1x_0 \quad (3.9)
\]

\[
x_ix_j = x_jx_i \quad |i - j| \geq 2 \quad (3.10)
\]

The isomorphism is given by $x_0 \mapsto T_{e,\{1\}}$, corresponding to the orbit $\{(F, F', v) | F = F', v \in F_1 \setminus \{0\}\}$. For $i \geq 1$, $x_i \mapsto T_{s_i,0}$, which are the simple reflections that generate $\mathcal{H}$ as a subalgebra of $\mathcal{R}$. In fact relations (3.6), (3.7) (3.10) are the same as (3.1).

3.2.3 Irreducible representations of $\mathcal{R}$

The Mirabolic Hecke algebra is a semisimple algebra of dimension $\sum_{k=0}^{d} k! \binom{d}{k}^2$, and its irreducible representations have been described by Siegel in [Si]. They are parametrized by pairs $(\theta, k)$, where $0 \leq k \leq d$ and $\theta$ is a partition of $k$. In [Si, Prop 3.10] there is an explicit formula for the action of $\mathcal{R}$ on the irreducible left module $M^{\theta, k}$, which, upon inspection, turns out to be equivalent to what is described in the following proposition.

Fix $k$ such that $0 \leq k \leq d$. Let $\mathcal{H}(k) \subset \mathcal{H}$ be the subalgebra generated by $T_{s_1}, \ldots, T_{s_{k-1}}$, and let $\mathcal{H}(d-k) \subset \mathcal{H}$ be the subalgebra generated by $T_{s_{k+1}}, \ldots, T_{s_{d-1}}$. They are isomorphic to the Hecke algebras of $S_k$ and $S_{d-k}$ respectively.

**Proposition 3.2.5.** There is an isomorphisms of left $\mathcal{H}$-modules

\[
M^{\theta, k} \simeq \mathcal{H} \otimes \mathcal{H}(k) \otimes \mathcal{H}(d-k) V^{\theta} \boxtimes U_{\text{sign}}; \quad (3.11)
\]
where $V^\theta$ is the irreducible representation of $\mathcal{H}_{(k)}$ corresponding to the partition $\theta \vdash k$ and $U_{\text{sign}}$ is the one dimensional sign representation of $\mathcal{H}_{(d-k)}$.

Moreover, consider the basis $\{T_w \otimes v_T\}_{w,T}$ of $M^{\theta,k}$, where $w \in S_d/(S_k \times S_{d-k})$ such that $w$ is the element with minimal length in the coset, and $T \in T_{\lambda}$. Then

$$x_0 \cdot (T_w \otimes v_T) = \begin{cases} (q - 1)T_w \otimes v_T & \text{if } w^{-1}(1) \in \{1, \ldots, k\} \\ -T_w \otimes v_T & \text{if } w^{-1}(1) \in \{k + 1, \ldots, d\} \end{cases} .$$

This explicit description, in particular, tells us that $\dim M^{\theta,k} = (\frac{d}{k}) f_{\theta}$, where $f_{\theta}$ is the number of standard tableaux of shape $\theta$. Also, Equation 3.11 allows us to give another proof of [T, Prop 8].

**Proposition 3.2.6 (Travkin).** If we extend the scalars to $\mathbb{C}(q^{\frac{1}{2}})$, $\mathcal{R}$ decomposes as a direct sum of irreducible $\mathcal{H}$-bimodules as

$$\mathcal{R} \simeq \bigoplus_{(\lambda',\theta',(\lambda')^t) \in T} \left(V^\lambda\right)^* \otimes V^{\lambda'}$$

where $V^\lambda$ is as in (3.11) and the sum is over the set $T$ described right after Remark 2.4.9.

**Proof.** When we extend the scalars to $\mathbb{C}(q^{\frac{1}{2}})$, we have the isomorphism $\mathcal{H} \simeq \mathbb{C}(q^{\frac{1}{2}})[S_d]$ with the group algebra of the symmetric group. We can then reinterpret (3.11) as the induced representation

$$M^{\theta,k} \simeq \text{Ind}_{S_k \times S_{d-k}}^{S_d} V^\theta \otimes U_{\text{sign}}$$

hence, by the Littlewood-Richardson’s rule we have an isomorphism of $S_d$-representations

$$M^{\theta,k} \simeq \bigoplus_{\{\lambda|\lambda_i \geq \theta_i \geq \lambda_i - 1\}} V^\lambda \quad (3.12)$$

where the sum is over all the partitions $\lambda \vdash d$ obtained from $\theta$ by adding $d - k$ boxes, with no two of them in the same row. In the notation of Remark 2.4.9, the sum is over $\lambda$ such that $(\lambda^t, \theta^t) \in \mathfrak{P}$.

Now, $\mathcal{R}$ is a semisimple finite dimensional algebra, with irreducible representations
\[ \{ M_{\theta,k} | 0 \leq k \leq d, \theta \vdash k \}, \text{ so we have isomorphisms, as } \mathcal{H}\text{-bimodules,} \]

\[
\mathcal{R} \cong \bigoplus_{\theta,k} \text{End}(M_{\theta,k}) \\
\cong \bigoplus_{\theta,k} (M_{\theta,k}^*) \otimes M_{\theta,k} \\
\text{by (3.12)} \cong \bigoplus_{\theta,k} \left( \bigoplus_{(\lambda,\lambda') \in \mathcal{P}} (V^\lambda)^* \otimes (V^{\lambda'}) \right)
\]

\[ \cong \bigoplus_{(\lambda,\lambda',(\lambda')^t) \in T} \left( (V^\lambda)^* \otimes V^{\lambda'} \right) \]

\[ \square \]

### 3.3 Affine Hecke Algebra

In this section, we will let \( k = \mathbb{F}_q((t)) \) be the field of Laurent series in one variable, \( V = k^d \) and \( G = \text{GL}(V) \), with Iwahori subgroup \( I \) as in (1.8). We will also let \( \mathcal{O} = \mathbb{F}_q[[t]] \subset k \) be the ring of formal power series. In Section 1.5 of the Introduction we briefly introduced the Affine Hecke Algebra as the convolution algebra of compactly supported, locally constant \( I \)-invariant functions on the affine flag variety \( G/I \). Let us see this more precisely.

We will think of an affine flag \( F \in G/I \) as a periodic complete flag of \( \mathcal{O} \)-sublattices in \( V \), that is \( F = (F_i)_{i \in \mathbb{Z}} \) such that, for all \( i \in \mathbb{Z} \),

\[ F_{i-1} \subset F_i; \quad tF_i \subset F_{i-d}; \quad \dim_{\mathbb{F}_q} F_i/F_{i-1} = 1. \]

The \( G \)-orbits on \( G/I \times G/I \) (which are the same as \( I \)-orbits on \( G/I \)) are parametrized by elements of the extended affine Weyl group \( S_d^{\text{aff}} \). We can see those as \textit{periodic} permutations of \( \mathbb{Z} \), i.e. bijections \( w : \mathbb{Z} \to \mathbb{Z} \) such that \( w(i + d) = w(i) + d \) for all \( i \in \mathbb{Z} \). This follows from the Iwahori-Bruhat decomposition \( G = IS_d^{\text{aff}}I \), for more details see, for example [Ho].
Remark 3.3.1. Let \( w \in S^\text{aff}_d \), we choose \( \{e_1, \ldots, e_d\} \) to be a \( k \)-basis of \( V \) and set \( e_{i+dj} := t^{-j}e_i \) for all \( i \in \{1, \ldots, d\} \), \( j \in \mathbb{Z} \). Then the pair of affine flags \((F, F')\) such that

\[
F_i = \langle e_i, e_{i-1}, e_{i-2}, \ldots \rangle, \quad F'_i = \langle e_{w(i)}, e_{w(i-1)}, e_{w(i-2)}, \ldots \rangle,
\]
is in the \( G \)-orbit corresponding to \( w \), which we denote by \( \Omega_w \). Moreover, given any two flags \((F, F') \in \Omega_w \), we can find such a basis of \( V \).

For \( w \in S^\text{aff}_d \), we denote the characteristic function of \( \Omega_w \) by \( T_w \). Then the collection \( \{T_w | w \in S^\text{aff}_d\} \) is a basis for the space \( \mathcal{H}^\text{aff}_q = C^\infty_c(G/I \times G/I)_G \) of locally constant compactly supported \( G \)-invariant functions on \( G/I \times G/I \). We define the convolution product in the usual way, for \( \alpha, \beta \in \mathcal{H}^\text{aff}_q \)

\[
\alpha \ast \beta(F, F') = \sum_{H \in G/I} \alpha(F, H) \beta(H, F').
\]

This is well defined because the sum is finite.

Definition 3.3.2. The structure constants are polynomials in \( q \) and so we will view \( \mathcal{H}^\text{aff}_q \) as the specialization at \( q \mapsto q \) of the \( \mathbb{C}[q^\frac{1}{2}, q^{-\frac{1}{2}}] \)-algebra \( \mathcal{H}^\text{aff} \). This is called the affine Hecke algebra.

For \( i = 1, \ldots, d - 1 \), let \( s_i \in G \) be the matrix of the transposition \((ii+1)\), also, we consider the elements \( s_0, \tau \in G \) defined by

\[
s_0 = \begin{pmatrix} 0 & t^{-1} \\ 1 & \ddots \\ t & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ \ddots & \ddots \\ t & 0 \end{pmatrix}.
\]

Then we have, for \( i = 0, \ldots, d - 1 \), that \( s_i : \mathbb{Z} \to \mathbb{Z} \) corresponds to the transposition \((ii+1)\) extended by periodicity to all of \( \mathbb{Z} \), hence

\[
\Omega_{s_i} = \{(F, F') | F_j \neq F'_j \text{ if and only if } j \equiv i \pmod{d}\}.
\]
Also, $\tau : \mathbb{Z} \to \mathbb{Z}$ is defined by $\tau(j) = j - 1$, therefore

$$\Omega_\tau = \{(F, F') | F_j = F'_{j+1} \text{ for all } j \in \mathbb{Z}\}.$$

**Theorem 3.3.3.** The algebra $\mathcal{H}^{\text{aff}}$ is generated by $T_{s_0}, T_{s_1}, \ldots, T_{s_{d-1}}, T_\tau, T^{-1}_\tau$ with relations

$$T_{s_i}^2 = (q - 1)T_{s_i} + q \quad i = 0, \ldots, d - 1$$

$$T_{s_i}T_{s_{i+1}}T_{s_i} = T_{s_{i+1}}T_{s_i}T_{s_{i+1}} \quad i = 0, \ldots, d - 1$$

$$T_{s_i}T_{s_j} = T_{s_j}T_{s_i} \quad |i - j| \geq 2$$

$$T_\tau T^{-1}_\tau = T^{-1}_\tau T_\tau = 1$$

$$T_\tau T_{s_i} = T_{s_{i+1}}T_\tau \quad i = 0, \ldots, d - 1$$

where $i + 1$ should be interpreted modulo $d$.

This is the Iwahori-Matsumoto presentation for the affine Hecke algebra, which allows us to view it as a deformation of the group algebra of the extended affine Weyl group $S^{\text{aff}}_d$ (which is not a Coxeter group).

### 3.4 Mirabolic Affine Hecke Algebra

The goal of this section is to define a new algebra, which we will call the *mirabolic affine Hecke algebra*. This will be an analogue of Solomon’s construction in the affine setting. We keep the same notation of Section 3.3. As we will see, there are some technical difficulties which force us to a somewhat complicated construction.

#### 3.4.1 G-orbits on $G/I \times G/I \times V$

The $G$-orbits on $G/I \times G/I \times V$ can be divided into two kinds, depending if the vector is taken to be zero or nonzero. In [FGT], the $G$-orbits on $G/I \times G/I \times (V \setminus \{0\})$ are parametrized by the set $RB^{\text{aff}}$ of pairs $(w, \beta)$, with $w \in S^{\text{aff}}_d$, $\beta \subset \mathbb{Z}$ such that if $j \in \beta$ and $i \notin \beta$, then either $i > j$ or $w(i) > w(j)$ and such that $\beta$ is bounded above and $\mathbb{Z} \setminus \beta$ is bounded below. A triple $(F, F', v)$ is in the orbit $\Omega_{w, \beta}$ if $(F, F')$ satisfy the conditions of
Remark 3.3.1 and \( v = \sum_{k \in \beta} e_k \), which is an infinite sum that makes sense in \( V \). We will also need to consider the \( G \)-orbits in \( G/I \times G/I \times \{0\} \), which are clearly parametrized by \( S^\text{aff}_d \) as in Section 3.3. To include those, we will expand the set \( RB^\text{aff} \) to allow \( \beta = \emptyset \) as a possibility.

3.4.2 Convolution

We would like to define a convolution product on the space of locally constant compactly supported functions \( \mathbb{C}_c^\infty(G/I \times G/I \times V) \) with a formula similar to (3.3), then take the subalgebra of \( G \)-invariant functions. However, there are some technical issues that need to be addressed. First of all, we cannot just write a formula like

\[
\alpha * \beta(F, F', v) = \sum_{H \in G/I, u \in V} \alpha(F, H, u) \beta(H, F', v - u)
\]

because in general the sum will not be finite, unless one or both of \( \alpha \) and \( \beta \) is supported in \( G/I \times G/I \times \{0\} \). (This condition is satisfied, though, if one is only trying to define an \( \mathcal{H}^\text{aff} \)-bimodule structure, which is what is done in [FGT]).

Since in general the sum will be over \( u \) in some sublattice of \( V \), the problem can be partly solved by taking the Haar measure of the locally compact additive group \( V \), for example the one that assigns measure 1 to \( \mathcal{O}^d \). This is a seemingly valid solution, but it is still not what we want. This is because the Haar measure on \( V \) gives measure 0 to single points, which implies that all functions supported on \( G/I \times G/I \times \{0\} \) would be supported in a set of measure zero, hence give no contribution to the convolution. But, in the spirit of Solomon’s construction, we would like for the functions supported on \( \{(F, F', v)|v = 0\} \) to form a copy of the Affine Hecke algebra sitting inside our algebra. This motivates the following definition.

**Definition 3.4.1.** We define a finitely additive, vector-valued measure \( \mu_V \) on \( V \) by setting

\[
\mu_V = \mu_C + \mu_H \cdot v
\]

where \( v \) is a formal parameter, \( \mu_H \) is the Haar measure (normalized so that \( \mu_H(\mathcal{O}^d) = 1 \)), and \( \mu_C \) is the counting measure for finite sets and is zero for infinite sets.
Remark 3.4.2. To have $\mu_V$ be a measure, we take the algebra of measurable sets to consist of all subsets of $V$ obtained with finitely many operations of union, intersection and taking complement from the set of all singletons and the set of all $\mathcal{O}$-sublattices in $V$. Clearly $\mu_V$ cannot be countably additive because, for example, it would be impossible to give a meaning to the measure of the set $\mathcal{O}^d$ minus countably many points.

Definition 3.4.3. We define a measure $\mu$ on $G/I \times V$ by $\mu = \mu_c \times \mu_V$, where $\mu_c$ is the counting measure on $G/I$ and $\mu_V$ is as in Definition 3.4.1. Now, we can define a convolution product: given two functions $\alpha, \beta$ we set

$$\alpha \ast \beta(F, F', v) = \int_{G/I \times V} \alpha(F, H, u) \beta(H, F', v - u) d\mu(H, u). \quad (3.14)$$

3.4.3 Semi-invariant Functions

Since it is not clear that this convolution is well defined on $C^\infty_c(G/I \times G/I \times V)$ and that it is actually false that the convolution of two $G$-invariant functions is still $G$-invariant, we will have to specify which space of functions we want to consider to get our algebra.

First of all, we will allow $G$ to act with a character. We have the absolute value on $k$, defined by $|x| = q^{-\nu(x)}$, where $\nu$ is the discrete valuation.

Definition 3.4.4. For each $m \in \mathbb{Z}$, we let

$$\chi_m : G \rightarrow q^\mathbb{Z} \quad g \mapsto |\det(g)^m|$$

and we consider the space of semi-invariant functions

$$E^{\chi_m} = \{ \alpha : G/I \times G/I \times V \rightarrow \mathbb{C} \mid g \cdot \alpha = \chi_m(g) \alpha \}$$

Now let $A = \bigoplus_{m \in \mathbb{Z}} E^{\chi_m}$. This is the space of functions on $G/I \times G/I \times V$ that are finite linear combinations of semi-invariant functions. In particular, if $G_0 := \ker \chi_1$, all the functions in $A$ are invariant under the $G_0$-action, i.e. they are constant on the $G_0$-orbits. Clearly $G/G_0 \simeq \mathbb{Z}$, so each $G$-orbit splits into a $\mathbb{Z}$-torsor of $G_0$-orbits.
Proposition 3.4.5. The $G$-orbit $\Omega_{w,\beta} \subset G/I \times G/I \times V$ decomposes into $G_0$-orbits as follows

$$\Omega_{w,\beta} = \coprod_{r \in \mathbb{Z}} \Omega_{w,\beta,r} := \coprod_{r \in \mathbb{Z}} \{(F, F', v) \in \Omega_{w,\beta} | \mu_V(F_1) = q^r v\}.$$ 

Proof. This is straightforward from the fact that $\mu_V(g \cdot F_1) = |\text{det}(g)| \mu_V(F_1)$, so two triples of flags that are in the same $G$-orbit, are also in the same $G_0$-orbit if and only if the lattices in the flags have the same measure. Since by our definition $\mu_V(O^d) = v$, then all other lattices have measures of the form $q^r v$ for some $r$. \hfill $\square$

Given an orbit $\Omega_{w,\beta}$, its characteristic function $T_{w,\beta}$ is $G$-invariant, but we have other functions supported on the same orbit that are in $A$ and are not just constant multiples of $T_{w,\beta}$.

For a given $m \in \mathbb{Z}$, if we want $\text{supp} \, \alpha \subset \Omega_{w,\beta}$ and $\alpha \in E^{\chi m}$, we will get that, for $\tau$ as in Section 3.3

$$\alpha(\tau F, \tau F', \tau v) = |\text{det}(\tau)^m| \alpha(F, F', v)$$
$$= |t^m| \alpha(F, F', v)$$
$$= q^{-m} \alpha(F, F', v). \quad (3.15)$$

Since $\alpha$ has to be constant on $G_0$-orbits, let’s assume that $\alpha|_{\Omega_{w,\beta,1}} = c$. Then if $(F, F', v) \in \Omega_{w,\beta,r}$, we have $(\tau F, \tau F', \tau v) \in \Omega_{w,\beta,r-1}$. Hence (3.15) shows that $\alpha|_{\Omega_{w,\beta,0}} = c q^{-m}$.

Iterating this computation leads to the following result.

Lemma 3.4.6. Every function in $E^{\chi m}$ supported on $\Omega_{w,\beta}$ is a constant multiple of the function that has value $q^{mr}$ on $\Omega_{w,\beta,r}$

Definition 3.4.7. Define a function

$$T_{\chi}(F, F', v) = \begin{cases} q^r & \text{if } F = F', \quad v = 0, \quad \mu_V(F_1) = q^r v \\ 0 & \text{else} \end{cases}$$

Then it is immediate that $(T_{\chi})^m \ast T_{w,\beta}$ is the function in Lemma 3.4.6.
3.4.4 Convolution Algebra

To get the convolution algebra we want, we need some finitess condition on the support of the functions in $A$ that we consider. There are many possible choices, but our main goal is to make sure that we always end up with (3.14) being a finite integral over a measurable set.

The first thing that comes to mind is taking functions that are supported on finitely many orbits, but that has to be ruled out because, for example, if we take $\beta_1 = \{i \in \mathbb{Z} | i \leq 1\}$, then

$$T_{e,\beta_1} * T_{e,\beta_1}(F, F', v) = \begin{cases} \frac{q-2}{q} \mu_V(F_1) & \text{if } (F, F', v) \in \Omega_{e,\beta_1} \\ \frac{q-1}{q} \mu_V(F_1) & \text{if } (F, F', v) \in \Omega_{e,\beta}, \beta \subset \beta_1 \\ 0 & \text{else} \end{cases}$$

(3.16)

Equivalently

$$T_{e,\beta_1} * T_{e,\beta_1} = \frac{q-2}{q} v T_{\chi} * T_{e,\beta_1} + \frac{q-1}{q} v \sum_{\beta \subset \beta_1} T_{\chi} * T_{e,\beta}.$$ 

Hence the convolution of functions supported in a single orbit can be supported on infinitely many orbits. Notice however that in (3.16), the result is a function that ‘stabilizes’ if we consider smaller and smaller orbits. In fact it is eventually the same constant on all small enough $G_0$-orbits.

We will need a new family of functions that we define now.

**Definition 3.4.8.** For a pair $(w, \beta)$ we let $D_{w,\beta}$ be the function defined by

$$D_{w,\beta} = \sum_{\beta' \subset \beta} T_{w,\beta'}.$$ 

Equivalently

$$D_{w,\beta}(F, F', v) = \begin{cases} 1 & \text{if } (F, F', v) \in \Omega_{w,\beta'} \text{ with } \beta' \subset \beta \\ 0 & \text{else} \end{cases}$$

These are functions that stabilize to the constant 1 for small enough $\beta$. We will show that this kind of finiteness condition is what we need for our algebra. Remark that from the definition it is clear that, for $\beta \neq \emptyset$, $D_{w,\beta}$ is not a finite linear combination of $T_{w,\beta}$'s.
Definition 3.4.9. We set \( \mathcal{R}^\text{aff} \subset A \) to be the set of functions that are finite linear combinations of functions of the form \( (T\chi)^m \ast D_{w,\beta} \). In particular, these are functions that are eventually constant on small enough \( G_0 \)-orbits.

Our goal is now to prove the following proposition.

Proposition 3.4.10. The space \( \mathcal{R}^\text{aff} \) is closed under convolution, hence it forms an algebra.

We will need some intermediate results to get there.

Lemma 3.4.11. For all \((w, \beta)\) we have that

(i) \( T\chi \ast T_{w,\beta} = |\det(w)|T_{w,\beta} \ast T\chi \).

(ii) \( T\chi \ast D_{w,\beta} = |\det(w)|D_{w,\beta} \ast T\chi \).

Proof. Part (i) is a straightforward computation, and part (ii) follows immediately from (i).

Proposition 3.4.12. Every \( T_{w,\beta} \) is a finite linear combination of \( D \)'s.

Proof. First of all, if \( \beta = \emptyset \), then \( T_{w,\beta} = T_{w,\emptyset} = D_{w,\emptyset} \) and the statement is obvious. So we can assume \( \beta \neq \emptyset \). We fix a \((w, \beta) \in \mathcal{R}^\text{aff}\). We will show that \( T_{w,\beta} \) is a finite combination of \( D_{w,\beta'} \) with the same \( w \) and different \( \beta' \)'s. Let \( \sigma := \sigma(\beta) \) as is discussed in [T, Lemma 2]. That is \( \sigma \subset \mathbb{Z} \) such that \( \{w(i)|i \in \sigma\} \) is a decreasing sequence that is related to \( \beta \) in the following way:

\[
\beta = \{j \in \mathbb{Z}|\exists i \in \sigma \text{ such that } j \leq i \text{ or } w(j) \leq w(i)\}.
\]

In particular, \( \sigma \) is a nonempty finite set (of cardinality at most \( d \)) and for all \( \kappa \subset \sigma \), we have that \((w, \beta \setminus \kappa) \in \mathcal{R}^\text{aff}\). We now let

\[
F(\beta) := \{\beta' \subset \beta|\beta \setminus \beta' \subset \sigma\}.
\]

That, is \( F(\beta) \) is the finite set consisting of all \( \beta' \) that are obtained from \( \beta \) by removing a subset of \( \sigma \). Hence, for all \( \beta' \in F(\beta) \) we have \((w, \beta') \in \mathcal{R}^\text{aff}\). Define now

\[
C(\beta) := \sum_{\beta' \in F(\beta)} (-1)^{\text{card}(\beta \setminus \beta')} D_{w,\beta'}.
\]
Claim 3.4.13. \( T_{w,\beta} = C(\beta) \).

In order to prove the claim, and hence the proposition, we will evaluate \( C(\beta)|_{\Omega_{w,\gamma}} \) for all possible \( \gamma \)'s and show that it equals 1 if \( \gamma = \beta \) and is zero otherwise.

First case, if \( \gamma \not\subset \beta \). Then \( \gamma \not\subset \beta' \), hence by Definition 3.4.8 we have \( D_{w,\beta'}|_{\Omega_{w,\gamma}} = 0 \) for all \( \beta' \in F(\beta) \). Therefore, \( C(\beta)|_{\Omega_{w,\gamma}} = 0 \).

Second, if \( \gamma = \beta \). We have then, \( D_{w,\beta'}|_{\Omega_{w,\beta}} = 0 \) for all \( \beta' \subsetneq \beta \). It follows that

\[
C(\beta)|_{\Omega_{w,\beta}} = (-1)^{\text{card}(\emptyset)} D_{w,\beta}|_{\Omega_{w,\beta}} = 1.
\]

Last case, if \( \gamma \subsetneq \beta \). This implies that \( \sigma \setminus \gamma \neq \emptyset \). Now let us consider the terms that are relevant in \( C(\beta) \). Those are the terms for which \( D_{w,\beta'}|_{\Omega_{w,\gamma}} \neq 0 \), i.e. the terms corresponding to \( \beta' \in F(\beta) \) such that \( \gamma \subset \beta' \). Now, after a moment of thinking, it is clear that we have the following equality of sets:

\[
\{ \beta' | (\beta \setminus \beta') \subset \sigma \text{ and } \gamma \subset \beta' \} = \{ \beta' | (\beta \setminus \beta') \subset (\sigma \setminus \gamma) \}.
\]

We then have

\[
C(\beta)|_{\Omega_{w,\gamma}} = \sum_{\beta' \in F(\beta)} (-1)^{\text{card}(\beta \setminus \beta')} D_{w,\beta'}|_{\Omega_{w,\gamma}}
\]

\[
= \sum_{\beta' \in F(\beta), \gamma \subset \beta'} (-1)^{\text{card}(\beta \setminus \beta')} D_{w,\beta'}|_{\Omega_{w,\gamma}}
\]

\[
= \sum_{\beta' | (\beta \setminus \beta') \subset (\sigma \setminus \gamma)} (-1)^{\text{card}(\beta \setminus \beta')} \cdot 1
\]

\[
= \sum_{\kappa \subset (\sigma \setminus \gamma)} (-1)^{\text{card}(\kappa)}
\]

\[
= (1 - 1)^{\text{card}(\sigma \setminus \gamma)} = 0.
\]

\[
\square
\]

Now, given Lemma 3.4.11 and Proposition 3.4.12, all we need to do to prove Proposition 3.4.10 is to check that the convolution of \( D \)'s is still an element of \( \mathcal{R}^{\text{aff}} \).
Proposition 3.4.14. For any two pairs \((w, \beta), (w', \beta')\), with \(\beta \neq \emptyset \neq \beta'\) we have that

\[
D_{w, \beta} \ast D_{w', \beta'} = \sum_{\text{finite}} c_{w'', \beta''} T_{\chi} \ast D_{w'', \beta''}.
\]

Proof. We want to compute the following convolution:

\[
D_{w, \beta} \ast D_{w', \beta'}(F, F', v) = \int D_{w, \beta}(F, H, u) D_{w', \beta'}(H, F', v - u) \, d\mu(H, u). \quad (3.17)
\]

We will start by expressing the result as a (possibly infinite) combination of \(T_{\chi} \ast T_{w'', \beta''}\). The term \(T_{\chi}\) will appear multiplying whatever our result will be, because \(\beta, \beta'\) are both not empty, hence the \(G\)-action will rescale the measure of \(u\) and \(v - u\) exactly by the action of the character \(\chi\).

Suppose that \(w''\) is such that there are \(\beta''\)'s with \(T_{\chi} \ast T_{w'', \beta''}\) appearing as a term in the result with a nonzero coefficient.

Claim 3.4.15. For all \(\beta''\) small enough, the same coefficient has to appear in front of all those \(T_{\chi} \ast T_{w'', \beta''}\).

Proof of Claim. Fix \(F, F'\) such that \((F, F') \in \Omega_{w''}\). The integral in (3.17) is over those periodic flags \(H\)'s that are in relative position \(w\) with \(F\) and \(w'\) with \(F'\), remark that this set of \(H\)'s is finite. Now, consider one of those periodic flags \(H\). By definition of \(G\)-orbits, \(D_{w, \beta}(F, H, u)\) is not zero (and in fact equals one) if and only if

\[
u \in V_{F, H}^\beta := F_{i_1} \cap H_{j_1} + \ldots + F_{i_k} \cap H_{j_k}
\]

with \(i_1 < \ldots < i_k, j_1 > \ldots j_k\) some indices depending on \(\beta\).

In the same way \(D_{w', \beta'}(H, F', v - u) = 1\) (and is not zero) if and only if

\[
v - u \in V_{F', H}^\beta := H_{i_1} \cap F'_{j_1} + \ldots + H_{i_m} \cap F'_{j_m}
\]

with the indices depending on \(\beta'\). Now, for some high enough powers \(\alpha_1, \alpha_2\), we have

\[
F_{i_1} - \alpha_1 d = t^{\alpha_1} F_{i_1} \subset H_{i_1}^\beta; \quad F'_{j_m} - \alpha_2 d = t^{\alpha_2} F'_{j_m} \subset H_{j_1}.
\]

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Hence
\[ F_{i_1 - \alpha_1 d} \cap F_{j_m - \alpha_2 d}^{j'} \subset V_{\beta}^{F,H} \cap V_{\beta}^{H,F'} . \]

So, for a given \( H \), we have found some indices \( z, z' \) such that if \( v \in F \cap F' \), then
\[
\mu\{u \mid u \in V_{\beta}^{F,H} \text{ and } v - u \in V_{\beta'}^{H,F'} \} = \mu\{u \mid u \in V_{\beta}^{F,H} \text{ and } u \in V_{\beta'}^{H,F'} \}
\]
\[
= \mu\{V_{\beta}^{F,H} \cap V_{\beta'}^{H,F'} \}. 
\]

Now, since there are only finitely many such \( H \)'s, we can take \( z_0 \) (resp. \( z'_0 \)) to be the minimum of all the \( z \)’s (resp. \( z' \)’s) that we defined.

Then, for all \( \beta'' \) small enough that \( V_{\beta''}^{F,F'} \subset F \cap F' \), we have that, for \((F,F',v) \in \Omega_{w',\beta''},
\[
D_{w,\beta} * D_{w',\beta'}(F,F',v) = \mu\{(H,u) \mid D_{w,\beta}(F,H,u) = 1 \text{ and } D_{w',\beta'}(H,F',v-u) = 1 \}
\]
\[
= \sum_{H} \mu\{u \mid u \in V_{\beta}^{F,H} \text{ and } v - u \in V_{\beta'}^{H,F'} \}
\]
\[
(\text{since } v \in F \cap F') = \sum_{H} \mu\{u \mid u \in V_{\beta}^{F,H} \text{ and } u \in V_{\beta'}^{H,F'} \}
\]
\[
= \sum_{H} \mu\{V_{\beta}^{F,H} \cap V_{\beta'}^{H,F'} \}.
\]

which does not depend on \( \beta'' \) because it does not depend on \( v \), which proves the claim. \( \square \)

Now, this implies that for a fixed \( w'' \), all those \( T_{w'',\beta''} \) factors in the result with \( \beta'' \) small enough can be expressed as a single \( D_{w'',\beta_0} \). There will possibly be a finite number of terms \( T_{w'',\beta''} \) left but, according to Proposition 3.4.12, those can be expressed as a finite combination of \( D \)'s. To conclude, we need only observe that there are only finitely many such \( w'' \) appearing. \( \square \)

**Corollary 3.4.16.** For any two pairs \((w,\beta), (w',\beta')\), with \( \beta, \beta' \) possibly empty, we have that
\[
D_{w,\beta} * D_{w',\beta'} = \sum_{\text{finite}} c^{w'',\beta''}_{w',\beta'} (T_{\chi})^\varepsilon * D_{w'',\beta''};
\]
with \( \varepsilon = 0 \) if at least one of \( \beta, \beta' \) is empty, and \( \varepsilon = 1 \) otherwise.
Proof. If both $\beta$ and $\beta'$ are nonempty, then $\varepsilon = 1$ and this is exactly Proposition 3.4.14. If $\beta = \beta' = \emptyset$, then this is just the same as the convolution of $T_{w, \emptyset} * T_{w', \emptyset}$ which is a finite combination of $T_{w'', \emptyset} = D_{w'', \emptyset}$ by the definition of convolution on $G/I \times G/I$. The only other possibility is (by symmetry) when $\beta \neq \emptyset$ and $\beta' = \emptyset$. In this case the argument is again entirely analogous to the proof of the Proposition 3.4.14. Indeed, the coefficient of $T_{w', \emptyset}$ in the result will stabilize for $\beta''$ small enough. We fix $w''$ and $(F, F') \in \Omega_{w''}$, we take $\beta''$ to be small enough that $V_{F, F', \beta''}$ for all the finitely many $H$'s appearing. Then, for $(F, F', v) \in \Omega_{w'', \beta''}$,

$$
D_{w, \beta} * D_{w', \beta'}(F, F', v) = \mu\{(H, u) | D_{w, \beta}(F, H, u) = 1 \text{ and } D_{w', \beta'}(H, F', v - u) = 1\} = \sum_{H} \mu\{u | u \in V_{\beta}^{F, H} \text{ and } v - u \in V_{\beta'}^{H, F'}\}
$$

(since $\beta' = \emptyset$) = \sum_{H} \mu\{u | u \in V_{\beta}^{F, H} \text{ and } v - u = 0\}

= \sum_{H} \mu\{u | u = v \text{ and } u \in V_{\beta}^{F, H}\}

(since $v \in V_{\beta}^{F, H}$) = \sum_{H} 1.

which does not depend on $\beta''$ because it does not depend on $v$, hence the result follows. \qed

With this, Proposition 3.4.10 has been proved.

**Remark 3.4.17.** In all of the constructions of this section (for example in the structure constants of the algebra, in the definition of the measure $\mu$ and in the function $T_{\chi}$) we get coefficients which are polynomials or rational functions in $q$. Therefore, as usual, we will consider the algebra we defined to be the specialization at $q = q$ of another algebra which by abuse of notation we will also call $\mathcal{R}^{\text{aff}}$. Remember also that we have another formal parameter $v$ in our algebra, which represents the Haar measure of $\mathcal{O}^d \subset V$.

**Definition 3.4.18.** We call the $\mathbb{C}(q, v)$-algebra $\mathcal{R}^{\text{aff}}$ the **Mirabolic Affine Hecke Algebra**.

### 3.4.5 Generators and Relations

We want to give a description of the Mirabolic Affine Hecke Algebra in terms of generators and relations. First of all, notice that if $\alpha, \beta \in \mathcal{R}^{\text{aff}}$ such that $\text{supp } \alpha, \text{supp } \beta \subset G/I \times G/I \times$
If both $\alpha$ and $\beta$ are $G$-invariant, then so will be the product. Hence we have the following result which is completely analogous to the finite setting.

**Lemma 3.4.19.** The algebra $H^{aff}$ is contained as a subalgebra of $R^{aff}$, with inclusion given by $T_w \mapsto T_w,\emptyset$.

Now, if $\alpha \in H^{aff} \subset R^{aff}$ and $\beta \in R^{aff}$ is $G$-invariant, then $\alpha \ast \beta$ will also be an invariant $G$-function and, by definition, this product is the same as the bimodule structure defined by Finkelberg, Ginzburg and Travkin in [FGT].

We consider the functions $T_s_i := T_{s_i,\emptyset}, i = 0, \ldots, d - 1, T_{\tau \pm 1} := T_{\tau \pm 1,\emptyset}$, which are the image in $R^{aff}$ of the generators of $H^{aff}$ that we defined in Section 3.3. In addition, we consider the function $T_\chi$ which we defined in Section 3.4.3. Finally, we will consider the function $D_{e,1} := D_{e,\beta_1}$ where $\beta_1 = \{i \in \mathbb{Z} | i \leq 1\}$. In fact $D_{e,1}$ is the characteristic function of the set

$$\{(F,F',v) \in G/I \times G/I \times V | \ F = F', \ v \in F_1\}.$$ 

**Theorem 3.4.20.** The functions $T_{s_0}, \ldots, T_{s_{d-1}}, T_\tau, T_{\tau-1}, T_\chi, (T_\chi)^{-1}, D_{e,1}$ generate $R^{aff}$ as a $\mathbb{C}(q,v)$-algebra.

**Proof.** Since the elements $(T_\chi)^m \ast D_{w,\beta}$ and $(T_\chi)^m \ast T_{w,\beta}$ form a basis for $R^{aff}$, it is enough to show that all such elements are contained in the algebra $A$ generated by $T_{s_0}, \ldots, T_{s_{d-1}}, T_\tau, T_{\tau-1}, T_\chi, (T_\chi)^{-1}, D_{e,1}$. Since we are taking $(T_\chi)^{\pm 1}$ as generators, it is then enough to show that we can get all $D_{w,\beta}$’s and $T_{w,\beta}$’s in $A$. Also, if $\beta = \emptyset$, then $D_{w,\emptyset} = D_{w,\emptyset} = T_{w,\emptyset}$ and all such elements are in $A$ because they are in the copy of $H^{aff}$ which is generated by $T_{s_0}, \ldots, T_{s_d}, T_{\tau \pm 1}$.

**Claim 3.4.21.** All the $T_{w,\beta}$’s are in $A$.

**Proof of claim.** For all $j \in \mathbb{Z}$, let $\beta_j := \{i \in \mathbb{Z} | i \leq j\}$. It is an immediate computation to show that, for all $j$, $T_{\tau - (j+1)} \ast D_{e,1} \ast T_{\tau j + 1} = D_{e,\beta_j}$, which is the characteristic function of the set

$$\{(F,F',v) \in G/I \times G/I \times V | \ F = F', \ v \in F_j\}.$$ 

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In particular,

\[ D_{e,1} - T_{\tau^{-1}} * D_{e,1} * T_{\tau} = D_{e,1} - D_{e,\beta} = T_{e,\beta_1} \].

So, \( T_{e,\beta_1} \in \mathcal{A} \). Then we also have that for all \( j \)

\[ T_{\tau^{-(j+1)}} * T_{e,\beta_1} * T_{\tau^j} = T_{e,\beta_j} \in \mathcal{A} \].

Also, in \( \mathcal{A} \) we have, for all \( i, j \), the elements

\[ T_{e,\beta_j} * T_{\tau^i} = T_{\tau^{i},\beta_j} \].

Remark that, with the notation used in [FGT], \( T_{\tau^{i},\beta_j} = \tilde{w}_{i,j} \), therefore [FGT, Lemma 1] implies the claim. \( \Box \)

Now, remark that for \( \beta, \beta' \) both non empty, the symmetric difference \( \beta \setminus \beta' \cup \beta' \setminus \beta \) is finite. This implies in particular that for a fixed \( w \) and for any two nonempty \( \beta, \beta' \), we have that \( D_{w,\beta} - D_{w,\beta'} \) is a finite linear combination of \( T_{w,\beta''} \) (with coefficients \( \pm 1 \)). In view of Claim 3.4.21, this reduces the problem of showing that \( D_{w,\beta} \in \mathcal{A} \) for all \( w \) and \( \beta \), to just the fact that for each \( w \) there exists some nonempty \( \beta \) with \( D_{w,\beta} \in \mathcal{A} \).

But clearly, for all \( w \),

\[ D_{e,1} * T_{w,\emptyset}(F,F',v) = \int D_{e,1}(F,H,u)T_{w,\emptyset}(H,F',v-u) \, d\mu \]

is the characteristic function of the set

\[ \{(F,F',v)|(F,F') \in \Omega_w, \quad v \in F_1\} \]

hence

\[ D_{w,\beta_1} = D_{e,1} * T_{w,\emptyset}. \]

It follows that \( D_{w,\beta_1} \in \mathcal{A} \) for all \( w \in S_{d}^{aff} \), which implies that \( \mathcal{R}^{aff} \subset \mathcal{A} \) and proves the theorem. \( \Box \)
Proposition 3.4.22. The generators of $R^{\text{aff}}$ satisfy the following relations:

\[
\begin{align*}
T_{s_i}^2 &= (q - 1)T_{s_i} + q & i &= 0, \ldots, d - 1 \\
T_{s_i}T_{s_{i+1}}T_{s_i} &= T_{s_{i+1}}T_{s_i}T_{s_{i+1}} & i &= 0, \ldots, d - 1 \\
T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} & |i - j| & \geq 2 \\
T_{\tau}T_{\tau}^{-1} &= T_{\tau}^{-1}T_{\tau} = 1 \\
T_{\tau}T_{s_i} &= T_{s_{i+1}}T_{\tau} & i &= 0, \ldots, d - 1 \\
D_{e,1}^2 &= \nu T_{\chi}D_{e,1} \\
D_{e,1}T_{s_i} &= T_{s_i}D_{e,1} & i &= 0, 2, \ldots, d - 1 \\
T_{s_1}D_{e,1}T_{s_1}D_{e,1} &= \nu T_{\chi}T_{\tau}D_{e,1}T_{s_1}T_{\tau}^{-1} + (q - 1)D_{e,1}T_{s_1}D_{e,1} \\
T_{s_1}D_{e,1}T_{s_1}D_{e,1} &= D_{e,1}T_{s_1}D_{e,1}T_{s_1} \\
T_{\chi}T_{s_i} &= T_{s_i}T_{\chi} & i &= 0, \ldots, d - 1 \\
T_{\chi}D_{e,1} &= D_{e,1}T_{\chi} \\
T_{\tau}T_{\chi} &= qT_{\chi}T_{\tau}
\end{align*}
\]

Proof. These can be checked directly. \qed

Conjecture 3.4.23. The relations in Proposition 3.4.22 are a complete list of relations for the Mirabolic Affine Hecke Algebra.

A possible strategy to prove this conjecture, and therefore to give a complete description in terms of generators and relations for the Mirabolic Affine Hecke Algebra, is to actually express the algebra in a different basis, showing that it is a finite rank module over a big (almost)commutative subalgebra. Then we could compare finite ranks of modules to check that all the relations have been found. This approach should be possible because the Affine Hecke Algebra has a big commutative subalgebra, which is however not evident in the presentation we gave in Section 3.3. The commutative subalgebra can be made explicit using Bernstein’s presentation of the Affine Hecke Algebra. This encourages to believe that a similar subalgebra should appear in the mirabolic case.

At the moment, this line of inquiry has not been developed sufficiently yet.
REFERENCES


