UNIVERSITY OF CALIFORNIA
RIVERSIDE

Ricci Flow Does Not Necessarily Preserve Positive Radial Sectional Curvature

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

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June 2022

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Acknowledgements

I thank my thesis advisor Frederick Wilhelm for guiding me in the doctoral program. I would not have been able to complete this dissertation without his assistance. Furthermore, I acknowledge him for believing in my ability to succeed in graduate school, even when I have doubted myself on many occasions and when I contemplated on quitting the program. Finally, I am grateful that he was extremely patient and always able to work things out with me, even when we have disputed with each other many times over certain matters.

I completed this dissertation entirely during the COVID-19 pandemic. In order to facilitate my work-life balance and maintain my sanity during my quarantine, I endeavored to make an animated short film at the same time of writing this dissertation. I attended several animation schools throughout Southern California, while all instruction was remote at the time. I thank animation instructors Allan Jacobsen, James Keeshen, and Mark Zöeller for all their invaluable feedback and encouragement on the production of my film. Also, I thank James Keeshen in particular for allowing me to share in his storytelling class my experience in graduate school here at UC Riverside and how it changed me in some way.

To my parents for all the support.

ABSTRACT OF THE DISSERTATION

Ricci Flow Does Not Necessarily Preserve Positive Radial Sectional Curvature

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2022
Dr. Frederick Henry Wilhelm, Jr., Chairperson

We exhibit a one-parameter family of smooth Riemannian metrics on the four-dimensional sphere with strictly positive radial sectional curvature that loses this property when evolved through the Ricci flow. In other words, while the radial sectional curvature of the four-dimensional sphere with any metric from our one-parameter family is strictly positive at initial time, there exists a tangent plane of the sphere such that the radial sectional curvature of that tangent plane is negative some time after when the metric is evolved through the Ricci flow.

For our approach, we initially construct a piecewise-smooth metric that has a nonnegative sectional curvature with a strictly negative temporal derivative of sectional curvature for some tangent plane at initial time. Then we will apply gluing, convolutions, and mollifications in order to obtain a smooth approximation of our piecewise-smooth metric, which is still a nonnegative radial sectional curvature of one tangent plane becoming negative when evolved through the Ricci flow. Furthermore, we will deform the metric slightly in

such a way that the nonnegative radial sectional curvature at initial time becomes positive, while one of them still becomes negative when evolved through the Ricci flow. From this, we will extract a one-parameter family of metrics that retain these properties.

Contents

Chapter 1

Introduction and background

Let M be a Riemannian manifold, and let G be a group acting on M. We say that M is cohomegeneity one if the dimension of the quotient space M/G is 1. A cohomogeneity one manifold introduces the existences of radial and tangential vector fields and, in turn, their corresponding radial and tangential sectional curvatures, as well as sectional curvatures which require taking into consideration the radial and tangential sectional curvatures simultaneously. The radial field has unit length and is perpendicular to the orbits of M, whereas a tangential field is tangent to the same orbits.

We will also define what it means for a Riemannian metric to be invariant under a group action.

Definition 1.0.1. *Let M be a Riemannian manifold, and let $p \in M$ be a point. Let G be a group acting on M. We say that a metric g is* invariant under G*, or G*-invariant*, if it satisfies*

$$g(V, W) = g(dk(V), dk(W)) \tag{1.0.1}$$

for all $k \in G$ and for any vectors $V, W \in T_pM$.

Recall that T_pM denotes the tangent space of M, which is a differentiable manifold that intersects M at only the point $p \in M$.

Now, we consider $\mathrm{SO}(3)$, the group of 3×3 orthogonal matrices with unit determinants. We will only consider one $\mathrm{SO}(3)$ action on the four-dimensional sphere S^4: the restriction to S^4 of the irreducible $\mathrm{SO}(3)$ action on the five-dimensional Euclidean space $\mathbb{R}^5$. We restate this as Definition 1.2.3 in Chapter 2. This is the only $\mathrm{SO}(3)$ action we will consider for this dissertation.

Our dissertation result will mostly focus on radial sectional curvatures. The reader can refer to, for instance, Definition 3.0.2 for the formal definition of the sectional curvature $\sec(X, Y)$ of the tangent plane $\mathrm{span}(X, Y)$ for any $X, Y \in T_pM$. Additionally, whenever we talk about sectional curvature of M itself, we are addressing the sectional curvatures for ALL tangent planes of M simultaneously.

Definition 1.0.2. *Let g be the Riemannian metric for the cohomegeneity one manifold M. Let $\frac{\partial}{\partial r}$ be the radial vector field on M, and let V be any tangential vector field on M. We say that the radial sectional curvature is* nonnegative *if the vector fields satisfy*

$$\sec\left(\frac{\partial}{\partial r}, V\right) \geq 0. \tag{1.0.2}$$

Likewise, we say that the radial sectional curvature is positive *if the vector fields satisfy*

$$\sec\left(\frac{\partial}{\partial r}, V\right) > 0. \tag{1.0.3}$$

In this dissertation, we will show that there exists a metric that has positive radial sectional curvature and flows to a metric with a negative radial sectional curvature through Ricci flow.

Theorem 1.0.3. *There exists an* $\mathrm{SO}(3)$*-invariant metric on the four-dimensional sphere S^4 with positive radial sectional curvature that evolves through a Ricci flow to a metric with a* *negative radial sectional curvature.*

A timely remark about Theorem 1.0.3 is in order. As the author of this dissertation, I have attempted to improve the result of [2] by Renato Bettiol and Anusha Krishnan in 2020 and 2021. Their theorem states that there exists a metric on S^4 with nonnegative sectional curvature that loses this property when evolved through a Ricci flow, whereas for this dissertation I have initially attempted to assert that we can prove the same result using a metric on S^4 with strictly positive sectional curvature. Because my research attempts for this over the course of two years have were unsuccessful, I have ultimately decided to settle on a weaker result of proving their theorem using a metric on S^4 with only positive radial sectional curvature, which we have stated in this dissertation as Theorem 1.0.3. Furthermore, my advisor informed me on February 25, 2022 that the authors of [2] have already established on December 25, 2021 that there exists a metric on S^4 with positive sectional curvature that loses this property when evolved through a Ricci flow. This means that Theorem 1.0.3 is now a special case of their main result in [3]. Despite this, the metric we have constructed in this dissertation contains properties that are substantially different from those of the metric that Renato Bettiol and Anusha Krishnan have discovered in their 2021 paper. For instance, the metric described in [3] achieves a negative sectional curvature when evolved through a Ricci flow on sufficiently small neighborhoods of principal orbits near $s = 0$ and near $s = \frac{\pi}{3}$, whereas the metric of this disseration achieves a negative radial sectional curvature when evolved through a Ricci flow on a sufficiently small neighborhood of principal orbits centered at $s = \frac{\pi}{6}$. Another difference is that our metric has boundary values described by (6.0.1), (6.0.2), (6.0.3), (6.0.4), (6.0.5), (6.0.6), whereas the metric described in [3] does not share the same properties. Also, neither [2] nor [3] have made explicit applications of our normal sectional curvature assumption given by (2.0.5), which not only ensures zero normal sectional curvatures but also has helped us simplify our expressions of covariant derivatives, sectional curvatures, and the system of partial differential equations associated with Ricci flow. Finally, it is worth noting that there are results in the literature concerning the positive lower bounds on sectional curvature that also hold when only the radial sectional curvature is positive; the reader can consult, for instance, [10], [11], [12], [14], [15].

There are two key lemmas that we will use to help us prove Theorem 1.0.3. We will give a proof following the statement of Lemma 1.0.4, but we will relegate the proof of Lemma 1.0.5 to Chapter 6.

Lemma 1.0.4. *Let $\{(M, g_\tau)\}_{\tau \geq 0}$ be a smooth family of Riemannian metrics. For simplicity of notation, we set $g := g_0$. Suppose there exists a tangent plane σ that satisfies*

$$\sec_g(\sigma) = 0, \tag{1.0.4}$$

$$(\sec_{g(t)}(\sigma))_t|_{t=0} < 0. \tag{1.0.5}$$

If $\tau > 0$ is sufficiently small, then (M, g_τ) evolves through a Ricci flow to a metric with a negative sectional curvature; that is, there exists $t_0 > 0$ that satisfies

$$\sec_{g_\tau(t_0)}(\sigma) < 0. \tag{1.0.6}$$

Proof. Suppose $t > 0$ is sufficiently small. Then (1.0.5) implies that there exist some fixed $\tau_0 > 0$ and a constant $C_{\tau_0} > 0$ and such that, for all $\tau \in (0, \tau_0)$, we have

$$\begin{aligned} (\sec_{g_\tau(t)}(\sigma))_t|_{t=0} &< -C_{\tau_0} \\ &< 0, \end{aligned} \tag{1.0.7}$$

which implies (1.0.6). □

Because of Lemma 1.0.4, all we need to do to prove Theorem 1.0.3 is to construct some family of metrics $\{(M, g_\tau)\}_{\tau \geq 0}$. We outline our process as follows:

1. Construct the piecewise smooth metric that has nonnegative radial sectional curvature and satisfies (1.0.4) and (1.0.5).
2. Deform the metric from Step 1 to one that is smooth and maintains nonnegative radial sectional curvature.
3. Deform the metric from Step 2 to a family of metrics that have positive radial sectional curvature.

Every SO(3)-invariant cohomogeneity one metric g on a four-dimensional manifold takes the form

$$g = ds^2 + \varphi^2\, dx^2 + \psi^2\, dy^2 + \xi^2\, dz^2. \tag{1.0.8}$$

Note that we would accomplish Step 1 if we prove Lemma 1.0.5, which we will write below.

Lemma 1.0.5. *There exists a continuous cohomogeneity one metric g of the form* (1.0.8) *on S^4 with the following properties:*

(1) *There exists a radial plane σ that satisfies* (1.0.4) *and* (1.0.5).

(2) *The functions φ, ψ, ξ that we define g in* (1.0.8) *are piecewise affine and concave. In particular, the first-order derivatives φ', ξ' have two discontinuities, while ψ' has four discontinuities, on the interval $[0, \frac{\pi}{3}]$.*

As we have previously stated, we will prove Lemma 1.0.5 in Chapter 6.

1.1 Objectives of the dissertation

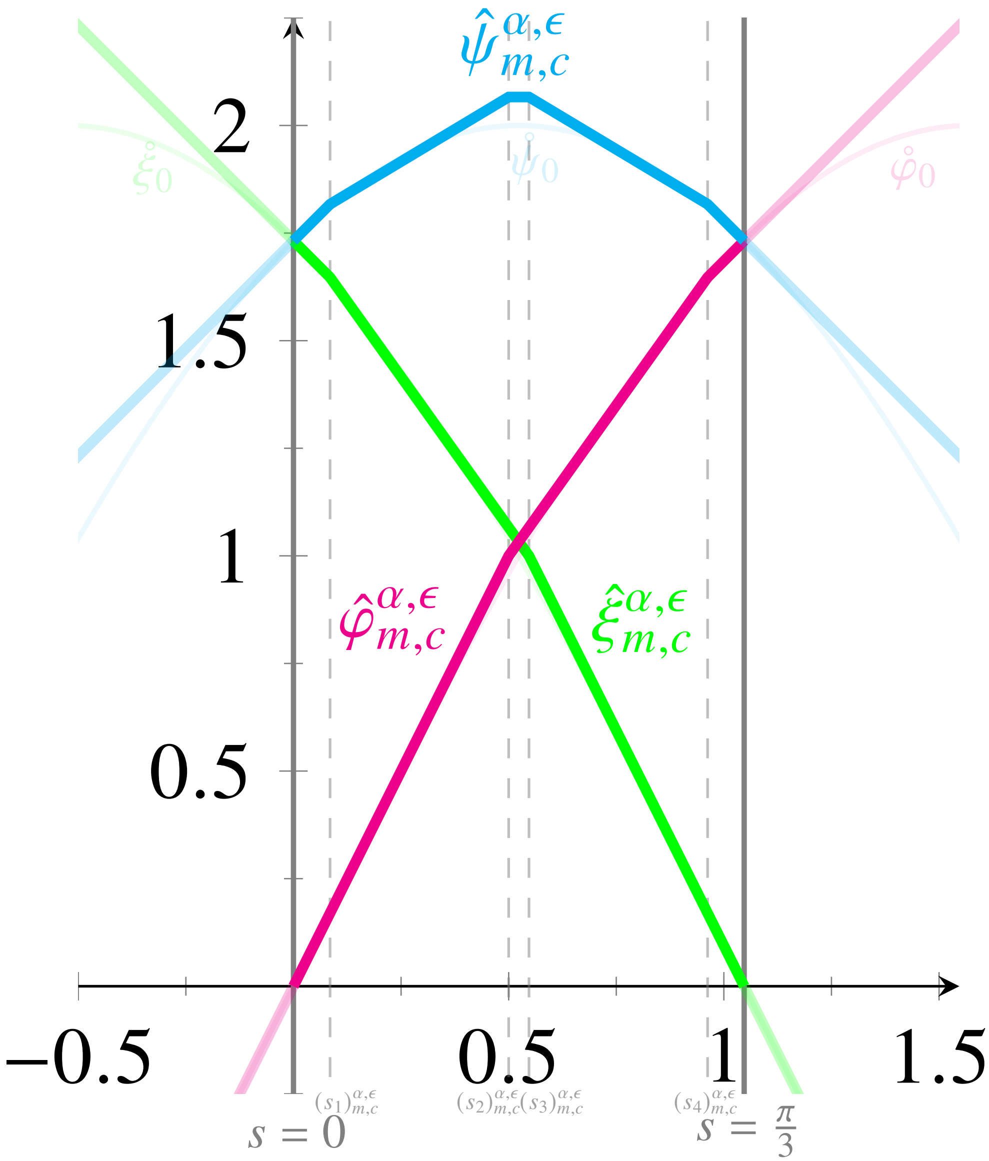
Richard Hamilton pioneered the subject of Ricci flow in 1984. A Ricci flow deforms Riemannian metrics on Riemannian manifolds by their associated Ricci tensor. The partial differential equation describing Ricci flow behaves like a nonlinear heat equation. Ricci flow facilitated Grigory Perelman's proof of the Poincairé conjecture, which states that any simply connected, compact three-dimensional manifold is homeomorphic to the three-dimensional sphere.

A Ricci flow does not necessarily preserve the nonnegativity of the sectional curvature of Riemannian metrics on compact manifolds. For instance, Renato Bettiol and Anusha Krishnan have demonstrated this for S^4 with their metric counterexample in [2]. Similarly, Ricci flow does not necessarily preserve the positivity of the sectional curvature of Riemannian metrics on compact manifolds. For instance, Christoph Böhm and Burkhard Wilking established this result for the flag manifold $\mathrm{Sp}(3)/(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1))$ with their metric counterexample in [4].

We will show that a Ricci flow does not preserve positive radial sectional curvature for the four-dimensional sphere. Positive radial sectional curvature is not the same as positive sectional curvature, which requires both positive radial and tangential sectional curvatures. Theorem 1.0.3 essentially states that we can exhibit a Riemannian metric on the four-dimensional sphere with positive sectional curvature at initial time that becomes negative a short time later as it evolves through a Ricci flow.

The goal of this dissertation is to construct an example of a smooth metric of the form given by (1.0.8) that satisfies the theorem mentioned at the beginning of this dissertation. Our theorem does NOT contradict the result by Böhm and Wilking because our metric in question has a negative sectional curvature for at least one of our tangential planes, and so our metric does NOT contain a positive sectional curvature operator. We will accomplish our goal by first constructing three piecewise linear functions, which we will do in Chapter 6, and proving that the metric that relies on these functions has nonnegative radial sectional curvature with a negative temporal derivative. Then we will deform these functions with perturbations, making them strictly concave, so that we can obtain a new metric metric that has strictly positive radial sectional curvature and maintains a negative temporal derivative. Then we will construct smooth analogs of these deformed functions by convolving them with a mollifier. Finally, we will employ smooth approximations of the Heaviside step function in order to glue the convolutions to the original piecewise smooth functions near the boundary points of the interval $[0, \frac{\pi}{3}]$. Executing this entire process results in new functions that are concave on $[0, \frac{\pi}{3}]$ and satisfy the assumptions of our theorem.

To illustrate our approach to proving our theorem, we present below the graphs of our three functions $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\psi}^{\alpha,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\epsilon}_{m,c}$ with $m = \frac{7}{5}$ and $c = \frac{3}{10}$, which are all continuous, piecewise, and linear.



As these graphs show, the piecewise linear functions are close to those of our actual example of our metric in the continuous topology and the C^1 topology, except at their cusps.

1.2 Group action by special orthogonal matrices on a four-dimensional sphere

In this section, we will introduce our group action on a Riemannian manifold by 3×3 special orthogonal matrices. We assume that the reader of this dissertation is already familiar with the subjects of differential topology and Riemannian geometry.

Let M be a four-dimensional Riemannian manifold, and let W, X, Y, Z be vector fields on M. We denote the area of the parallelogram formed by X and Y by the $(0, 2)$ metric tensor $g(X, Y)$, and we define the length of X by

$$|X| = \sqrt{g(X, X)} \tag{1.2.1}$$

Let ∇ denote the Levi-Civita connection, and let $\nabla_X Y$ represent the covariant derivative of Y in the direction of X. We define the Lie bracket of X, Y by

$$[X, Y] := \nabla_X Y - \nabla_Y X. \tag{1.2.2}$$

Note that all of these quantities become functions of s whenever we restrict X to γ.

Definition 1.2.1. *We say that the action by G on M is with **cohomogeneity one** if the quotient space M/G is one-dimensional.*

Since M is compact, it follows that the quotient space M/G must be isometric to either a circle S^1 or a closed interval $[0, L]$ for some $L > 0$. In this presentation, we will focus only on the closed interval $[0, L]$. Let $\pi : M \to M/G$ be the canonical projection map. Consider for each $s \in [0, L]$ the preimage $\pi^{-1}(\{s\}) \subseteq S^4$.

Definition 1.2.2. *We say that the preimage $\pi^{-1}(\{s\})$ is a **principal orbit** if it satisfies*

$$\operatorname{codim}(\pi^{-1}(\{s\})) = 1. \tag{1.2.3}$$

*Otherwise, we say that $\pi^{-1}(\{s\})$ is a **singular orbit** if it satisfies*

$$\operatorname{codim}(\pi^{-1}(\{s\})) \geq 2. \tag{1.2.4}$$

We need to separate $s \in [0, L]$ into two distinct cases: the interior points $s \in (0, L)$ and the endpoints $s = 0, L$. By the Slice Theorem, every $\pi^{-1}(\{s\})$ for any $s \in (0, L)$ is principal, whereas $\pi^{-1}(\{0\})$ and $\pi^{-1}(\{L\})$ are singular. Now let $\gamma : [0, L] \to M$ be a minimal geodesic that connects the points $\gamma(0) \in \pi^{-1}(\{0\})$ and $\gamma(L) \in \pi^{-1}(\{L\})$. Let

$$G_{\gamma(s)} := \{h \in G \mid h \cdot \gamma(s) = \gamma(s)\} \tag{1.2.5}$$

be the isotropy group at $\gamma(s)$ for all $s \in [0, L]$. Then there exists a decomposition of M by orbit type:

$$\begin{aligned} M &= \bigcup_{0 \leq r \leq L} (G/G_{\gamma(s)}) \\ &= (G/G_{\gamma(0)}) \cup \left(\bigcup_{s \in (0,L)} (G/G_{\gamma(s)}) \right) \cup (G/G_{\gamma(L)}). \end{aligned} \tag{1.2.6}$$

Notice that the geodesic γ intersects each orbit $G/G_{\gamma(s)}$ orthogonally.

From this point on, we will specialize to a specific $\mathrm{SO}(3)$ action on S^4. We will describe this as follows. We define the vector space of symmetric, traceless 3×3 matrices

$$V := \{A \in \mathbb{R}^{3 \times 3} \mid A = A^T, \operatorname{tr}(A) = 0\}. \tag{1.2.7}$$

Then V is five-dimensional, and its natural inner product is

$$g(A, B) := \operatorname{tr}(A^T B) \tag{1.2.8}$$

for all $A, B \in V$.

Definition 1.2.3. *We define the* **SO(3)** ***action on*** $\mathbf{S^4}$ *to be the restriction to the unit sphere in* V *of the group action of* $\mathrm{SO}(3)$ *on* V *by conjugation, which is the map* $\sigma_P : V \to V$ *defined by*

$$\sigma_P(A) := PAP^{-1} \tag{1.2.9}$$

for all invertible matrices $P \in \mathrm{SO}(3)$ *with inverse* P^{-1}.

According to the Spectral Theorem, every real symmetric matrix is diagonalizable by matrices in SO(3). So every orbit $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$ orthogonally intersects the great circle

$$\mathcal{F} := \left\{ \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \;\middle|\; \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, \sum_{i=1}^{3} \lambda_i = 0, \sum_{i=1}^{3} \lambda_i^2 = 1 \right\}. \tag{1.2.10}$$

The geodesic $\gamma : [0, \frac{\pi}{3}] \to V$ defined by

$$\gamma(s) := \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} \tag{1.2.11}$$

runs orthogonally through all the principal orbits $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$ of S^4 and joins the endpoints

$$\gamma(0) = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}, \tag{1.2.12}$$

$$\gamma\left(\frac{\pi}{3}\right) = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \end{bmatrix}. \tag{1.2.13}$$

Recall the real projective space $\mathbb{RP}^2$, the space of lines in $\mathbb{R}^3$ passing through the origin. The singular orbit $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(0)} \cong \mathbb{RP}^2$ of the $\mathrm{SO}(3)$ action on S^4 is the set of matrices with two equal positive eigenvalues. Likewise, the singular orbit $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(\frac{\pi}{3})} \cong \mathbb{RP}^2$ of the $\mathrm{SO}(3)$ action on S^4 is the set of matrices with two equal negative eigenvalues.

The Spectral Theorem asserts that every real symmetric matrix is diagonalizable by matrices in $\mathrm{SO}(3)$. So every orbit $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$ orthogonally intersects the great circle

$$F := \left\{ \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \middle| \; \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, \sum_{i=1}^{3} \lambda_i = 0, \sum_{i=1}^{3} \lambda_i^2 = 1 \right\}. \tag{1.2.14}$$

The geodesic $\gamma : [0, \frac{\pi}{3}] \to V$ for the round metric $\mathring{g}_0$ defined by

$$\gamma(s) := \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} \tag{1.2.15}$$

runs orthogonally through all the principal orbits $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$ of S^4 and serves as a continuous path between the endpoints

$$\gamma(0) = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}, \tag{1.2.16}$$

$$\gamma\left(\frac{\pi}{3}\right) = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \end{bmatrix}. \tag{1.2.17}$$

Recall the real projective space $\mathbb{RP}^2$, the space of lines in $\mathbb{R}^3$ passing through the origin. The singular orbit $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(0)} \cong \mathbb{RP}^2$ of the $\mathrm{SO}(3)$ action on S^4 is the set of matrices with two equal positive eigenvalues. Likewise, the singular orbit $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(\frac{\pi}{3})} \cong \mathbb{RP}^2$ of the $\mathrm{SO}(3)$ action on S^4 is the set of matrices with two equal negative eigenvalues.

Finally, we introduce the Killing fields $X, Y, Z \in \mathrm{SO}(3)$ defined at any point $p \in M$ by

$$X(p) := \frac{d}{ds} \exp(sE_{23}) \cdot p \bigg|_{s=0}, \tag{1.2.18}$$

$$Y(p) := \frac{d}{ds} \exp(sE_{31}) \cdot p \bigg|_{s=0}, \tag{1.2.19}$$

$$Z(p) := \frac{d}{ds} \exp(sE_{12}) \cdot p \bigg|_{s=0}, \tag{1.2.20}$$

where

$$\exp : T_p(M \setminus (\pi^{-1}(\{0\}) \cup \pi^{-1}(\{L\}))) \to M \setminus (\pi^{-1}(\{0\}) \cup \pi^{-1}(\{L\})) \tag{1.2.21}$$

is the Lie group exponential map.

Proposition 1.2.4. *Consider, for all $\theta \in \mathbb{R}$, the rotation matrices $R_x, R_y, R_z \in \mathrm{SO}(3)$ given* *by*

$$R_z(\theta) := \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_y(\theta) := \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, \tag{1.2.22}$$

$$R_x(\theta) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Then the Killing fields $X, Y, Z \in \mathrm{SO}(3)$*, which are generated by the first-order derivatives* R'_x, R'_z, R'_y*, satisfy*

$$[X, Y] = -Z, \tag{1.2.23}$$

$$[Y, Z] = -X, \tag{1.2.24}$$

$$[Z, X] = -Y, \tag{1.2.25}$$

$$\left[\frac{\partial}{\partial r}, X\right] = 0, \tag{1.2.26}$$

$$\left[\frac{\partial}{\partial r}, Y\right] = 0, \tag{1.2.27}$$

$$\left[\frac{\partial}{\partial r}, Z\right] = 0. \tag{1.2.28}$$

Proof. The proofs of (1.2.23), (1.2.24), (1.2.25) are all analogous to each other. Without loss of generality, we choose to only prove (1.2.23). By the naturality of the Lie bracket, it suffices to check (1.2.23) for the matrices in SO(3). Consider the great circle

$$\mathcal{F} := \left\{ \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \in \mathbb{R}^{3\times 3} \;\middle|\; \sum_{i=1}^{3} \lambda_i^2 = 1, \sum_{i=1}^{3} \lambda_i = 0 \right\} \subseteq S^4 \tag{1.2.29}$$

and its tangent bundle $T\mathcal{F}$. We define the orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\} \subseteq T\mathcal{F}$ by

$$e_1 := \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}, \tag{1.2.30}$$

$$e_2 := \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.2.31}$$

$$e_3 := \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.2.32}$$

$$e_4 := \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}, \tag{1.2.33}$$

$$e_5 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}. \tag{1.2.34}$$

Consider the geodesic γ for S^4 with the round metric $\mathring{g}_0$ defined by

$$\begin{aligned}
\gamma(s) &:= e_1 \cos(s) + e_2 \sin(s) \\
&= \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \cos(s) + \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \sin(s) \\
&= \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} + \begin{bmatrix} \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix}.
\end{aligned} \tag{1.2.35}$$

Consider for all $\theta \in \mathbb{R}$ the rotation matrices $R_x, R_y, R_z \in \mathrm{SO}(3)$ about the x, y, z-axes, respectively, defined by

$$R_x(\theta) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{1.2.36}$$

$$R_y(\theta) := \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, \tag{1.2.37}$$

$$R_z(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.2.38}$$

and their respective inverse elements $R_x^{-1}, R_y^{-1}, R_z^{-1} \in \mathrm{SO}(3)$ given by

$$R_x^{-1}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{1.2.39}$$

$$R_y^{-1}(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, \tag{1.2.40}$$

$$R_z^{-1}(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.2.41}$$

At $\theta = 0$, the rotation matrices of SO(3) and their inverses are all equal to the identity matrix; in other words, we have

$$R_x(0) = R_y(0) = R_z(0) = R_x^{-1}(0) = R_y^{-1}(0) = R_z^{-1}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.2.42}$$

Also, their derivatives at $\theta = 0$ are

$$
\begin{aligned}
R_x'(\theta)|_{\theta=0} &= \frac{d}{d\theta}\left.\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}\right|_{\theta=0} \\
&= \left.\begin{bmatrix} \frac{d}{d\theta}(1) & \frac{d}{d\theta}(0) & \frac{d}{d\theta}(0) \\ \frac{d}{d\theta}(0) & \frac{d}{d\theta}(\cos(\theta)) & \frac{d}{d\theta}(-\sin(\theta)) \\ \frac{d}{d\theta}(0) & \frac{d}{d\theta}(\sin(\theta)) & \frac{d}{d\theta}(\cos(\theta)) \end{bmatrix}\right|_{\theta=0} \\
&= \left.\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin(\theta) & -\cos(\theta) \\ 0 & \cos(\theta) & -\sin(\theta) \end{bmatrix}\right|_{\theta=0} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin(0) & -\cos(0) \\ 0 & \cos(0) & -\sin(0) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned} \tag{1.2.43}
$$

and

$$
\begin{aligned}
R'_y(\theta)|_{\theta=0} &= \frac{d}{d\theta}\left.\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}\right|_{\theta=0} \\
&= \left.\begin{bmatrix} \frac{d}{d\theta}(\cos(\theta)) & \frac{d}{d\theta}(0) & \frac{d}{d\theta}(-\sin(\theta)) \\ \frac{d}{d\theta}(0) & \frac{d}{d\theta}(1) & \frac{d}{d\theta}(0) \\ \frac{d}{d\theta}(\sin(\theta)) & \frac{d}{d\theta}(0) & \frac{d}{d\theta}(\cos(\theta)) \end{bmatrix}\right|_{\theta=0} \\
&= \left.\begin{bmatrix} -\sin(\theta) & 0 & -\cos(\theta) \\ 0 & 0 & 0 \\ \cos(\theta) & 0 & -\sin(\theta) \end{bmatrix}\right|_{\theta=0} \\
&= \begin{bmatrix} -\sin(0) & 0 & -\cos(0) \\ 0 & 0 & 0 \\ \cos(0) & 0 & -\sin(0) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{1.2.44}
$$

and

$$
\begin{aligned}
R_z'(\theta)|_{\theta=0} &= \frac{d}{d\theta}\left.\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|_{\theta=0} \\
&= \left.\begin{bmatrix} \frac{d}{d\theta}(\cos(\theta)) & \frac{d}{d\theta}(-\sin(\theta)) & \frac{d}{d\theta}(0) \\ \frac{d}{d\theta}(\sin(\theta)) & \frac{d}{d\theta}(\cos(\theta)) & \frac{d}{d\theta}(0) \\ \frac{d}{d\theta}(0) & \frac{d}{d\theta}(0) & \frac{d}{d\theta}(1) \end{bmatrix}\right|_{\theta=0} \\
&= \left.\begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix}\right|_{\theta=0} \\
&= \begin{bmatrix} -\sin(0) & -\cos(0) & 0 \\ \cos(0) & -\sin(0) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{1.2.45}
$$

and

$$
\begin{aligned}
(R_x')^{-1}(\theta)|_{\theta=0} &= \frac{d}{d\theta}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}\Bigg|_{\theta=0} \\
&= \begin{bmatrix} \frac{d}{d\theta}(1) & \frac{d}{d\theta}(0) & \frac{d}{d\theta}(0) \\ \frac{d}{d\theta}(0) & \frac{d}{d\theta}(\cos(\theta)) & \frac{d}{d\theta}(\sin(\theta)) \\ \frac{d}{d\theta}(0) & \frac{d}{d\theta}(-\sin(\theta)) & \frac{d}{d\theta}(\cos(\theta)) \end{bmatrix}\Bigg|_{\theta=0} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin(\theta) & \cos(\theta) \\ 0 & -\cos(\theta) & -\sin(\theta) \end{bmatrix}\Bigg|_{\theta=0} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin(0) & \cos(0) \\ 0 & -\cos(0) & -\sin(0) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}
\end{aligned} \tag{1.2.46}
$$

and

$$
\begin{aligned}
(R_y')^{-1}(\theta)|_{\theta=0} &= \frac{d}{d\theta}\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}\Bigg|_{\theta=0} \\
&= \begin{bmatrix} \frac{d}{d\theta}(\cos(\theta)) & \frac{d}{d\theta}(0) & \frac{d}{d\theta}(\sin(\theta)) \\ \frac{d}{d\theta}(0) & \frac{d}{d\theta}(1) & \frac{d}{d\theta}(0) \\ \frac{d}{d\theta}(-\sin(\theta)) & \frac{d}{d\theta}(0) & \frac{d}{d\theta}(\cos(\theta)) \end{bmatrix}\Bigg|_{\theta=0} \\
&= \begin{bmatrix} -\sin(\theta) & 0 & \cos(\theta) \\ 0 & 0 & 0 \\ -\cos(\theta) & 0 & -\sin(\theta) \end{bmatrix}\Bigg|_{\theta=0} \\
&= \begin{bmatrix} -\sin(0) & 0 & \cos(0) \\ 0 & 0 & 0 \\ -\cos(0) & 0 & -\sin(0) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{1.2.47}
$$

and

$$
\begin{aligned}
(R_z')^{-1}(\theta)|_{\theta=0} &= \frac{d}{d\theta}\left.\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|_{\theta=0} \\
&= \left.\begin{bmatrix} \frac{d}{d\theta}(\cos(\theta)) & \frac{d}{d\theta}(\sin(\theta)) & \frac{d}{d\theta}(0) \\ \frac{d}{d\theta}(-\sin(\theta)) & \frac{d}{d\theta}(\cos(\theta)) & \frac{d}{d\theta}(0) \\ \frac{d}{d\theta}(0) & \frac{d}{d\theta}(0) & \frac{d}{d\theta}(1) \end{bmatrix}\right|_{\theta=0} \\
&= \left.\begin{bmatrix} -\sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix}\right|_{\theta=0} \\
&= \begin{bmatrix} -\sin(0) & \cos(0) & 0 \\ -\cos(0) & -\sin(0) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{1.2.48}
$$

So we obtain the Lie bracket relations

$$\begin{aligned}
[R_z'(\theta)|_{\theta=0}, R_y'(\theta)|_{\theta=0}] &= (R_z'(\theta)|_{\theta=0})(R_y'(\theta)|_{\theta=0}) - (R_y'(\theta)|_{\theta=0})(R_z'(\theta)|_{\theta=0}) \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
&= -R_x'(\theta)\big|_{\theta=0}
\end{aligned} \tag{1.2.49}$$

and

$$
\begin{aligned}
[R'_y(\theta)|_{\theta=0}, R'_x(\theta)|_{\theta=0}] &= (R'_y(\theta)|_{\theta=0})(R'_x(\theta)|_{\theta=0}) - (R'_x(\theta)|_{\theta=0})(R'_y(\theta)|_{\theta=0}) \\
&= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= -R'_z(\theta)\big|_{\theta=0}
\end{aligned}
\tag{1.2.50}
$$

and

$$
\begin{aligned}
[R_x'(\theta)|_{\theta=0}, R_z'(\theta)|_{\theta=0}] &= (R_x'(\theta)|_{\theta=0})(R_z'(\theta)|_{\theta=0}) - (R_z'(\theta)|_{\theta=0})(R_x'(\theta)|_{\theta=0}) \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\
&= -R_y'(\theta)\big|_{\theta=0}\,.
\end{aligned} \tag{1.2.51}
$$

Because the associated Lie algebra $\mathfrak{so}(3)$ generates the Killing fields X, Y, Z, and because $\{R_x'(0), R_y'(0), R_z'(0)\}$ is a basis of $\mathfrak{so}(3)$, it follows that, according to [13], the naturality of the Lie bracket implies

$$[X, Y] = -Z, \tag{1.2.52}$$

which is (1.2.23).

The proofs of (1.2.26), (1.2.27), (1.2.28) are all analogous to each other. Without loss of generality, we choose to only prove (1.2.26). Again, consider the Killing fields X, Y, Z.

Since the field $\frac{\partial}{\partial s}$ is SO(3)-invariant, the Lie derivatives of X, Y, Z along $\frac{\partial}{\partial s}$ are all zero; that is, we have

$$L_{\frac{\partial}{\partial s}} X = 0. \tag{1.2.53}$$

Also recall that the Lie derivative of a vector field in the direction of another vector field is given by the Lie bracket

$$L_{\frac{\partial}{\partial s}} X = \left[\frac{\partial}{\partial s}, X \right]. \tag{1.2.54}$$

So we conclude

$$\begin{aligned} \left[\frac{\partial}{\partial s}, X \right] &= L_{\frac{\partial}{\partial s}} X \\ &= 0, \end{aligned} \tag{1.2.55}$$

which is (1.2.26). □

Calculating the Lie bracket on SO(3) gives the Lie Bracket of the corresponding Killing fields X, Y, Z on S^4 via the naturality of the Lie Bracket.

According to [2], we can write the Riemannian metric g in the form

$$g = ds^2 + \varphi(s)^2\, dx^2 + \psi^2\, dy^2 + \xi^2\, dz^2, \tag{1.2.56}$$

where dx, dy, dz are the covectors of the Killing fields X, Y, Z along the geodesic γ in M, respectively, and $\varphi, \psi, \xi : [0, L] \times [0, \infty) \to \mathbb{R}$ are smooth functions defined by the lengths of the Killing fields

$$\varphi(s) := |X(\gamma(s))| = |X|, \tag{1.2.57}$$

$$\psi(s) := |Y(\gamma(s))| = |Y|, \tag{1.2.58}$$

$$\xi(s) := |Z(\gamma(s))| = |Z|. \tag{1.2.59}$$

Finally, we note that $\{\frac{\partial}{\partial s}, \frac{X}{\varphi}, \frac{Y}{\psi}, \frac{Z}{\xi}\}$ is a g-orthonormal frame at some point $p \in M$, meaning that we have

$$\begin{aligned} g\left(\frac{\partial}{\partial s}, X\right) &= 0, \\ g\left(\frac{\partial}{\partial s}, Y\right) &= 0, \\ g\left(\frac{\partial}{\partial s}, Z\right) &= 0, \\ g(X, Y) &= 0, \\ g(X, Z) &= 0, \\ g(Y, Z) &= 0. \end{aligned} \tag{1.2.60}$$

The SO(3)-invariant action by group conjugation on S^4 generates the Killing fields

X, Y, Z along the geodesic γ. In other words, for all $s \in [0, \sqrt{3}]$, we have

$$
\begin{aligned}
X(\gamma(s)) &= (R_z(\theta)\gamma(s)R_z^{-1}(\theta))'|_{\theta=0} \\
&= (R_z'(\theta)\gamma(s)R_z^{-1}(\theta))|_{\theta=0} + (R_z(\theta)\gamma(s)(R_z^{-1})'(\theta))|_{\theta=0} \\
&= R_z'(0)\gamma(s)R_z^{-1}(0) + R_z(0)\gamma(s)(R_z^{-1})'(0) \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 \\ \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 & \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 \\ -\frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{2\sin(s)}{\sqrt{2}} & 0 \\ \frac{2\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{1.2.61}
$$

and

$$
\begin{aligned}
Y(\gamma(s)) &= (R_y(\theta)\gamma(s)R_y^{-1}(\theta))'|_{\theta=0} \\
&= (R_y'(\theta)\gamma(s)R_y^{-1}(\theta))|_{\theta=0} + (R_y(\theta)\gamma(s)(R_y^{-1})'(\theta))|_{\theta=0} \\
&= R_y'(0)\gamma(s)R_y^{-1}(0) + R_y(0)\gamma(s)(R_y^{-1})'(0) \\
&= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \frac{2\cos(s)}{\sqrt{6}} \\ 0 & 0 & 0 \\ \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{2\cos(s)}{\sqrt{6}} & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix}
\end{aligned} \tag{1.2.62}
$$

and

$$
\begin{aligned}
Z(\gamma(s)) &= (R_x(\theta)\gamma(s)R_x^{-1}(\theta))'|_{\theta=0} \\
&= (R_x'(\theta)\gamma(s)R_x^{-1}(\theta))|_{\theta=0} + (R_x(\theta)\gamma(s)(R_x^{-1})'(\theta))|_{\theta=0} \\
&= R_x'(0)\gamma(s)R_x^{-1}(0) + R_x(0)\gamma(s)(R_x^{-1})'(0) \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{2\cos(s)}{\sqrt{6}} \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & \frac{2\cos(s)}{\sqrt{6}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix}.
\end{aligned} \tag{1.2.63}
$$

The associated lengths of X, Y, Z are

$$\begin{aligned}\mathring{\varphi}(s) &= |X(\gamma(s))| \\ &= \left\| \begin{bmatrix} 0 & \frac{2\sin(s)}{\sqrt{2}} & 0 \\ \frac{2\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\| \\ &= \sqrt{\left(\frac{2\sin(s)}{\sqrt{2}}\right)^2 + \left(\frac{2\sin(s)}{\sqrt{2}}\right)^2} \\ &= 2\sin(s)\end{aligned} \tag{1.2.64}$$

and

$$\begin{aligned}\mathring{\psi}(s) &= |Y(\gamma(s))| \\ &= \left\| \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix} \right\| \\ &= \sqrt{\left(\frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}}\right)^2 + \left(\frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}}\right)^2} \\ &= \sqrt{3}\cos(s) + \sin(s)\end{aligned} \tag{1.2.65}$$

and

$$
\begin{aligned}
\mathring{\xi}(s) &= |Z(\gamma(s))| \\
&= \left\| \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \right\| \\
&= \sqrt{\left(\frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}}\right)^2 + \left(\frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}}\right)^2} \\
&= \sqrt{3}\cos(s) - \sin(s).
\end{aligned} \tag{1.2.66}
$$

We remark that $\mathring{\varphi}, \mathring{\psi}, \mathring{\xi}$ are functions of the round metric $\mathring{g}_0$, which we will discuss in more detail later in Chapter 5.

1.3 Smoothness conditions at the singular orbits of a four-dimensional sphere

We devote this section to establishing abstract conditions that guarantee smoothness of the SO(3)-invariant cohomogeneity one metric g at the singular orbit for $s = 0$. Similarly, the reader can also establish analogous abstract conditions for the other singular orbit at $s = L$.

The following theorem is a result by Luigi Verdiani and Wolfgang Ziller. The reader can find the original statement of this theorem, as well as its proof, in [18].

Theorem 1.3.1 (Theorem A of [18])**.** *Let G act by cohomogeneity one on $(G \times V)/K$ and g be a smooth cohomogeneity one metric defined on the set of regular points in $(G \times V)/K$. Then g has a smooth extension to the singular orbit if and only if it is smooth when restricted to every* 2 *plane in the slice V containing $\gamma'(0)$.*

So we need to show that g is smooth when restricted to $\mathbb{D}^2$ containing $\gamma'(0)$. To achieve this goal, we need to apply Theorem 1.3.1 for

$$V := \mathbb{D}^2, \tag{1.3.1}$$

$$G := \mathrm{SO}(3), \tag{1.3.2}$$

$$K := \mathrm{S}(\mathrm{O}(1)\mathrm{O}(2)) \cong \mathrm{SO}(3)_{\gamma(0)}, \tag{1.3.3}$$

$$H := \mathrm{S}(\mathrm{O}(1)\mathrm{O}(1)\mathrm{O}(1)). \tag{1.3.4}$$

Theorem 1.3.2. *Let γ be a geodesic such that $\mathbb{D}^2$ contains $\gamma'(0)$. The* SO(3)*-invariant cohomogeneity one metric g is smooth on the singular orbit $(\mathrm{SO}(3) \times \mathbb{D}^2)/\mathrm{SO}(3)_{\gamma(0)}$ if and only if the metric*

$$g|_{\mathbb{D}^2} := ds^2 + \varphi^2\, dx^2 \tag{1.3.5}$$

is smooth on $\mathbb{D}^2$ *and the extended functions* $\psi_{ext}, \xi_{ext} : [-L, L] \to \mathbb{R}$ *defined by*

$$\psi_{ext}(s) := \begin{cases} \psi(s) & \textit{for } s \geq 0, \\ \xi(-s) & \textit{for } s < 0 \end{cases} \tag{1.3.6}$$

$$\xi_{ext}(s) := \begin{cases} \xi(s) & \textit{for } s \geq 0, \\ \psi(-s) & \textit{for } s < 0 \end{cases} \tag{1.3.7}$$

are also smooth on $[-L, L]$.

The interested reader can state and prove the counterpart of Theorem 1.3.2 for the other singular orbit $(\mathrm{SO}(3) \times \mathbb{D}^2)/\mathrm{SO}(3)_{\gamma(\frac{\pi}{3})}$.

We will provide a proof of Theorem 1.3.2 at the end of this section. For now, we will discuss the Weyl group $\mathcal{W}$ and its subset $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$. To achieve this, we will need to construct an orthonormal basis of $\mathrm{SO}(3)_{\gamma(0)}$, the singular orbit of S^4.

Proposition 1.3.3. *Define the matrices* $x_- \in K_-$ *and* $x_+ \in K_+$ *by*

$$x_- := \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}, \tag{1.3.8}$$

$$x_+ := \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \end{bmatrix}. \tag{1.3.9}$$

Then $\{e_1, e_2\}$*, with* e_1, e_2 *defined by* (1.2.30) *and* (1.2.31)*, respectively, is an orthonormal basis of the singular orbit* $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(0)} \subseteq S^4$.

Proof. We apply the Gram-Schmidt process to obtain an orthonormal basis of $\mathrm{S}(\mathrm{O}(1)\mathrm{O}(2))$.

Set

$$u_1 := x_-, \tag{1.3.10}$$

$$u_2 := x_- - \frac{g(x_-, x_+)}{|x_-|^2} x_-, \tag{1.3.11}$$

where $g(\cdot, \cdot)$ denotes the inner product of two matrices defined by the trace of their matrix product; that is, we define

$$g(A, B) := \mathrm{tr}(A^T B) \tag{1.3.12}$$

for any 3×3 matrices A, B.

The product of x_- and x_+ is

$$\begin{aligned} x_- x_+ &= \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \end{aligned} \tag{1.3.13}$$

which implies that the inner product of x_- and x_+ is

$$\begin{aligned}
g(x_-, x_+) &= \operatorname{tr}(x_- x_+) \\
&= \operatorname{tr}\left(\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\right) \\
&= \frac{1}{3} + \left(-\frac{1}{6}\right) + \frac{1}{3} \\
&= \frac{1}{2}.
\end{aligned} \tag{1.3.14}$$

Also, the magnitude of x_- is

$$\begin{aligned}
|x_-| &= \sqrt{\left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(-\frac{2}{\sqrt{6}}\right)^2} \\
&= \sqrt{\frac{1}{6} + \frac{1}{6} + \frac{2}{3}} \\
&= \sqrt{1} \\
&= 1.
\end{aligned} \tag{1.3.15}$$

So we have

$$\begin{aligned}
u_1 &= x_- \\
&= \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}
\end{aligned} \tag{1.3.16}$$

and

$$\begin{aligned}
u_2 &= x_+ - \frac{g(x_-, x_+)}{|x_-|^2} x_- \\
&= \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \end{bmatrix} - \frac{\frac{1}{2}}{1^2} \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \end{bmatrix} - \begin{bmatrix} \frac{1}{2\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{2\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{2\sqrt{6}} & 0 & 0 \\ 0 & -\frac{3}{2\sqrt{6}} & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{1.3.17}$$

Their magnitudes are

$$\begin{aligned} |u_1| &= |x_-| \\ &= 1 \end{aligned} \tag{1.3.18}$$

and

$$\begin{aligned} |u_2| &= \sqrt{\left(\frac{3}{2\sqrt{6}}\right)^2 + \left(-\frac{3}{2\sqrt{6}}\right)^2} \\ &= \frac{\sqrt{3}}{2}. \end{aligned} \tag{1.3.19}$$

So our orthonormal basis of the singular orbit of S^4 is $\{e_1, e_2\}$, where we define $e_1, e_2 \in T\mathcal{F}$

by

$$
\begin{aligned}
e_1 &:= \frac{u_1}{|u_1|} \\
&= \frac{x_-}{1} \\
&= \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}
\end{aligned} \tag{1.3.20}
$$

and

$$
\begin{aligned}
e_2 &= \frac{u_2}{|u_2|} \\
&= \frac{1}{\frac{\sqrt{3}}{2}} \begin{bmatrix} \frac{3}{2\sqrt{6}} & 0 & 0 \\ 0 & -\frac{3}{2\sqrt{6}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{1.3.21}
$$

which are (1.2.30) and (1.2.31), respectively. □

Now, we define the geodesic $\gamma : [0, \frac{\pi}{3}] \to \mathbb{R}$ along the singular orbit K_- by

$$\begin{aligned}
\gamma(s) &:= e_1 \cos(s) + e_2 \sin(s) \\
&= \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \cos(s) + \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \sin(s) \\
&= \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} + \begin{bmatrix} \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix}.
\end{aligned} \tag{1.3.22}$$

Then the first derivative of γ is given by

$$\begin{aligned}
\gamma'(s) &= \frac{d}{ds} \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{d}{ds}(\frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}}) & 0 & 0 \\ 0 & \frac{d}{ds}(\frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}}) & 0 \\ 0 & 0 & \frac{d}{ds}(-\frac{2\cos(s)}{\sqrt{6}}) \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\sin(s)}{\sqrt{6}} + \frac{\cos(s)}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{\sin(s)}{\sqrt{6}} - \frac{\cos(s)}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{2\sin(s)}{\sqrt{6}} \end{bmatrix}.
\end{aligned} \tag{1.3.23}$$

In particular, at $s = 0$, we have

$$\begin{aligned}\gamma'(0) &= \begin{bmatrix} -\frac{\sin(0)}{\sqrt{6}} + \frac{\cos(0)}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{\sin(0)}{\sqrt{6}} - \frac{\cos(0)}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{2\sin(0)}{\sqrt{6}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}.\end{aligned} \tag{1.3.24}$$

For some geodesic γ in the Riemannian manifold M, let $G_{\gamma(0)} \subseteq \mathrm{O}(3)$ be the isotropy group, or stabilizer group, of the point $\gamma(0)$, which is a subgroup of $\mathrm{O}(3)$ that fixes $\gamma(0)$. Consequently, we can write

$$G_{\gamma(0)} = \left\{ \begin{bmatrix} A & 0 \\ 0 & \det(A) \end{bmatrix} \in O(3) \;\middle|\; A \in O(2) \right\}. \tag{1.3.25}$$

Then the *Weyl group* of some geodesic γ is the group $\mathcal{W}$ of all isometries of γ that are induced by the elements of $\mathrm{SO}(3)$ through the $\mathrm{SO}(3)$ action of conjugation. Consequently, we can write

$$\mathcal{W} = \{P \in \mathrm{SO}(3) \mid P\gamma(s)P^{-1} = \gamma(\pm s + b), b \in \mathbb{R}\}. \tag{1.3.26}$$

The intersection of the Weyl group $\mathcal{W}$ and the isotropy group $\mathrm{SO}(3)_{\gamma(0)}$ is

$$\begin{aligned}\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}} &:= \mathcal{W} \cap \mathrm{SO}(3)_{\gamma(0)} \\ &= \{P \in \mathrm{SO}(3) \mid P\gamma(s)P^{-1} = \gamma(\pm s)\}.\end{aligned} \tag{1.3.27}$$

Also, observe that, for all $P \in \mathcal{W}$, we have

$$
\begin{aligned}
P\gamma'(s)P^{-1} &= \frac{d}{ds}(P\gamma(s)P^{-1}) \\
&= \frac{d}{ds}(\gamma(\pm s + b)) \\
&= \pm\gamma'(\pm s + b).
\end{aligned}
\tag{1.3.28}
$$

In particular, at $s = 0$ and $b = 0$, we have

$$
P\gamma'(0)P^{-1} = \pm\gamma'(0). \tag{1.3.29}
$$

So we can also write

$$
\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}} := \{P \in G_{\gamma(0)} \mid P\gamma'(0)P^{-1} = \pm\gamma'(0)\}. \tag{1.3.30}
$$

This is particularly useful for determining whether the functions $\mathring{\varphi}, \mathring{\psi}, \mathring{\xi}$ for the round metric $\mathring{g}_0$ are smooth on a neighborhood of $s = 0$.

Define the generalized rotation matrix $R_z^{\pm} \in \mathrm{SO}(3)$ about the z-axis by

$$
R_z^{\pm}(\theta) := \begin{bmatrix} \pm\cos(\theta) & \mp\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}. \tag{1.3.31}
$$

Note that their inverses are $(R_z^{\pm})^{-1} \in \mathrm{SO}(3)$ given by

$$
(R_z^{\pm})^{-1}(\theta) = \begin{bmatrix} \cos(\theta) & \pm\sin(\theta) & 0 \\ -\sin(\theta) & \pm\cos(\theta) & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}. \tag{1.3.32}
$$

Alternatively, we write

$$R_z^+(\theta) = R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.3.33}$$

$$(R_z^+)^{-1}(\theta) = R_z^{-1}(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.3.34}$$

$$R_z^-(\theta) = \begin{bmatrix} -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{1.3.35}$$

$$(R_z^-)^{-1}(\theta) = R_z^-(\theta) = \begin{bmatrix} -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{1.3.36}$$

Proposition 1.3.4. *Consider the isotropy group*

$$\mathrm{SO}(3)_{\gamma(0)} := \{R_z^{\pm}(\theta) \mid \theta \in [0, 2\pi)\} \tag{1.3.37}$$

equipped with the binary operation of matrix multiplication. Then the intersection of the

Weyl group $\mathcal{W}$ and the isotropy group $\mathrm{SO}(3)_{\gamma(0)}$ *is*

$$\begin{aligned}\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}} &= \mathcal{W} \cap \mathrm{SO}(3)_{\gamma(0)}\\ &= \Big\{R_z^+(0), R_z^-(0), R_z^+\left(\frac{\pi}{2}\right),\\ &\quad R_z^-\left(\frac{\pi}{2}\right), R_z^+(\pi), R_z^-(\pi), R_z^+\left(\frac{3\pi}{2}\right) R_z^-\left(\frac{3\pi}{2}\right)\Big\}\\ &= \left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix},\right.\\ &\quad \left.\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}\right\}.\end{aligned} \tag{1.3.38}$$

Notice that $R_z^+(0), R_z^-(0), R_z^+(\pi), R_z^-(\pi)$ *all fix* $\gamma(s)$ *through the* $\mathrm{SO}(3)$ *action of conjugation, whereas* $R_z^+(\frac{\pi}{2}), R_z^-(\frac{\pi}{2}), R_z^+(\frac{3\pi}{2}), R_z^-(\frac{3\pi}{2})$ *all send* $\gamma(s)$ *to* $\gamma(-s)$ *through the same action.*

Proof. According to (1.3.30), any element $R_z^{\pm}(\theta) \in \mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$ for suitable values of θ satisfies the condition

$$R_z^{\pm}(\theta)\gamma'(0)(R_z^{\pm})^{-1}(\theta) = \gamma'(0), \tag{1.3.39}$$

$$R_z^{\pm}(\theta)\gamma'(0)(R_z^{\pm})^{-1}(\theta) = -\gamma'(0), \tag{1.3.40}$$

which are equivalent to

$$R_z^{\pm}(\theta)\gamma'(0) = \gamma'(0)R_z^{\pm}(\theta), \tag{1.3.41}$$

$$R_z^{\pm}(\theta)\gamma'(0) = -\gamma'(0)R_z^{\pm}(\theta), \tag{1.3.42}$$

respectively. We have

$$
\begin{aligned}
R_z^{\pm}(\theta)\gamma'(0) &= \begin{bmatrix} \pm\cos(\theta) & \mp\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \pm\frac{\cos(\theta)}{\sqrt{2}} & \pm\frac{\sin(\theta)}{\sqrt{2}} & 0 \\ \frac{\sin(\theta)}{\sqrt{2}} & -\frac{\cos(\theta)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{1.3.43}
$$

and

$$
\begin{aligned}
\gamma'(0)R_z^{\pm}(\theta) &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pm\cos(\theta) & \mp\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \\
&= \begin{bmatrix} \pm\frac{\cos(\theta)}{\sqrt{2}} & \mp\frac{\sin(\theta)}{\sqrt{2}} & 0 \\ -\frac{\sin(\theta)}{\sqrt{2}} & -\frac{\cos(\theta)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{1.3.44}
$$

and

$$
\begin{aligned}
-\gamma'(0)R_z^{\pm}(\theta) &= -\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pm\cos(\theta) & \mp\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \\
&= -\begin{bmatrix} \pm\frac{\cos(\theta)}{\sqrt{2}} & \mp\frac{\sin(\theta)}{\sqrt{2}} & 0 \\ -\frac{\sin(\theta)}{\sqrt{2}} & -\frac{\cos(\theta)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mp\frac{\cos(\theta)}{\sqrt{2}} & \pm\frac{\sin(\theta)}{\sqrt{2}} & 0 \\ \frac{\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{1.3.45}
$$

So (1.3.41) and (1.3.42) are equivalent to the matrix equations

$$
\begin{bmatrix} \pm\frac{\cos(\theta)}{\sqrt{2}} & \pm\frac{\sin(\theta)}{\sqrt{2}} & 0 \\ \frac{\sin(\theta)}{\sqrt{2}} & -\frac{\cos(\theta)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \pm\frac{\cos(\theta)}{\sqrt{2}} & \mp\frac{\sin(\theta)}{\sqrt{2}} & 0 \\ -\frac{\sin(\theta)}{\sqrt{2}} & -\frac{\cos(\theta)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.3.46}
$$

$$
\begin{bmatrix} \pm\frac{\cos(\theta)}{\sqrt{2}} & \pm\frac{\sin(\theta)}{\sqrt{2}} & 0 \\ \frac{\sin(\theta)}{\sqrt{2}} & -\frac{\cos(\theta)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mp\frac{\cos(\theta)}{\sqrt{2}} & \pm\frac{\sin(\theta)}{\sqrt{2}} & 0 \\ \frac{\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1.3.47}
$$

respectively. Equating the entries of (1.3.46) tells us that we require θ to satisfy

$$
\sin(\theta) = -\sin(\theta). \tag{1.3.48}
$$

Likewise, equating the entries of (1.3.47) tells us that we require θ to satisfy

$$\cos(\theta) = -\cos(\theta). \tag{1.3.49}$$

The solutions of (1.3.48) on $[0, 2\pi)$ are $\theta = 0$ and $\theta = \pi$, and the solutions of (1.3.49) on $[0, 2\pi)$ are $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. So we substitute $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ into R_x^+ in order to obtain

$$R_z^+(0) = \begin{bmatrix} \cos(0) & -\sin(0) & 0 \\ \sin(0) & \cos(0) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.3.50}$$

$$R_z^+\left(\frac{\pi}{2}\right) = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) & 0 \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.3.51}$$

$$R_z^+(\pi) = \begin{bmatrix} \cos(\pi) & -\sin(\pi) & 0 \\ \sin(\pi) & \cos(\pi) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.3.52}$$

$$R_z^+\left(\frac{3\pi}{2}\right) = \begin{bmatrix} \cos(\frac{3\pi}{2}) & -\sin(\frac{3\pi}{2}) & 0 \\ \sin(\frac{3\pi}{2}) & \cos(\frac{3\pi}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.3.53}$$

and into R_x^- in order to obtain

$$R_z^-(0) = \begin{bmatrix} -\cos(0) & \sin(0) & 0 \\ \sin(0) & \cos(0) & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{1.3.54}$$

$$R_z^-\left(\frac{\pi}{2}\right) = \begin{bmatrix} -\cos(\frac{\pi}{2}) & \sin(\frac{\pi}{2}) & 0 \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{1.3.55}$$

$$R_z^-(\pi) = \begin{bmatrix} -\cos(\pi) & \sin(\pi) & 0 \\ \sin(\pi) & \cos(\pi) & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{1.3.56}$$

$$R_z^-\left(\frac{3\pi}{2}\right) = \begin{bmatrix} -\cos(\frac{3\pi}{2}) & \sin(\frac{3\pi}{2}) & 0 \\ \sin(\frac{3\pi}{2}) & \cos(\frac{3\pi}{2}) & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{1.3.57}$$

which are precisely all the elements of $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$, as we claimed in (1.3.38). □

Proposition 1.3.5. *Every element of $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$ does the following:*

(1) *Every $P \in \mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$ sends $Y(\gamma(s))$ to $\pm Y(\gamma(s))$ or $\pm\mathring{Z}(\gamma(-s))$ through the* SO(3) *action of conjugation on S^4; that is, we have any one of*

$$PY(\gamma(s))P^{-1} = \pm Y(\gamma(s)), \tag{1.3.58}$$

$$PY(\gamma(s))P^{-1} = \pm Z(\gamma(-s)) \tag{1.3.59}$$

for all $P \in \mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$. Elements of $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$ that fix $\gamma(s)$ imply (1.3.58)*, and elements of $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$ that send $\gamma(s)$ to $\gamma(-s)$ imply* (1.3.59)*.*

(2) *Every* $P \in \mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$ *sends* $Z(\gamma(s))$ *to* $\pm Y(\gamma(s))$ *or* $\pm \mathring{Z}(\gamma(-s))$ *through the* SO(3) *action of conjugation on* S^4*; that is, we have any one of*

$$PZ(\gamma(s))P^{-1} = \pm Z(\gamma(s)), \tag{1.3.60}$$

$$PZ(\gamma(s))P^{-1} = \pm Y(\gamma(-s)) \tag{1.3.61}$$

for all $P \in \mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$*. Elements of* $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$ *that fix* $\gamma(s)$ *imply* (1.3.60)*, and elements of* $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$ *that send* $\gamma(s)$ *to* $\gamma(-s)$ *imply* (1.3.61).

Proof. We will use the rotation matrices about the z-axis for $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$; in other words, we will use $R_z^+(0), R_z^+(\frac{\pi}{2}), R_z^+(\pi), R_z^+(\frac{3\pi}{2}), R_z^-(0), R_z^-(\frac{\pi}{2}), R_z^-(\pi), R_z^-(\frac{3\pi}{2})$, which we defined by (1.3.50), (1.3.51), (1.3.52), (1.3.53), (1.3.54), (1.3.55), (1.3.56), (1.3.57), respectively, as well as their respective inverses

$$(R_z^+)^{-1}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.3.62}$$

$$(R_z^+)^{-1}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.3.63}$$

$$(R_z^+)^{-1}(\pi) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.3.64}$$

$$(R_z^+)^{-1}\left(\frac{3\pi}{2}\right) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.3.65}$$

and

$$(R_z^-)^{-1}(0) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{1.3.66}$$

$$(R_z^-)^{-1}\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{1.3.67}$$

$$(R_z^-)^{-1}(\pi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{1.3.68}$$

$$(R_z^-)^{-1}\left(\frac{3\pi}{2}\right) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{1.3.69}$$

in our computations.

Now, we will prove (1). In order to prove either (1.3.58) or (1.3.59) for all $P \in$

$\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$, it suffices to establish

$$R_z^+(0)Y(\gamma(s))(R_z^+)^{-1}(0) = Y(\gamma(s)), \tag{1.3.70}$$

$$R_z^-(0)Y(\gamma(s))(R_z^-)^{-1}(0) = Y(\gamma(s)), \tag{1.3.71}$$

$$R_z^+\left(\frac{\pi}{2}\right)Y(\gamma(s))(R_z^+)^{-1}\left(\frac{\pi}{2}\right) = Z(\gamma(-s)), \tag{1.3.72}$$

$$R_z^-\left(\frac{\pi}{2}\right)Y(\gamma(s))(R_z^-)^{-1}\left(\frac{\pi}{2}\right) = -Z(\gamma(-s)), \tag{1.3.73}$$

$$R_z^+(\pi)Y(\gamma(s))(R_z^+)^{-1}(\pi) = -Y(\gamma(s)), \tag{1.3.74}$$

$$R_z^-(\pi)Y(\gamma(s))(R_z^-)^{-1}(\pi) = -Y(\gamma(s)), \tag{1.3.75}$$

$$R_z^+\left(\frac{3\pi}{2}\right)Y(\gamma(s))(R_z^+)^{-1}\left(\frac{3\pi}{2}\right) = -Z(\gamma(-s)), \tag{1.3.76}$$

$$R_z^-\left(\frac{3\pi}{2}\right)Y(\gamma(s))(R_z^-)^{-1}\left(\frac{3\pi}{2}\right) = Z(\gamma(-s)), \tag{1.3.77}$$

according to (1.3.38). To achieve this, we have

$$\begin{aligned}
&R_z^+(0)Y(\gamma(s))(R_z^+)^{-1}(0)\\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix}\\
&= \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix}\\
&= Y(\gamma(s))
\end{aligned} \tag{1.3.78}$$

and

$$
\begin{aligned}
&R_z^-(0)Y(\gamma(s))(R_z^-)^{-1}(0)\\
&=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}0 & 0 & \frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\ 0 & 0 & 0\\ \frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}} & 0 & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\\
&=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}0 & 0 & -\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\ 0 & 0 & 0\\ -\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}} & 0 & 0\end{bmatrix}\\
&=\begin{bmatrix}0 & 0 & \frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\ 0 & 0 & 0\\ \frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}} & 0 & 0\end{bmatrix}\\
&=Y(\gamma(s))
\end{aligned}
\tag{1.3.79}
$$

and

$$
\begin{aligned}
&R_z^+\left(\frac{\pi}{2}\right)Y(\gamma(s))(R_z^+)^{-1}\left(\frac{\pi}{2}\right) \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(-s)}{\sqrt{6}} - \frac{\sin(-s)}{\sqrt{2}} \\ 0 & \frac{3\cos(-s)}{\sqrt{6}} - \frac{\sin(-s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&= Z(\gamma(-s))
\end{aligned}
\tag{1.3.80}
$$

and

$$
\begin{aligned}
&R_z^-\left(\frac{\pi}{2}\right) Y(\gamma(s)) (R_z^-)^{-1}\left(\frac{\pi}{2}\right) \\
&\quad = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & -\frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \qquad (1.3.81) \\
&\quad = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&\quad = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(-s)}{\sqrt{6}} - \frac{\sin(-s)}{\sqrt{2}} \\ 0 & \frac{3\cos(-s)}{\sqrt{6}} - \frac{\sin(-s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&\quad = -Z(\gamma(-s))
\end{aligned}
$$

and

$$
\begin{aligned}
&R_z^+(\pi)Y(\gamma(s))(R_z^+)^{-1}(\pi)\\
&=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\\
&=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\-\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&=\begin{bmatrix}0&0&-\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\-\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&=-\begin{bmatrix}0&0&\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&=-Y(\gamma(s))
\end{aligned}
\tag{1.3.82}
$$

and

$$
\begin{aligned}
&R_z^-(\pi)Y(\gamma(s))(R_z^-)^{-1}(\pi) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -\frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ -\frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix} \\
&= - \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix} \\
&= -Y(\gamma(s))
\end{aligned}
\tag{1.3.83}
$$

and

$$
\begin{aligned}
&R_z^+\left(\frac{3\pi}{2}\right)Y(\gamma(s))(R_z^+)^{-1}\left(\frac{3\pi}{2}\right)\\
&=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}\\
&=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\0&-\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\\
&=\begin{bmatrix}0&0&0\\0&0&-\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&-\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\\
&=-\begin{bmatrix}0&0&0\\0&0&\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\\
&=-\begin{bmatrix}0&0&0\\0&0&\frac{3\cos(-s)}{\sqrt{6}}-\frac{\sin(-s)}{\sqrt{2}}\\0&\frac{3\cos(-s)}{\sqrt{6}}-\frac{\sin(-s)}{\sqrt{2}}&0\end{bmatrix}\\
&=-Z(\gamma(-s))
\end{aligned}
\tag{1.3.84}
$$

and

$$
\begin{aligned}
&R_z^-\left(\frac{3\pi}{2}\right)Y(\gamma(s))(R_z^-)^{-1}\left(\frac{3\pi}{2}\right)\\
&=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&0&\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}\\
&=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&0&-\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\0&-\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\\
&=\begin{bmatrix}0&0&0\\0&0&\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\\
&=\begin{bmatrix}0&0&0\\0&0&\frac{3\cos(-s)}{\sqrt{6}}-\frac{\sin(-s)}{\sqrt{2}}\\0&\frac{3\cos(-s)}{\sqrt{6}}-\frac{\sin(-s)}{\sqrt{2}}&0\end{bmatrix}\\
&=Z(\gamma(-s)),
\end{aligned}
\tag{1.3.85}
$$

which are (1.3.70), (1.3.71), (1.3.72), (1.3.73), (1.3.74), (1.3.75), (1.3.76), (1.3.77), respectively. This completes our proof of (1).

Next, we will prove (2). In order to prove either (1.3.60) or (1.3.61) for all $P \in$

$\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$, it suffices to establish

$$R_z^+(0)Z(\gamma(s))(R_z^+)^{-1}(0) = Z(\gamma(s)), \tag{1.3.86}$$

$$R_z^-(0)Z(\gamma(s))(R_z^-)^{-1}(0) = -Z(\gamma(s)), \tag{1.3.87}$$

$$R_z^+\left(\frac{\pi}{2}\right)Z(\gamma(s))(R_z^+)^{-1}\left(\frac{\pi}{2}\right) = -\mathring{Y}(\mathring{\gamma}(-s)), \tag{1.3.88}$$

$$R_z^-\left(\frac{\pi}{2}\right)Z(\gamma(s))(R_z^-)^{-1}\left(\frac{\pi}{2}\right) = -\mathring{Y}(\mathring{\gamma}(-s)), \tag{1.3.89}$$

$$R_z^+(\pi)Z(\gamma(s))(R_z^+)^{-1}(\pi) = -Z(\gamma(s)), \tag{1.3.90}$$

$$R_z^-(\pi)Z(\gamma(s))(R_z^-)^{-1}(\pi) = Z(\gamma(s)), \tag{1.3.91}$$

$$R_z^+\left(\frac{3\pi}{2}\right)Z(\gamma(s))(R_z^+)^{-1}\left(\frac{3\pi}{2}\right) = \mathring{Y}(\mathring{\gamma}(-s)), \tag{1.3.92}$$

$$R_z^-\left(\frac{3\pi}{2}\right)Z(\gamma(s))(R_z^-)^{-1}\left(\frac{3\pi}{2}\right) = \mathring{Y}(\mathring{\gamma}(-s)), \tag{1.3.93}$$

according to (1.3.38). To achieve this, we have

$$\begin{aligned}
&R_z^+(0)Z(\gamma(s))(R_z^+)^{-1}(0)\\
&\quad= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\\
&\quad= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\\
&\quad= \begin{bmatrix}0&0&0\\0&0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\\
&\quad= Z(\gamma(s))
\end{aligned} \tag{1.3.94}$$

and

$$
\begin{aligned}
&R_z^-(0)Z(\gamma(s))(R_z^-)^{-1}(0)\\
&=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\\
&=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&-\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\\
&=\begin{bmatrix}0&0&0\\0&0&-\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&-\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\\
&=-\begin{bmatrix}0&0&0\\0&0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\\
&=-Z(\gamma(s))
\end{aligned}
\tag{1.3.95}
$$

and

$$
\begin{aligned}
&R_z^+\left(\frac{\pi}{2}\right)Z(\gamma(s))(R_z^+)^{-1}\left(\frac{\pi}{2}\right)\\
&=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\ 0 & \frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\\
&=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\ -\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}} & 0 & 0\end{bmatrix}\\
&=\begin{bmatrix}0 & 0 & -\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\ 0 & 0 & 0\\ -\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}} & 0 & 0\end{bmatrix}\\
&=-\begin{bmatrix}0 & 0 & \frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\ 0 & 0 & 0\\ \frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}} & 0 & 0\end{bmatrix}\\
&=-\begin{bmatrix}0 & 0 & \frac{3\cos(-s)}{\sqrt{6}}+\frac{\sin(-s)}{\sqrt{2}}\\ 0 & 0 & 0\\ \frac{3\cos(-s)}{\sqrt{6}}+\frac{\sin(-s)}{\sqrt{2}} & 0 & 0\end{bmatrix}\\
&=-Y(\gamma(-s))
\end{aligned}
\tag{1.3.96}
$$

and

$$
\begin{aligned}
&R_z^-\left(\frac{\pi}{2}\right)Z(\gamma(s))(R_z^-)^{-1}\left(\frac{\pi}{2}\right)\\
&\quad=\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}\\
&\quad=\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&-\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&\quad=\begin{bmatrix}0&0&-\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\-\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&\quad=-\begin{bmatrix}0&0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&\quad=-\begin{bmatrix}0&0&\frac{3\cos(-s)}{\sqrt{6}}+\frac{\sin(-s)}{\sqrt{2}}\\0&0&0\\\frac{3\cos(-s)}{\sqrt{6}}+\frac{\sin(-s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&\quad=-Y(\gamma(-s))
\end{aligned}
\tag{1.3.97}
$$

and

$$
\begin{aligned}
&R_z^+(\pi)Z(\gamma(s))(R_z^+)^{-1}(\pi) \\
&= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & -\frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & -\frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&= -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&= -Z(\gamma(s))
\end{aligned}
\tag{1.3.98}
$$

and

$$
\begin{aligned}
&R_z^-(\pi)Z(\gamma(s))(R_z^-)^{-1}(\pi) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} \\ 0 & -\frac{3\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \\
&= Z(\gamma(s))
\end{aligned}
\tag{1.3.99}
$$

and

$$\begin{aligned}
&R_z^+\left(\frac{3\pi}{2}\right) Z(\gamma(s)) (R_z^+)^{-1}\left(\frac{3\pi}{2}\right)\\
&= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix}\\
&= \begin{bmatrix} 0 & 0 & \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \end{bmatrix}\\
&= \begin{bmatrix} 0 & 0 & \frac{3\cos(-s)}{\sqrt{6}} + \frac{\sin(-s)}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{3\cos(-s)}{\sqrt{6}} + \frac{\sin(-s)}{\sqrt{2}} & 0 & 0 \end{bmatrix}\\
&= Y(\gamma(-s))
\end{aligned} \tag{1.3.100}$$

and

$$
\begin{aligned}
&R_z^-\left(\frac{3\pi}{2}\right)Z(\gamma(s))(R_z^-)^{-1}\left(\frac{3\pi}{2}\right)\\
&=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0\end{bmatrix}\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}\\
&=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&-\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\-\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&=\begin{bmatrix}0&0&0\\0&0&-\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}\\-\frac{3\cos(s)}{\sqrt{6}}+\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&=\begin{bmatrix}0&0&\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}\\0&0&0\\\frac{3\cos(s)}{\sqrt{6}}-\frac{\sin(s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&=\begin{bmatrix}0&0&\frac{3\cos(-s)}{\sqrt{6}}+\frac{\sin(-s)}{\sqrt{2}}\\0&0&0\\\frac{3\cos(-s)}{\sqrt{6}}+\frac{\sin(-s)}{\sqrt{2}}&0&0\end{bmatrix}\\
&=Y(\gamma(-s)),
\end{aligned}
\tag{1.3.101}
$$

which are (1.3.86), (1.3.87), (1.3.88), (1.3.89), (1.3.90), (1.3.91), (1.3.92), (1.3.93), respectively. This completes our proof of (2). □

Now we will show that any metric g that is smooth on $(\mathrm{SO}(3)\times\mathbb{D}^2)/\mathrm{SO}(3)_{\gamma(0)}$ also extends uniquely to an $\mathrm{SO}(3)_{\gamma(0)}$-invariant metric on $\mathbb{D}^2$. In particular, the Killing fields X, Y, Z do not depend on our choice of g.

Proposition 1.3.6. *Let K be a group that acts transitively on $\mathbb{D}^2$. If $g|_\gamma$ is smooth and $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$-invariant, then $g|_\gamma$ extends uniquely to an $\mathrm{SO}(3)_{\gamma(0)}$-invariant metric on $\mathbb{D}^2$.*

Proof. Since K acts transitively on the unit sphere in $\mathbb{D}^2$, for all $p \in \mathbb{D}^2$ and for all $V, W \in T_p((\mathrm{SO}(3) \times \mathbb{D}^2)/\mathrm{SO}(3)_{\gamma(0)})$, there exists $k \in K$ that satisfies

$$kp = \gamma(s) \tag{1.3.102}$$

for some $s \in [-1, 1]$. Now, we set

$$g(V, W) := g|_\gamma(dk(V), dk(W))), \tag{1.3.103}$$

which means this g is $\mathrm{SO}(3)_{\gamma(0)}$-invariant, provided that g is well-defined. To prove that g is well-defined, let $\tilde{k} \in K$ satisfy

$$\tilde{k}p = \gamma(\tilde{s}). \tag{1.3.104}$$

Since K also acts on $\mathbb{D}^2$ by isometries and every $k \in K$ fixes $\gamma(0)$, the group action must preserve distances, and so we conclude $\tilde{s} = \pm s$. For the case $\tilde{s} = s$, we obtain

$$\begin{aligned} \gamma(s) &= \tilde{k}p \\ &= \tilde{k}(k^{-1}k)p \\ &= \tilde{k}k^{-1}(kp) \\ &= \tilde{k}k^{-1}\gamma(s), \end{aligned} \tag{1.3.105}$$

which implies $\tilde{k}k^{-1} \in \mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$, according to (1.3.27). Likewise, for the case $\tilde{s} = -s$, we obtain

$$\begin{aligned}\gamma(s) &= \tilde{k}p \\ &= \tilde{k}(k^{-1}k)p \\ &= \tilde{k}k^{-1}(kp) \\ &= \tilde{k}k^{-1}\gamma(-s),\end{aligned} \tag{1.3.106}$$

which implies $\tilde{k}k^{-1} \in \mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$, according to (1.3.27). So we conclude

$$g(V, W) := g|_{\gamma}(d\tilde{k}(V), d\tilde{k}(W))), \tag{1.3.107}$$

and so g, as defined by (1.3.103), does not depend on any choice of $k \in K$. Therefore, g is well-defined, and so (1.3.103) implies that g is $\mathrm{SO}(3)_{\gamma(0)}$-invariant and also unique. □

Finally, we will write a proof of Theorem 1.3.2.

Proof of Theorem 1.3.2. According to Theorem 1.3.1, it suffices to show that $g|_{\mathbb{D}^2}$ defined by (1.3.5) is smooth. Furthermore, by Proposition 1.3.6, it suffices to show that the restriction of $g|_{\mathbb{D}^2}$ to the geodesic γ is smooth, and we let $g|_{\gamma}$ denote such a restriction. Since $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$ leaves γ invariant, the metric $g|_{\gamma}$ must be $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$-invariant. Also, as we have demonstrated in Proposition 1.3.5, elements of $\mathcal{W}_{\mathrm{SO}(3)_{\gamma(0)}}$ that send $\gamma(s)$ to $\gamma(-s)$ must also send $Y(\gamma(s))$ to $\pm Z(\gamma(-s))$ and $Z(\gamma(s))$ to $\pm Y(\gamma(-s))$. So we have

$$\begin{aligned}\psi(s) &= |Y(\gamma(s))| \\ &= |Z(\gamma(-s))| \\ &= \xi(-s)\end{aligned} \tag{1.3.108}$$

and

$$\begin{aligned}\xi(s) &= |Z(\gamma(s))| \\ &= |Y(\gamma(-s))| \\ &= \psi(-s).\end{aligned} \tag{1.3.109}$$

So we conclude that $g|_\gamma$ is smooth if and only if the functions ψ, ξ satisfy (5.1.20) and (5.1.21). □

Chapter 2

Applications of covariant derivatives

In this chapter, we will obtain expressions of covariant derivatives in terms of the three functions φ, ψ, ξ associated with the SO(3)-invariant Riemannian metric g on S^4 as defined by (1.0.8). To do this, we will invoke the Fundamental Theorem of Riemannian Geometry, which states that the affine connection of nearby tangent spaces is metric and torsion-free; in other words, for any vector fields X, Y, Z that are tangent to any four-dimensional Riemannian manifold M, the affine connection ∇ satisfies

$$D_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X, Z), \tag{2.0.1}$$

$$[X,Y] = \nabla_X Y - \nabla_Y X. \tag{2.0.2}$$

We will also apply the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) = D_X(g(Y,Z)) + D_Y(g(X,Z)) - D_Z(g(X,Y)) \\ + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X), \end{aligned} \tag{2.0.3}$$

which is used to prove the Fundamental Theorem of Riemannian Geometry.

Using the covariant derivative, we are also able to introduce the second fundamental form.

Definition 2.0.1. *Let* X, Y *are vector fields tangent to* $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$*. We define the* second fundamental form *of* $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)} \subseteq M$ *for all* $s \in [0, L]$ *by*

$$\mathrm{II}(X, Y) := (\nabla_X Y)|_{\frac{\partial}{\partial r}}, \tag{2.0.4}$$

where $(\nabla_X Y)|_{\frac{\partial}{\partial r}}$ *denotes the restriction of* $\nabla_X Y$ *to the radial field* $\frac{\partial}{\partial r}$*.*

We consider the radial vector field $\frac{\partial}{\partial s}$, which is orthogonal to $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)} \subseteq S^4$ for all $s \in (0, \frac{\pi}{3})$, and the Killing fields X, Y, Z, which are tangent to $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$. According to (1.2.57), (1.2.58), (1.2.59), we denote by φ, ψ, ξ the respective lengths of X, Y, Z.

For further details on this exposition, the interested reader can consult, for instance, Chapters 2 and 3 of [17]. The remainder of this chapter will focus on our results that make use of the covariant derivative. Wherever possible, we will also apply our condition

$$\psi = \varphi + \xi \tag{2.0.5}$$

into our expressions in order to simplify them as much as we can.

2.1 Covariant derivatives along the radial vector field

We will obtain expressions of covariant derivatives along the radial vector field $\frac{\partial}{\partial s}$ in terms of the functions φ, ψ, ξ and the vector fields X, Y, Z.

Lemma 2.1.1. *The covariant derivatives associated with a family of* $\mathrm{SO}(3)$*-invariant Riemannian metric* g *are*

$$\nabla_{\frac{\partial}{\partial s}} X = \frac{\varphi'}{\varphi} X, \tag{2.1.1}$$

$$\nabla_{\frac{\partial}{\partial s}} Y = \frac{\psi'}{\psi} Y, \tag{2.1.2}$$

$$\nabla_{\frac{\partial}{\partial s}} Z = \frac{\xi'}{\xi} Z. \tag{2.1.3}$$

Proof. The proofs of (2.1.1), (2.1.2), (2.1.3) are all analogous to each other. Without loss of generality, we choose to only prove (2.1.1). We apply (2.0.1) and (2.0.2) in order to obtain

$$
\begin{aligned}
g\left(\nabla_{\frac{\partial}{\partial s}} X, \frac{\partial}{\partial s}\right) &= g\left(\left[\frac{\partial}{\partial s}, X\right] + \nabla_X \frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) \\
&= g\left(0 + \nabla_X \frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) \\
&= g\left(\nabla_X \frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) \\
&= \frac{1}{2}\left(g\left(\nabla_X \frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) + g\left(\frac{\partial}{\partial s}, \nabla_X \frac{\partial}{\partial s}\right)\right) \\
&= \frac{1}{2} D_X g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) \\
&= \frac{1}{2} D_X \left(\left|\frac{\partial}{\partial s}\right|^2\right) \\
&= \frac{1}{2} D_X ((1)^2) \\
&= \frac{1}{2} \cdot 0 \\
&= 0
\end{aligned}
\tag{2.1.4}
$$

and

$$
\begin{aligned}
g(\nabla_{\frac{\partial}{\partial s}} X, X) &= \frac{1}{2}(g(\nabla_{\frac{\partial}{\partial s}} X, X) + g(X, \nabla_{\frac{\partial}{\partial s}} X)) \\
&= \frac{1}{2} D_{\frac{\partial}{\partial s}} g(X, X) \\
&= \frac{1}{2} D_{\frac{\partial}{\partial s}} |X|^2 \\
&= \frac{1}{2} D_{\frac{\partial}{\partial s}} \varphi^2 \\
&= \frac{(\varphi^2)'}{2} \\
&= \varphi\varphi'.
\end{aligned}
\tag{2.1.5}
$$

We apply (2.0.3) in order to obtain

$$
\begin{aligned}
g(\nabla_{\frac{\partial}{\partial s}} X, Y) &= D_{\frac{\partial}{\partial s}}(g(X,Y)) + D_X\left(g\left(\frac{\partial}{\partial s}, Y\right)\right) - D_Y\left(g\left(\frac{\partial}{\partial s}, X\right)\right) \\
&\quad + g\left(\left[\frac{\partial}{\partial s}, X\right], Y\right) - g\left(\left[\frac{\partial}{\partial s}, Y\right], X\right) - g\left([X,Y], \frac{\partial}{\partial s}\right) \\
&= 0 + D_X(0) - D_Y(0) + g(0,Y) - g(0,X) - g\left(-Z, \frac{\partial}{\partial s}\right) \qquad (2.1.6) \\
&= g\left(Z, \frac{\partial}{\partial s}\right) \\
&= 0
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
g(\nabla_{\frac{\partial}{\partial s}} X, Z) &= D_{\frac{\partial}{\partial s}}(g(X,Z)) + D_X\left(g\left(\frac{\partial}{\partial s}, Z\right)\right) - D_Z\left(g\left(\frac{\partial}{\partial s}, X\right)\right) \\
&\quad + g\left(\left[\frac{\partial}{\partial s}, X\right], Z\right) - g\left(\left[\frac{\partial}{\partial s}, Z\right], X\right) - g\left([X,Z], \frac{\partial}{\partial s}\right) \\
&= 0 + D_X(0) - D_Z(0) + g(0,Z) - g(0,X) - g\left(-Y, \frac{\partial}{\partial s}\right) \qquad (2.1.7) \\
&= g\left(Y, \frac{\partial}{\partial s}\right) \\
&= 0.
\end{aligned}
$$

Now, (2.1.4), (2.1.5), (2.1.6), (2.1.7) all imply that that $\nabla_{\frac{\partial}{\partial s}} X$ is orthogonal to $\frac{\partial}{\partial s}, Y, Z$ and therefore parallel to X. So $\nabla_{\frac{\partial}{\partial s}} X$ must be the same as its projection vector field along $\frac{\partial}{\partial s}$.

In other words, we have

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial s}} X &= \mathrm{proj}_X(\nabla_{\frac{\partial}{\partial s}} X) \\
&= g\left(\nabla_{\frac{\partial}{\partial s}} X, \frac{X}{\varphi}\right)\frac{X}{\varphi} \\
&= \frac{g(\nabla_{\frac{\partial}{\partial s}} X, X)}{\varphi^2} X \\
&= \frac{\varphi\varphi'}{\varphi^2} X \\
&= \frac{\varphi'}{\varphi} X,
\end{aligned} \tag{2.1.8}$$

which is (2.1.1). □

2.2 Covariant derivatives along tangential vector fields

We will obtain expressions of covariant derivatives along the tangential vector fields X, Y, Z in terms of the functions φ, ψ, ξ and the vector fields $\frac{\partial}{\partial s}, X, Y, Z$.

Lemma 2.2.1. *The covariant derivatives associated with a family of* $\mathrm{SO}(3)$*-invariant Riemannian metrics* g *are*

$$\nabla_X X = -\varphi\varphi' \frac{\partial}{\partial s}, \tag{2.2.1}$$

$$\nabla_Y Y = -\psi\psi' \frac{\partial}{\partial s}, \tag{2.2.2}$$

$$\nabla_Z Z = -\xi\xi' \frac{\partial}{\partial s}. \tag{2.2.3}$$

In particular, when we restrict $\nabla_X X, \nabla_Y Y, \nabla_Z Z$ *to* $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$*, we obtain their corresponding intrinsic covariant derivatives*

$$(\nabla_X X)|_{\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}} = 0, \tag{2.2.4}$$

$$(\nabla_Y Y)|_{\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}} = 0, \tag{2.2.5}$$

$$(\nabla_Z Z)|_{\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}} = 0. \tag{2.2.6}$$

Proof. The proofs of (2.2.1), (2.2.2), (2.2.3) are all analogous to each other. Without loss of generality, we choose to only prove (2.2.1). Using the symmetry, metric, and torsion-free properties of the covariant derivative, as well as the first assertion, we obtain

$$\begin{aligned}
g\left(\nabla_X X, \frac{\partial}{\partial s}\right) &= D_X g\left(X, \frac{\partial}{\partial s}\right) - g\left(X, \nabla_X\left(\frac{\partial}{\partial s}\right)\right) \\
&= D_X(0) - g\left(X, \nabla_X\left(\frac{\partial}{\partial s}\right)\right) \\
&= -g\left(X, \nabla_X\left(\frac{\partial}{\partial s}\right)\right) \\
&= -g\left(X, \left[X, \frac{\partial}{\partial s}\right] + \nabla_{\frac{\partial}{\partial s}} X\right) \\
&= -g(X, 0 + \nabla_{\frac{\partial}{\partial s}} X) \\
&= -g(\nabla_{\frac{\partial}{\partial s}} X, X) \\
&= -\varphi\varphi'.
\end{aligned} \tag{2.2.7}$$

which imply that $\nabla_X X$ is orthogonal to X, Y, Z and therefore parallel to $\frac{\partial}{\partial s}$. So $\nabla_X X$ must be the same as its projection vector field along $\frac{\partial}{\partial s}$. In other words, the covariant derivative is

$$\begin{aligned}
\nabla_X X &= \mathrm{proj}_{\frac{\partial}{\partial s}}(\nabla_X X) \\
&= g\left(\nabla_X X, \frac{\partial}{\partial s}\right)\frac{\partial}{\partial s} \\
&= -\varphi\varphi'\frac{\partial}{\partial s} \\
&= -\varphi\varphi'\frac{\partial}{\partial s},
\end{aligned} \tag{2.2.8}$$

which is (2.2.1).

The proofs of (2.2.4), (2.2.5), (2.2.6) are all analogous to each other. Without loss of generality, we choose to only prove (2.2.4). Since the metric g is left invariant on M, we have

$$D_Y\left(g\left(X, \frac{\partial}{\partial s}\right)\right) = 0, \tag{2.2.9}$$

$$D_X\left(g\left(\frac{\partial}{\partial s}, Y\right)\right) = 0, \tag{2.2.10}$$

$$D_{\frac{\partial}{\partial s}}(g(Y, X)) = 0. \tag{2.2.11}$$

Also, by the definition of the Lie bracket, we have

$$\begin{aligned}[X, X] &= XX - XX \\ &= 0.\end{aligned} \tag{2.2.12}$$

Finally, we note the fact that the Lie bracket of two vector fields in the same tangent plane is also a vector field in that tangent plane. This implies in particular that, since X, Y, Z are perpendicular to $\frac{\partial}{\partial s}$, it follows that the Lie brackets $[X, Y]$, $[X, Z]$, $[Y, Z]$ are also perpendicular to $\frac{\partial}{\partial s}$; in other words, we have

$$g\left(\frac{\partial}{\partial s}, [X, Y]\right) = 0, \tag{2.2.13}$$

$$g\left(\frac{\partial}{\partial s}, [X, Z]\right) = 0, \tag{2.2.14}$$

$$g\left(\frac{\partial}{\partial s}, [Y, Z]\right) = 0. \tag{2.2.15}$$

So we apply (2.0.3) in order to obtain

$$
\begin{aligned}
& g\left(\nabla_X Y, \frac{\partial}{\partial s}\right) \\
& \quad = \frac{1}{2} g\left(D_Y g\left(X, \frac{\partial}{\partial s}\right) + D_X g\left(\frac{\partial}{\partial s}, Y\right) - D_{\frac{\partial}{\partial s}} g(Y, X)\right. \\
& \qquad\qquad \left. - g\left([Y, X], \frac{\partial}{\partial s}\right) - g\left(\left[X, \frac{\partial}{\partial s}\right], Y\right) + g\left(\left[\frac{\partial}{\partial s}, Y\right], X\right)\right) \\
& \quad = \frac{1}{2}\left(D_X(0) + D_X(0) - D_{\frac{\partial}{\partial s}}(0) - g\left(0, \frac{\partial}{\partial s}\right) - g\,(0, X) + g\,(0, X)\right) \\
& \quad = \frac{1}{2}(0 - 0 - 0 - 0 - 0 + 0) \\
& \quad = 0.
\end{aligned}
\tag{2.2.16}
$$

Using the metric and torsion-free properties of the covariant derivative, we also obtain

$$
\begin{aligned}
g(\nabla_X Y, X) &= g([X, Y] + \nabla_Y X, X) \\
&= g(-Z + \nabla_Y X, X) \\
&= -g(Z, X) + g(\nabla_Y X, X) \\
&= -0 + g(\nabla_Y X, X) \\
&= g(\nabla_Y X, X) \\
&= g(\nabla_Y X, X) \\
&= \frac{1}{2}(g(\nabla_Y X, X) + g(X, \nabla_Y X)) \\
&= \frac{1}{2} D_Y g(X, X) \\
&= \frac{1}{2} D_Y(|X|^2) \\
&= \frac{1}{2} D_Y(\varphi^2) \\
&= \frac{1}{2} \cdot 0 \\
&= 0
\end{aligned}
\tag{2.2.17}
$$

and

$$\begin{aligned} g(\nabla_X Y, Y) &= \frac{1}{2}(g(\nabla_X Y, Y) + g(Y, \nabla_X Y)) \\ &= \frac{1}{2} D_X g(Y, Y) \\ &= \frac{1}{2} D_X(\psi^2) \\ &= \frac{1}{2} \cdot 0 \\ &= 0. \end{aligned} \tag{2.2.18}$$

Furthermore, by (2.2.17) we have

$$\begin{aligned} 0 &= g(\nabla_X Y, X) \\ &= D_X g(Y, X) - g(Y, \nabla_X X) \\ &= 0 - g(Y, \nabla_X X) \\ &= -g(Y, \nabla_X X) \end{aligned} \tag{2.2.19}$$

and

$$\begin{aligned} 0 &= \frac{1}{2} D_X g(X, X) \\ &= \frac{1}{2}(g(\nabla_X X, X) + g(X, \nabla_X X)) \\ &= \frac{1}{2}(g(\nabla_X X, X) + g(\nabla_X X, X) \\ &= g(\nabla_X X, X). \end{aligned} \tag{2.2.20}$$

Since X is nonzero, we conclude (2.2.4). □

Up to this point, we have not required the assumption given by (2.0.5). We will now apply this assumption for the first time to our next lemma in order to further simplify our expressions.

Lemma 2.2.2. *If φ, ψ, ξ satisfy* (2.0.5)*, then the covariant derivatives associated with the* SO(3)*-invariant Riemannian metric g are*

$$\nabla_X Y = -\frac{\psi}{\xi} Z, \tag{2.2.21}$$

$$\nabla_Y Z = \frac{\xi}{\varphi} X, \tag{2.2.22}$$

$$\nabla_Z X = -\frac{\varphi}{\psi} Y, \tag{2.2.23}$$

$$\nabla_Y X = -\frac{\varphi}{\xi} Z, \tag{2.2.24}$$

$$\nabla_Z Y = \frac{\psi}{\varphi} X, \tag{2.2.25}$$

$$\nabla_X Z = \frac{\xi}{\psi} Y. \tag{2.2.26}$$

Proof. Using the Koszul formula again,

$$\begin{aligned}
g(\nabla_X Y, Z) &= \frac{1}{2}(D_Y g(X,Z) + D_X g(Z,Y) - D_Z g(Y,X) \\
&\qquad - g([Y,X],Z) - g([X,Z],Y) + g([Z,Y],X)) \\
&= \frac{1}{2}(0 + 0 - 0 - g([Y,X],Z) - g([X,Z],Y) + g([Z,Y],X)) \\
&= \frac{1}{2}(-g(Z,Z) - g(Y,Y) + g(X,X)) \\
&= \frac{1}{2}(-|Z|^2 - |Y|^2 + |X|^2) \\
&= \frac{1}{2}(-\xi^2 - \psi^2 + \varphi^2) \\
&= \frac{\varphi^2 - \psi^2 - \xi^2}{2} \\
&= \frac{\varphi^2 - (\varphi + \xi)^2 - \xi^2}{2} \\
&= \frac{\varphi^2 - (\varphi^2 + 2\varphi\xi + \xi^2) - \xi^2}{2} \\
&= \frac{-2\varphi\xi - 2\xi^2}{2} \\
&= -\frac{2\xi(\varphi + \xi)}{2} \\
&= -\frac{2\xi\psi}{2} \\
&= -\psi\xi
\end{aligned} \tag{2.2.27}$$

and

$$
\begin{aligned}
g(\nabla_Y Z, X) &= \frac{1}{2}(D_Z g(Y,X) + D_Y g(X,Z) - D_X g(Z,Y) \\
&\qquad - g([Z,Y],X) - g([Y,X],Z) + g([X,Z],Y)) \\
&= \frac{1}{2}(0 + 0 - 0 - g([Z,Y],X) - g([Y,X],Z) + g([X,Z],Y)) \\
&= \frac{1}{2}(-g(X,X) - g(Z,Z) + g(Y,Y)) \\
&= \frac{1}{2}(-|X|^2 - |Z|^2 + |Y|^2) \\
&= \frac{\psi^2 - \xi^2 - \varphi^2}{2} \\
&= \frac{(\varphi + \xi)^2 - \xi^2 - \varphi^2}{2} \\
&= \frac{(\varphi^2 + 2\varphi\xi + \xi^2) - \xi^2 - \varphi^2}{2} \\
&= \frac{2\varphi\xi}{2} \\
&= \varphi\xi
\end{aligned}
\tag{2.2.28}
$$

and

$$
\begin{aligned}
g(\nabla_Z X, Y) &= \frac{1}{2}(D_X g(Z,Y) + D_Z g(Y,X) - D_Y g(X,Z) \\
&\qquad - g([X,Z],Y) - g([Z,Y],X) + g([Y,X],Z)) \\
&= \frac{1}{2}(0 + 0 - 0 - g([X,Z],Y) - g([Z,Y],X) + g([Y,X],Z)) \\
&= \frac{1}{2}(-g(Y,Y) - g(X,X) + g(Z,Z)) \\
&= \frac{1}{2}(-|Y|^2 - |X|^2 + |Z|^2) \\
&= \frac{\xi^2 - \varphi^2 - \psi^2}{2} \\
&= \frac{(\xi+\varphi)(\xi-\varphi) - \psi^2}{2} \\
&= \frac{\psi(\xi-\varphi) - \psi^2}{2} \\
&= \frac{\psi((\xi-\varphi) - \psi)}{2} \\
&= \frac{\psi((\xi-\varphi) - (\varphi+\xi))}{2} \\
&= \frac{\psi(\xi - \varphi - \varphi - \xi)}{2} \\
&= \frac{-2\varphi\psi}{2} \\
&= -\varphi\psi.
\end{aligned} \tag{2.2.29}
$$

So we conclude that the covariant derivatives are

$$
\begin{aligned}
\nabla_X Y &= \operatorname{proj}_Z(\nabla_X Y) \\
&= g\left(\nabla_X Y, \frac{Z}{\xi}\right)\frac{Z}{\xi} \\
&= \frac{g(\nabla_X Y, Z)}{\xi^2} Z \\
&= \frac{-\psi\xi}{\xi^2} Z \\
&= -\frac{\psi}{\xi} Z
\end{aligned} \tag{2.2.30}
$$

and

$$
\begin{aligned}
\nabla_Y Z &= \mathrm{proj}_X(\nabla_Y Z) \\
&= g\left(\nabla_Y Z, \frac{X}{\varphi}\right)\frac{X}{\varphi} \\
&= \frac{g(\nabla_Y Z, X)}{\varphi^2}X \\
&= \frac{\varphi\xi}{\varphi^2}X \\
&= \frac{\xi}{\varphi}X
\end{aligned}
\tag{2.2.31}
$$

and

$$
\begin{aligned}
\nabla_Z X &= \mathrm{proj}_Y(\nabla_Z X) \\
&= g\left(\nabla_Z X, \frac{Y}{\psi}\right)\frac{Y}{\psi} \\
&= \frac{g(\nabla_Z X, Y)}{\psi^2}Y \\
&= \frac{-\varphi\psi}{\psi^2}Y \\
&= -\frac{\varphi}{\psi}Y,
\end{aligned}
\tag{2.2.32}
$$

which are (2.2.21), (2.2.22), (2.2.23), respectively.

We also obtain the remaining covariant derivatives

$$
\begin{aligned}
\nabla_Y X &= \nabla_X Y + [Y, X] \\
&= -\frac{\psi}{\xi} Z + Z \\
&= \left(-\frac{\psi}{\xi} + 1 \right) Z \\
&= \left(-\frac{\psi}{\xi} + \frac{\xi}{\xi} \right) Z \\
&= \frac{-\psi + \xi}{\xi} Z \\
&= \frac{-(\varphi + \xi) + \xi}{\xi} Z \\
&= -\frac{\varphi}{\xi} Z
\end{aligned}
\tag{2.2.33}
$$

and

$$
\begin{aligned}
\nabla_Z Y &= \nabla_Y Z + [Z, Y] \\
&= \frac{\xi}{\varphi} X + X \\
&= \left(\frac{\xi}{\varphi} + 1 \right) X \\
&= \left(\frac{\xi}{\varphi} + \frac{\varphi}{\varphi} \right) X \\
&= \frac{\varphi + \xi}{\varphi} X \\
&= \frac{\psi}{\varphi} X
\end{aligned}
\tag{2.2.34}
$$

and

$$\begin{aligned}\nabla_X Z &= \nabla_Z X + [X, Z] \\ &= -\frac{\varphi}{\psi}Y + Y \\ &= \left(-\frac{\varphi}{\psi} + 1\right)Y \\ &= \left(-\frac{\varphi}{\psi} + \frac{\psi}{\psi}\right)Y \\ &= \frac{-\varphi + \psi}{\psi}Y \\ &= \frac{-\varphi + (\varphi + \xi)}{\psi}Y \\ &= \frac{\xi}{\psi}Y,\end{aligned} \tag{2.2.35}$$

which are (2.2.24), (2.2.25), (2.2.26), respectively. □

2.3 Second fundamental form on a hypersurface

In this section, will list one result that makes use of the second fundamental form of the orbits $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)} \subseteq M$ for all $s \in [0, L]$.

Corollary 2.3.1. *The* SO(3)*-invariant Riemannian metrics g satisfies the second fundamental forms*

$$\mathrm{II}(X, X) = \varphi\varphi', \tag{2.3.1}$$

$$\mathrm{II}(Y, Y) = \psi\psi', \tag{2.3.2}$$

$$\mathrm{II}(Z, Z) = \xi\xi', \tag{2.3.3}$$

$$\mathrm{II}(X, Y) = 0, \tag{2.3.4}$$

$$\mathrm{II}(X, Z) = 0, \tag{2.3.5}$$

$$\mathrm{II}(Y, Z) = 0. \tag{2.3.6}$$

Proof. The proofs of (2.3.1), (2.3.2), (2.3.3) are all analogous to each other. Without loss of generality, we choose to only prove (2.3.1). We note

$$\begin{aligned}\nabla_X\left(\frac{\partial}{\partial s}\right) &= \left[X, \frac{\partial}{\partial s}\right] + \nabla_{\frac{\partial}{\partial s}} X \\ &= -\left[\frac{\partial}{\partial s}, X\right] + \nabla_{\frac{\partial}{\partial s}} X \\ &= -0 + \nabla_{\frac{\partial}{\partial s}} X \\ &= \nabla_{\frac{\partial}{\partial s}} X.\end{aligned} \tag{2.3.7}$$

So we have

$$
\begin{aligned}
\mathrm{II}(X,X) &= g\left(\nabla_X\left(\frac{\partial}{\partial s}\right), X\right) \\
&= g(\nabla_{\frac{\partial}{\partial s}} X, X) \\
&= g\left(\frac{\varphi'}{\varphi} X, X\right) \\
&= \frac{\varphi'}{\varphi} g(X,X) \\
&= \frac{\varphi'}{\varphi} |X|^2 \\
&= \frac{\varphi'}{\varphi} \varphi^2 \\
&= \varphi\varphi',
\end{aligned}
\tag{2.3.8}
$$

which is (2.3.1).

The proofs of (2.3.4), (2.3.5), (2.3.6) are all analogous to each other. Without loss of generality, we choose to only prove (2.3.4). We have

$$
\begin{aligned}
\mathrm{II}(X,Y) &= g(\nabla_{\frac{\partial}{\partial s}} X, Y) \\
&= g\left(\frac{\varphi'}{\varphi} X, Y\right) \\
&= \frac{\varphi'}{\varphi} g(X,Y) \\
&= \frac{\varphi'}{\varphi} \cdot 0 \\
&= 0,
\end{aligned}
\tag{2.3.9}
$$

which is (2.3.4). □

Chapter 3

Applications of Riemann and Ricci curvatures

Let M be any four-dimensional Riemannian manifold equipped with the Riemannian metric g, and let ∇ be the Riemannian connection. We review the usual tensors from the literature of Riemannian geometry that makes use of this connection. Throughout this chapter, we let W, X, Y, Z be vector fields that are tangent to M.

Definition 3.0.1. *The* Riemannian curvature tensor *is the* $(1, 3)$*-tensor defined by*

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{3.0.1}$$

where we recall that $[X, Y]$ *is the Lie bracket of* X *and* Y *and is defined by* (1.2.2). *The metric* g *allows us to change the* $(1, 3)$*-tensor* R *into the* $(0, 4)$*-tensor given by*

$$R(X,Y,Z,W) := g(R(X,Y)Z, W). \tag{3.0.2}$$

According to Proposition 3.1.1 of [17], the Riemannian curvature tensor $R(X, Y, Z, W)$

satisfies the following properties:

(1) R is skew-symmetric in the first two and last two entries:

$$\begin{aligned} R(X,Y,Z,W) &= -R(Y,X,Z,W) \\ &= R(Y,X,W,Z). \end{aligned} \tag{3.0.3}$$

(2) R is symmetric between the first two and last two entries:

$$R(X,Y,Z,W) = R(Z,W,X,Y). \tag{3.0.4}$$

(3) R satisfies a cyclic permutation property called the *first Bianchi identity*:

$$R(X,Y,Z,W) + R(Z,X,Y,W) + R(Y,Z,X,W) = 0. \tag{3.0.5}$$

Furthermore, if X, Y, Z are Killing fields on M, and if $\frac{\partial}{\partial s}$ denotes the radial field on M, then we have the Jacobi equation

$$\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial s}} X + R\left(X, \frac{\partial}{\partial s}\right)\left(\frac{\partial}{\partial s}\right) = 0. \tag{3.0.6}$$

We can use the Jacobi equation here because any Killing field on a Riemannian manifold restricted to a geodesic is also Jacobi field.

The following biquadratic forms are immediate consequences of the Riemannian curvature tensor.

Definition 3.0.2. *The* unnormalized sectional curvature *of the tangent plane* $\mathrm{span}(X,Y)$ *is*

$$\mathrm{curv}(X,Y) := R(X,Y,Y,X). \tag{3.0.7}$$

The normalized sectional curvature*, or* sectional curvature*, of* $\mathrm{span}(X, Y)$ *is*

$$\begin{aligned}\sec(X,Y) &:= \frac{\mathrm{curv}(X,Y)}{|X|^2|Y|^2 - g(X,Y)^2} \\ &= \frac{R(X,Y,Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}.\end{aligned} \tag{3.0.8}$$

where $|X|$ denotes the length of the Killing field and is defined by (1.2.1).

The last tensor we will need here will be central to the topic of Ricci flow, which we will discuss in the context of our dissertation problem in Chapter 4.

Definition 3.0.3. *For any point $p \in M$, let T_pM be the tangent space to M, and let $\{e_1, e_2, e_3, e_4\} \subseteq T_pM$ be an orthonormal basis. Then the* Ricci curvature *is a contraction of R defined by*

$$\begin{aligned}\mathrm{Ric}(X,Y) &:= \sum_{i=1}^{4} R(e_i, X, Y, e_i) \\ &= \sum_{i=1}^{4} g(R(e_i, X)Y, e_i).\end{aligned} \tag{3.0.9}$$

We remark that, for the metric g on S^4 that takes the form given by (1.0.8), our choice of an orthonormal basis of T_pS^4 for any $p \in S^4$ is $\{\frac{\partial}{\partial s}, \frac{X}{\varphi}, \frac{Y}{\psi}, \frac{Z}{\xi}\} \subseteq T_pS^4$.

Finally, we will also need to make use of the Normal Curvature Equation

$$R(X,Y,Z,N) = -(\nabla_X \mathrm{II})(Y,Z) + (\nabla_Y \mathrm{II})(X,Z), \tag{3.0.10}$$

which is also printed in, for instance, Theorem 3.2.5 of [17].

For further details on this exposition, the interested reader can consult, for instance, Chapter 3 of [17]. The remainder of this chapter will focus on our results that make use of the Riemannian curvature tensor. Wherever possible, we will continue to apply (2.0.5) into our expressions in order to simplify them as much as we can. As we will see later on in this chapter, (2.0.5) also ensures that the normal Riemannian curvatures of S^4 with the metric we will construct are zero.

Finally, throughout Chapter 3, we assume that any metric g on S^4 is SO(3)-invariant. In particular, as we have previously stated in Chapter 1, any metric g takes the form given by (1.0.8), which allows us to express any curvature term as either zero or some expression using the functions φ, ψ, ξ.

3.1 Riemannian curvatures of tangent planes generated by basis vector fields

This section consists of a series of propositions that assert all the tangential, radial, and mixed curvature terms for (S^4, g).

Here, we will work with unnormalized radial and tangential sectional curvature terms, as opposed to normalized ones, and refer them to just “radial sectional curvatures” and “tangential sectional curvatures” for brevity.

Proposition 3.1.1. *The radial sectional curvatures of* (S^4, g) *are*

$$R\left(\frac{\partial}{\partial s}, X, X, \frac{\partial}{\partial s}\right) = -\varphi\varphi'', \tag{3.1.1}$$

$$R\left(\frac{\partial}{\partial s}, Y, Y, \frac{\partial}{\partial s}\right) = -\psi\psi'', \tag{3.1.2}$$

$$R\left(\frac{\partial}{\partial s}, Z, Z, \frac{\partial}{\partial s}\right) = -\xi\xi''. \tag{3.1.3}$$

Proof. The proofs of (3.1.1), (3.1.2), (3.1.3) are all analogous to each other. Without loss of generality, we choose to only prove (3.1.1). Using the first covariant derivative (2.1.1), we obtain

$$
\begin{aligned}
\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial s}}X &= \nabla_{\frac{\partial}{\partial s}}\left(\frac{\varphi'}{\varphi}X\right) \\
&= \nabla_{\frac{\partial}{\partial s}}\left(\frac{\varphi'}{\varphi}\right)X + \frac{\varphi'}{\varphi}\nabla_{\frac{\partial}{\partial s}}X \\
&= \nabla_{\frac{\partial}{\partial s}}\left(\frac{\varphi'}{\varphi}\right)X + \frac{\varphi'}{\varphi}\nabla_{\frac{\partial}{\partial s}}X \\
&= \left(\frac{\varphi'}{\varphi}\right)' X + \frac{\varphi'}{\varphi}\left(\frac{\varphi'}{\varphi}X\right) \\
&= \frac{\varphi''\varphi - (\varphi')^2}{\varphi^2}X + \frac{(\varphi')^2}{\varphi^2}X \qquad (3.1.4) \\
&= \left(\frac{\varphi''\varphi - (\varphi')^2}{\varphi^2} + \frac{(\varphi')^2}{\varphi^2}\right)X \\
&= \left(\frac{\varphi''\varphi - (\varphi')^2 + (\varphi')^2}{\varphi^2}\right)X \\
&= \frac{\varphi''\varphi}{\varphi^2}X \\
&= \frac{\varphi''}{\varphi}X.
\end{aligned}
$$

By (3.0.6) with $i = 1$ and (3.1.4), we obtain

$$
\begin{aligned}
R\left(\frac{\partial}{\partial s}, X, X, \frac{\partial}{\partial s}\right) &= R\left(X, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}, X\right) \\
&= g\left(R\left(X, \frac{\partial}{\partial s}\right)\left(\frac{\partial}{\partial s}\right), X\right) \\
&= g(-\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial s}}X, X) \\
&= g\left(-\frac{\varphi''}{\varphi}X, X\right) \\
&= -\frac{\varphi''}{\varphi}g(X, X) \\
&= -\frac{\varphi''}{\varphi}|X|^2 \\
&= -\frac{\varphi''}{\varphi}\varphi^2 \\
&= -\varphi\varphi'',
\end{aligned}
\tag{3.1.5}
$$

which is (3.1.1). □

Proposition 3.1.2. *The mixed radial sectional curvatures of* (S^4, g) *are*

$$R\left(\frac{\partial}{\partial s}, X, Y, \frac{\partial}{\partial s}\right) = 0, \tag{3.1.6}$$

$$R\left(\frac{\partial}{\partial s}, X, Z, \frac{\partial}{\partial s}\right) = 0, \tag{3.1.7}$$

$$R\left(\frac{\partial}{\partial s}, Y, X, \frac{\partial}{\partial s}\right) = 0, \tag{3.1.8}$$

$$R\left(\frac{\partial}{\partial s}, Y, Z, \frac{\partial}{\partial s}\right) = 0, \tag{3.1.9}$$

$$R\left(\frac{\partial}{\partial s}, Z, X, \frac{\partial}{\partial s}\right) = 0, \tag{3.1.10}$$

$$R\left(\frac{\partial}{\partial s}, Z, Y, \frac{\partial}{\partial s}\right) = 0. \tag{3.1.11}$$

Proof. The proofs of (3.1.6), (3.1.7), (3.1.8), (3.1.9), (3.1.10), (3.1.11) are all analogous to each other. Without loss of generality, we choose to only prove (3.1.6). We have

$$R\left(\frac{\partial}{\partial s}, X, Y, \frac{\partial}{\partial s}\right) = 0 \tag{3.1.12}$$

for all distinct indices $i, j = 1, 2, 3$. We can use (3.0.6) with $i = 1$ to obtain

$$\begin{aligned} R\left(\frac{\partial}{\partial s}, X, Y, \frac{\partial}{\partial s}\right) &= R\left(X, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}, Y\right) \\ &= g\left(R\left(X, \frac{\partial}{\partial s}\right)\left(\frac{\partial}{\partial s}\right), Y\right) \\ &= g\left(-\frac{\varphi''}{\varphi} X, Y\right) \\ &= -\frac{\varphi''}{\varphi} g(X, Y) \\ &= -\frac{\varphi''}{\varphi} \cdot 0 \\ &= 0, \end{aligned} \tag{3.1.13}$$

which is (3.1.6). ◻

Proposition 3.1.3. *The tangential sectional curvatures of* (S^4, g) *that satisfies* (2.0.5) *are*

$$R(X, Y, Y, X) = \varphi\psi(2 - \varphi'\psi'), \tag{3.1.14}$$

$$R(X, Z, Z, X) = -\varphi\xi(2 + \varphi'\xi'), \tag{3.1.15}$$

$$R(Y, Z, Z, Y) = \psi\xi(2 - \psi'\xi') \tag{3.1.16}$$

for all distinct indices $i, j, k = 1, 2, 3$.

Proof. First, we will prove (3.1.14). We obtain

$$\begin{aligned} R(X,Y,Y,X) &= g(R(X,Y)Y,X) \\ &= g(\nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y, X) \\ &= -g(\nabla_Y \nabla_X Y, X) - g(\nabla_{[X,Y]} Y, X) + g(\nabla_X \nabla_Y Y, X). \end{aligned} \tag{3.1.17}$$

We will work with each of the terms in the final expression of (3.1.17) separately. We have the intrinsic curvature terms

$$\begin{aligned} -g(\nabla_Y \nabla_X Y, X) &= -(D_Y(\nabla_X g(Y,X) - g(\nabla_Y X, \nabla_X Y)) \\ &= -(0 - g(\nabla_Y X, \nabla_X Y)) \\ &= g(\nabla_Y X, \nabla_X Y) \\ &= g\left(-\frac{\varphi}{\xi} Z, -\frac{\psi}{\xi} Z\right) \\ &= \left(-\frac{\varphi}{\xi}\right)\left(-\frac{\psi}{\xi}\right) g(Z,Z) \\ &= \frac{\varphi\psi}{\xi^2} |Z|^2 \\ &= \frac{\varphi\psi}{\xi^2} \xi^2 \\ &= \varphi\psi \end{aligned} \tag{3.1.18}$$

and

$$\begin{aligned}
-g(\nabla_{[X,Y]}Y, X) &= -g(\nabla_{-Z}Y, X) \\
&= -g(-\nabla_Z Y, X) \\
&= g(\nabla_Z Y, X) \\
&= g\left(\frac{\psi}{\varphi}X, X\right) \\
&= \frac{\psi}{\varphi}g(X, X) \\
&= \frac{\psi}{\varphi}|X|^2 \\
&= \frac{\psi}{\varphi}\varphi^2 \\
&= \varphi\psi,
\end{aligned} \tag{3.1.19}$$

as well as the Gaussian curvature term

$$\begin{aligned}
g(\nabla_X \nabla_Y Y, X) &= D_X g(\nabla_Y Y, X) - g(\nabla_Y Y, \nabla_X X) \\
&= 0 - g(\nabla_Y Y, \nabla_X X) \\
&= -g(\nabla_Y Y, \nabla_X X) \\
&= -g\left(-\psi\psi'\frac{\partial}{\partial s}, -\varphi\varphi'\frac{\partial}{\partial s}\right) \\
&= -\varphi\varphi'\psi\psi' g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) \\
&= -\varphi\varphi'\psi\psi'\left|\frac{\partial}{\partial s}\right|^2 \\
&= -\varphi\varphi'\psi\psi' \cdot 1^2 \\
&= -\varphi\varphi'\psi\psi'.
\end{aligned} \tag{3.1.20}$$

So our final expression is

$$
\begin{aligned}
R(X,Y,Y,X) &= -g(\nabla_Y \nabla_X Y, X) - g(\nabla_{[X,Y]} Y, X) + g(\nabla_X \nabla_Y Y, X) \\
&= \varphi\psi + \varphi\psi - \varphi\varphi'\psi\psi' \\
&= 2\varphi\psi - \varphi\varphi'\psi\psi' \\
&= \varphi\psi(2 - \varphi'\psi'),
\end{aligned}
\tag{3.1.21}
$$

which is (3.1.14).

Next, we will prove (3.1.15). We obtain

$$
\begin{aligned}
R(X,Z,Z,X) &= g(R(X,Z)Z, X) \\
&= g(\nabla_X \nabla_Z Z - \nabla_Z \nabla_X Z - \nabla_{[X,Z]} Z, X) \\
&= -g(\nabla_Z \nabla_X Z, X) - g(\nabla_{[X,Z]} Z, X) + g(\nabla_X \nabla_Z Z, X).
\end{aligned}
\tag{3.1.22}
$$

We will work with each of the terms in the final expression of (3.1.22) separately. We have the intrinsic curvature terms

$$
\begin{aligned}
-g(\nabla_Z \nabla_X Z, X) &= -(D_Z(\nabla_X g(Z,X) - g(\nabla_Z X, \nabla_X Z)) \\
&= -(0 - g(\nabla_Z X, \nabla_X Z)) \\
&= g(\nabla_Z X, \nabla_X Z) \\
&= g\left(-\frac{\varphi}{\psi} Y, \frac{\xi}{\psi} Y\right) \\
&= \left(-\frac{\varphi}{\psi}\right)\left(\frac{\xi}{\psi}\right) g(Y,Y) \\
&= -\frac{\varphi\xi}{\psi^2}\psi^2 \\
&= -\varphi\xi
\end{aligned}
\tag{3.1.23}
$$

and

$$
\begin{aligned}
-g(\nabla_{[X,Z]}Z, X) &= -g(\nabla_Y Z, X) \\
&= -g(\nabla_Y Z, X) \\
&= -g\left(\frac{\xi}{\varphi}X, X\right) \\
&= -\frac{\xi}{\varphi}g(X, X) \\
&= -\frac{\xi}{\varphi}|X|^2 \\
&= -\frac{\xi}{\varphi}\varphi^2 \\
&= -\varphi\xi,
\end{aligned}
\tag{3.1.24}
$$

as well as the Gaussian curvature term

$$
\begin{aligned}
g(\nabla_X \nabla_Z Z, X) &= D_X g(\nabla_Z Z, X) - g(\nabla_Z Z, \nabla_X X) \\
&= 0 - g(\nabla_Z Z, \nabla_X X) \\
&= -g(\nabla_Z Z, \nabla_X X) \\
&= -g\left(-\xi\xi' \frac{\partial}{\partial s}, -\varphi\varphi' \frac{\partial}{\partial s}\right) \\
&= -\varphi\varphi'\xi\xi' g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) \\
&= -\varphi\varphi'\xi\xi' \left|\frac{\partial}{\partial s}\right|^2 \\
&= -\varphi\varphi'\xi\xi' \cdot 1^2 \\
&= -\varphi\varphi'\xi\xi'.
\end{aligned}
\tag{3.1.25}
$$

So our final expression is

$$
\begin{aligned}
R(X,Z,Z,X) &= -g(\nabla_Y\nabla_X Y, X) - g(\nabla_{[X,Y]}Y, X) + g(\nabla_X\nabla_Y Y, X) \\
&= -\varphi\xi - \varphi\xi - \varphi\varphi'\xi\xi' \\
&= -2\varphi\xi - \varphi\varphi'\xi\xi' \\
&= -\varphi\xi(2+\varphi'\xi'),
\end{aligned}
\tag{3.1.26}
$$

which is (3.1.15).

Next, we will prove (3.1.16). We obtain

$$
\begin{aligned}
R(Y,Z,Z,Y) &= g(R(Y,Z)Z,Y) \\
&= g(\nabla_Y\nabla_Z Z - \nabla_Z\nabla_Y Z - \nabla_{[Y,Z]}Z, Y) \\
&= -g(\nabla_Z\nabla_Y Z, Y) - g(\nabla_{[Y,Z]}Z, Y) + g(\nabla_Y\nabla_Z Z, Y).
\end{aligned}
\tag{3.1.27}
$$

We will work with each of the terms in the final expression of (3.1.27) separately. We have the intrinsic curvature terms

$$
\begin{aligned}
-g(\nabla_Z\nabla_Y Z, Y) &= -(D_Z(\nabla_Y g(Z,Y) - g(\nabla_Z Y, \nabla_Y Z)) \\
&= -(0 - g(\nabla_Z Y, \nabla_Y Z)) \\
&= g(\nabla_Z Y, \nabla_Y Z) \\
&= g\left(\frac{\psi}{\varphi}X, \frac{\xi}{\varphi}X\right) \\
&= \frac{\psi}{\varphi}\frac{\xi}{\varphi}g(X,X) \\
&= \frac{\psi\xi}{\varphi^2}|X|^2 \\
&= \frac{\psi\xi}{\varphi^2}\varphi^2 \\
&= \psi\xi
\end{aligned}
\tag{3.1.28}
$$

and

$$
\begin{aligned}
-g(\nabla_{[Y,Z]}Z, Y) &= -g(\nabla_Y Z, Y) \\
&= -g(\nabla_{-X} Z, Y) \\
&= -g(-\nabla_X Z, Y) \\
&= g(\nabla_X Z, Y) \\
&= g\left(\frac{\xi}{\psi}Y, Y\right) \\
&= \frac{\xi}{\psi} g(Y, Y) \\
&= \frac{\xi}{\psi}|Y|^2 \\
&= \frac{\xi}{\psi}\psi^2 \\
&= \psi\xi,
\end{aligned}
\tag{3.1.29}
$$

as well as the Gaussian curvature term

$$
\begin{aligned}
g(\nabla_Y \nabla_Z Z, Y) &= D_Y g(\nabla_Z Z, Y) - g(\nabla_Z Z, \nabla_Y Y) \\
&= 0 - g(\nabla_Z Z, \nabla_Y Y) \\
&= -g(\nabla_Z Z, \nabla_Y Y) \\
&= -g\left(-\xi\xi'\frac{\partial}{\partial s}, -\psi\psi'\frac{\partial}{\partial s}\right) \\
&= -\psi\psi'\xi\xi' g\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) \\
&= -\psi\psi'\xi\xi'\left|\frac{\partial}{\partial s}\right|^2 \\
&= -\psi\psi'\xi\xi' \cdot 1^2 \\
&= -\psi\psi'\xi\xi'.
\end{aligned}
\tag{3.1.30}
$$

So our final expression is

$$\begin{aligned}
R(Y,Z,Z,Y) &= -g(\nabla_Z \nabla_Y Z, Y) - g(\nabla_{[Y,Z]} Z, Y) + g(\nabla_Y \nabla_Z Z, Y) \\
&= \psi\xi + \psi\xi - \psi\psi'\xi\xi' \\
&= 2\psi\xi - \psi\psi'\xi\xi' \\
&= \psi\xi(2 - \psi'\xi'),
\end{aligned} \tag{3.1.31}$$

which is (3.1.16). □

Proposition 3.1.4. *If g satisfies* (2.0.5)*, then the mixed tangential sectional curvatures of* (S^4, g) *are*

$$R(X,Y,Z,W) = 0, \tag{3.1.32}$$

$$R(Y,Z,X,W) = 0, \tag{3.1.33}$$

$$R(Z,X,Y,W) = 0, \tag{3.1.34}$$

$$R(X,Z,Y,W) = 0, \tag{3.1.35}$$

$$R(Y,X,Z,W) = 0, \tag{3.1.36}$$

$$R(Z,Y,X,W) = 0. \tag{3.1.37}$$

Proof. The proofs of (3.1.32), (3.1.33), (3.1.34) are all analogous to each other. Without loss of generality, we choose to only prove (3.1.32). We have the $(1,3)$-tensor

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \nabla_X \left(\frac{\xi}{\varphi} X \right) - \nabla_Y \left(\frac{\xi}{\psi} Y \right) - \nabla_{-Z} Z \\ &= \frac{\xi}{\varphi} \nabla_X X - \frac{\xi}{\psi} \nabla_Y Y + \nabla_Z Z \\ &= \frac{\xi}{\varphi} \left(-\varphi\varphi' \frac{\partial}{\partial s} \right) - \frac{\xi}{\psi} \left(-\psi\psi' \frac{\partial}{\partial s} \right) - \xi\xi' \frac{\partial}{\partial s} \\ &= -\varphi'\xi \frac{\partial}{\partial s} + \psi'\xi \frac{\partial}{\partial s} - \xi\xi' \frac{\partial}{\partial s} \\ &= (-\varphi' + \psi')\xi \frac{\partial}{\partial s} - \xi\xi' \frac{\partial}{\partial s} \\ &= (-\varphi' + (\varphi + \xi)')\xi \frac{\partial}{\partial s} - \xi\xi' \frac{\partial}{\partial s} \\ &= (-\varphi' + (\varphi' + \xi'))\xi \frac{\partial}{\partial s} - \xi\xi' \frac{\partial}{\partial s} \\ &= \xi'\xi \frac{\partial}{\partial s} - \xi\xi' \frac{\partial}{\partial s} \\ &= (\xi'\xi - \xi\xi') \frac{\partial}{\partial s} \\ &= 0 \frac{\partial}{\partial s} \\ &= 0. \end{aligned} \tag{3.1.38}$$

So we conclude

$$\begin{aligned} R(X,Y,Z,W) &= g(R(X,Y)Z, W) \\ &= g(0, W) \\ &= 0, \end{aligned} \tag{3.1.39}$$

which is (3.1.32).

The proofs of (3.1.35), (3.1.36), (3.1.37) are all analogous to each other. Without loss of generality, we choose to only prove (3.1.35). We can invoke the first Bianchi identity, which we have written as (3.0.5), in order to write a shorter proof using only (3.1.32) and (3.1.33). Indeed, by applying (3.0.5), we obtain

$$\begin{aligned} R(Z,X,Y,W) &= (R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W)) \\ &\qquad - R(X,Y,Z,W) - R(Y,Z,X,W) \\ &= (0) - 0 - 0 \\ &= 0, \end{aligned} \tag{3.1.40}$$

which is (3.1.35). □

We will also see that (2.0.5) is not only good for simplifying many expressions but also it leads to zero normal sectional curvature.

Proposition 3.1.5. *If g satisfies* (2.0.5)*, then the mixed radial and tangential sectional curvatures of* (S^4, g) *are*

$$R\left(X,Y,X,\frac{\partial}{\partial s}\right) = 0, \tag{3.1.41}$$

$$R\left(Y,Z,Y,\frac{\partial}{\partial s}\right) = 0, \tag{3.1.42}$$

$$R\left(Z,X,Z,\frac{\partial}{\partial s}\right) = 0, \tag{3.1.43}$$

$$R\left(X,Y,Z,\frac{\partial}{\partial s}\right) = 0, \tag{3.1.44}$$

$$R\left(Y,Z,X,\frac{\partial}{\partial s}\right) = 0, \tag{3.1.45}$$

$$R\left(Z,X,Y,\frac{\partial}{\partial s}\right) = 0. \tag{3.1.46}$$

Proof. The proofs of (3.1.41), (3.1.42), (3.1.43) are all analogous to each other. Without loss of generality, we choose to only prove (3.1.41). Since we assume (2.0.5), the expressions given by (2.2.21) for $\nabla_X Y$ and by (2.2.24) for $\nabla_Y X$ apply here. We have

$$
\begin{aligned}
-(\nabla_X \,\mathrm{II})(Y, X) &= -(\nabla_X(\mathrm{II}(Y, X)) - \mathrm{II}(\nabla_X Y, X) - \mathrm{II}(Y, \nabla_X X)) \\
&= -\nabla_X(0) + \mathrm{II}(\nabla_X Y, X) + \mathrm{II}(Y, \nabla_X X) \\
&= \mathrm{II}\left(-\frac{\psi}{\xi} Z, X\right) + \mathrm{II}\left(Y, -\varphi\varphi' \frac{\partial}{\partial s}\right) \\
&= -\frac{\psi}{\xi}\,\mathrm{II}(Z, X) - \frac{\varphi\varphi'}{\zeta^2}\,\mathrm{II}\left(Y, \frac{\partial}{\partial r}\right) \\
&= -\frac{\psi}{\xi} \cdot 0 - \frac{\varphi\varphi'}{\zeta^2} \cdot 0 \\
&= 0
\end{aligned}
\tag{3.1.47}
$$

and

$$
\begin{aligned}
(\nabla_Y \,\mathrm{II})(X, X) &= \nabla_Y(\mathrm{II}(X, X)) - \mathrm{II}(\nabla_Y X, X) - \mathrm{II}(X, \nabla_Y X) \\
&= \nabla_Y(\varphi\varphi') - 2\,\mathrm{II}(\nabla_Y X, X) \\
&= 0 - 2\,\mathrm{II}\left(-\frac{\varphi}{\xi} Z, X\right) \\
&= \frac{2\varphi}{\xi}\,\mathrm{II}\,(Z, X) \\
&= \frac{2\varphi}{\xi} \cdot 0 \\
&= 0.
\end{aligned}
\tag{3.1.48}
$$

We substitute (3.1.47) and (3.1.48) into (3.0.10) in order to conclude

$$
\begin{aligned}
R\left(X, Y, X, \frac{\partial}{\partial s}\right) &= -(\nabla_X \,\mathrm{II})(Y, X) + (\nabla_Y \,\mathrm{II})(X, X) \\
&= 0 + 0 \\
&= 0,
\end{aligned}
\tag{3.1.49}
$$

which is (3.1.41).

Next, we will prove (3.1.44). By applying (3.0.10) to the radial vector field $\frac{\partial}{\partial s}$ and the Killing fields X, Y, Z, we have

$$R\left(X, Y, Z, \frac{\partial}{\partial s}\right) = -(\nabla_X \,\mathrm{II})(Y, Z) + (\nabla_Y \,\mathrm{II})(X, Z). \tag{3.1.50}$$

So we obtain

$$\begin{aligned} -(\nabla_X \,\mathrm{II})(Y, Z) &= -(\nabla_X(\mathrm{II}(Y, Z)) - \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(Y, \nabla_X Z)) \\ &= -\nabla_X(0) + \mathrm{II}\,(\nabla_X Y, Z) + \mathrm{II}(Y, \nabla_X Z) \\ &= -0 + \mathrm{II}\left(-\frac{\psi}{\xi} Z, Z\right) + \mathrm{II}\left(Y, \frac{\xi}{\psi} Y\right) \\ &= -\frac{\psi}{\xi}\,\mathrm{II}(Z, Z) + \frac{\xi}{\psi}\,\mathrm{II}(Y, Y) \\ &= -\frac{\psi}{\xi}\xi\xi' + \frac{\xi}{\psi}\psi\psi' \\ &= -\psi\xi' + \psi'\xi \end{aligned} \tag{3.1.51}$$

and

$$\begin{aligned} (\nabla_Y \,\mathrm{II})(X, Z) &= \nabla_Y(\mathrm{II}(X, Z)) - \mathrm{II}(\nabla_Y X, Z) - \mathrm{II}(X, \nabla_Y Z) \\ &= \nabla_Y(0) - \mathrm{II}\left(-\frac{\xi}{\varphi} Z, Z\right) - \mathrm{II}\left(X, \frac{\xi}{\varphi} X\right) \\ &= 0 - \left(-\frac{\varphi}{\xi}\right)\mathrm{II}(Z, Z) - \frac{\xi}{\varphi}\,\mathrm{II}(X, X) \\ &= \frac{\varphi}{\xi}\,\mathrm{II}(Z, Z) - \frac{\xi}{\varphi}\,\mathrm{II}(X, X) \\ &= \frac{\varphi}{\xi}\xi\xi' - \frac{\xi}{\varphi}\varphi\varphi' \\ &= \varphi\xi' - \varphi'\xi. \end{aligned} \tag{3.1.52}$$

So our final expression is

$$
\begin{aligned}
R\left(X, Y, Z, \frac{\partial}{\partial s}\right) &= -(\nabla_X \mathrm{II})(Y, Z) + (\nabla_Y \mathrm{II})(X, Z) \\
&= (-\psi\xi' + \psi'\xi) + (\varphi\xi' - \varphi'\xi) \\
&= (\psi' - \varphi')\xi + (\varphi - \psi)\xi' \\
&= ((\varphi + \xi)' - \varphi')\xi + (\varphi - (\varphi + \xi))\xi' \\
&= (\varphi' + \xi' - \varphi')\xi - \xi\xi' \\
&= \xi'\xi - \xi\xi' \\
&= 0,
\end{aligned}
\tag{3.1.53}
$$

which is (3.1.44).

Next, we will prove (3.1.45). By applying (3.0.10) to the radial vector field $\frac{\partial}{\partial s}$ and the Killing fields X, Y, Z, we have

$$
R\left(Y, Z, X, \frac{\partial}{\partial s}\right) = -(\nabla_Y \mathrm{II})(Z, X) + (\nabla_Z \mathrm{II})(Y, X). \tag{3.1.54}
$$

So we obtain

$$
\begin{aligned}
-(\nabla_Y \mathrm{II})(Z, X) &= -(\nabla_Y(\mathrm{II}(Z, X)) - \mathrm{II}(\nabla_Y Z, X) - \mathrm{II}(Z, \nabla_Y X)) \\
&= -\nabla_Y(0) + \mathrm{II}\,(\nabla_Y Z, X) + \mathrm{II}(Z, \nabla_Y X) \\
&= -0 + \mathrm{II}\left(\frac{\xi}{\varphi}X, X\right) + \mathrm{II}\left(Z, -\frac{\varphi}{\xi}Z\right) \\
&= \frac{\xi}{\varphi}\,\mathrm{II}(X, X) - \frac{\varphi}{\xi}\,\mathrm{II}(Z, Z) \\
&= \frac{\xi}{\varphi}\varphi\varphi' - \frac{\varphi}{\xi}\xi\xi' \\
&= \varphi'\xi - \varphi\xi'
\end{aligned}
\tag{3.1.55}
$$

and

$$
\begin{aligned}
(\nabla_Z \mathrm{II})(Y, X) &= \nabla_Z(\mathrm{II}(Y, X)) - \mathrm{II}(\nabla_Z Y, X) - \mathrm{II}(Y, \nabla_Z X) \\
&= \nabla_Z(0) - \mathrm{II}\left(\frac{\psi}{\varphi} X, X\right) - \mathrm{II}\left(Y, -\frac{\varphi}{\psi} Y\right) \\
&= 0 - \frac{\psi}{\varphi} \mathrm{II}(X, X) - \left(-\frac{\varphi}{\psi}\right) \mathrm{II}(Y, Y) \\
&= -\frac{\psi}{\varphi} \mathrm{II}(X, X) + \frac{\varphi}{\psi} \mathrm{II}(Y, Y) \\
&= -\frac{\psi}{\varphi} \varphi\varphi' + \frac{\varphi}{\psi} \psi\psi' \\
&= -\varphi'\psi + \varphi\psi'.
\end{aligned} \tag{3.1.56}
$$

So our final expression is

$$
\begin{aligned}
R\left(Y, Z, X, \frac{\partial}{\partial s}\right) &= -(\nabla_Y \mathrm{II})(Z, X) + (\nabla_Z \mathrm{II})(Y, X) \\
&= (\varphi'\xi - \varphi\xi') + (-\varphi'\psi + \varphi\psi') \\
&= (-\psi + \xi)\varphi' + \varphi(\psi' - \xi') \\
&= (-(\varphi + \xi) + \xi)\varphi' + \varphi((\varphi + \xi)' - \xi') \\
&= (-\varphi - \xi + \xi)\varphi' + \varphi(\varphi' + \xi' - \xi') \\
&= -\varphi\varphi' + \varphi\varphi' \\
&= 0,
\end{aligned} \tag{3.1.57}
$$

which is (3.1.45).

Finally, we will prove (3.1.46). While we can certainly prove (3.1.46) in a similar way that we did for (3.1.44) and (3.1.45), we do not need to do so. Instead, we can invoke the first Bianchi identity, which we have written as (3.0.5), in order to write a shorter proof using only (3.1.44) and (3.1.45). Indeed, by applying (3.0.5) to the radial vector field $\frac{\partial}{\partial s}$

and the Killing fields X, Y, Z, we have

$$R\left(X, Y, Z, \frac{\partial}{\partial s}\right) + R\left(Z, X, Y, \frac{\partial}{\partial s}\right) + R\left(Y, Z, X, \frac{\partial}{\partial s}\right) = 0. \tag{3.1.58}$$

So we obtain

$$\begin{aligned} R\left(Z, X, Y, \frac{\partial}{\partial s}\right) &= \left(R\left(X, Y, Z, \frac{\partial}{\partial s}\right) + R\left(Y, Z, X, \frac{\partial}{\partial s}\right) + R\left(Z, X, Y, \frac{\partial}{\partial s}\right)\right) \\ &\quad - R\left(X, Y, Z, \frac{\partial}{\partial s}\right) - R\left(Y, Z, X, \frac{\partial}{\partial s}\right) \\ &= (0) - 0 - 0 \\ &= 0, \end{aligned} \tag{3.1.59}$$

which is (3.1.46). □

3.2 Riemannian curvatures of all tangent planes

In this section, we will apply our expressions of Riemannian curvatures from the previous section in order to obtain a convenient formula for the Riemannian curvature of any tangent plane of S^4, provided that we assume sufficient conditions.

We begin this section with a lemma.

Lemma 3.2.1 (Span Lemma)**.** *For any plane σ that is tangent to S^4, there exist $g(t)$-perpendicular vector fields V, W that are tangent to the orbit of* $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$ *for all* $s \in [0, \sqrt{3}]$ *and satisfy* $\sigma = \mathrm{span}(\frac{\partial}{\partial s} + V, W)$.

Proof. Since the interior orbits $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)} \subseteq M$ for all $r \in (0, L)$ have codimension 1, every tangent plane of $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$ must take the form

$$\begin{aligned} \mathrm{span}\left(\frac{\partial}{\partial s} + V, \lambda\frac{\partial}{\partial r} + W\right) &= \mathrm{span}\left(\frac{\partial}{\partial r} + V, \left(\lambda\frac{\partial}{\partial s} + W\right) - \lambda\left(\frac{\partial}{\partial s} + V\right)\right) \\ &= \mathrm{span}\left(\frac{\partial}{\partial s} + V, W - \lambda V\right) \end{aligned} \tag{3.2.1}$$

for all scalars $\lambda \in \mathbb{R} \setminus \{0\}$. Furthermore, it will be convenient for us to define

$$\tilde{W} := W - \lambda V, \tag{3.2.2}$$

which allows us to write (3.2.3) as

$$\begin{aligned} \mathrm{span}\left(\frac{\partial}{\partial s} + V, \lambda\frac{\partial}{\partial r} + W\right) &= \mathrm{span}\left(\frac{\partial}{\partial s} + V, W - \lambda V\right) \\ &= \mathrm{span}\left(\frac{\partial}{\partial s} + V, \tilde{W}\right) \end{aligned} \tag{3.2.3}$$

for all $\lambda \in \mathbb{R} \setminus \{0\}$. Notice that, since V, W are tangent to $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$, it follows that $\tilde{W}$ is also tangent to $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)}$, which implies in particular that $\tilde{W}$ and $\frac{\partial}{\partial s}$ are

$g(t)$-perpendicular to each other. Also notice that the first equality of (3.2.3),

$$\operatorname{span}\left(\frac{\partial}{\partial s}+V,\lambda\frac{\partial}{\partial s}+W\right)=\operatorname{span}\left(\frac{\partial}{\partial s}+V,\left(\lambda\frac{\partial}{\partial s}+W\right)-\lambda\left(\frac{\partial}{\partial s}+V\right)\right),\tag{3.2.4}$$

exemplifies a general rule from linear algebra: for any nonzero scalar $\lambda \in \mathbb{R} \setminus \{0\}$ and for any vectors u_1, u_2 that span an entire vector space, we have

$$\operatorname{span}(u_1,u_2)=\operatorname{span}(u_1,u_1-\lambda u_2).\tag{3.2.5}$$

Indeed, (3.2.5) holds because we can write the elements of the two sets $\operatorname{span}(u_1, u_2)$ and $\operatorname{span}(u_1, u_1 - \lambda u_2)$ as

$$\begin{aligned}\mu_1u_1+\mu_2u_2&=\left(\mu_1+\frac{\mu_2}{\lambda}\right)u_1-\frac{\mu_2}{\lambda}(u_1-\lambda u_2),\\ \mu_1u_1+\mu_2(u_1-\lambda u_2)&=(\mu_1+\mu_2)u_1-\lambda\mu_2u_2\end{aligned}\tag{3.2.6}$$

for all scalars $\mu_1, \mu_2 \in \mathbb{R}$.

We will now decompose the vector field $\tilde{W}$ into its components that are tangent to and perpendicular to V, writing

$$\tilde{W}=\tilde{W}^V+\tilde{W}^\perp,\tag{3.2.7}$$

where $\tilde{W}^V$ is the tangential component of $\tilde{W}$ in the direction of V and $\tilde{W}^\perp$ is the component of $\tilde{W}$ that is perpendicular to $\tilde{W}^V$. In particular for the tangential component of $\tilde{W}$, we can write

$$V=\frac{1}{|\tilde{W}|^2}\tilde{W}^V.\tag{3.2.8}$$

Following the Gram–Schmidt process of constructing an orthonormal basis, as well as invoking our above result and the same general rule from linear algebra, we have

$$\begin{aligned}\text{span}\left(\frac{\partial}{\partial s}+V,\lambda\frac{\partial}{\partial r}+W\right)&=\text{span}\left(\frac{\partial}{\partial s}+V,\tilde{W}\right)\\&=\text{span}\left(\frac{\partial}{\partial s}+V-\frac{g(V,\tilde{W})}{|\tilde{W}|^2}\tilde{W},\tilde{W}\right)\\&=\text{span}\left(\frac{\partial}{\partial s}+\tilde{V},\tilde{W}\right),\end{aligned}\tag{3.2.9}$$

where we define

$$\tilde{V}:=V-\frac{g(V,\tilde{W})}{|\tilde{W}|^2}\tilde{W}.\tag{3.2.10}$$

Finally, we have

$$\begin{aligned}g\left(\frac{\partial}{\partial s}+\tilde{V},\tilde{W}\right)&=g\left(\frac{\partial}{\partial s},\tilde{W}\right)+g(\tilde{V},\tilde{W})\\&=0+g\left(V-\frac{g(V,\tilde{W})}{|\tilde{W}|^2}\tilde{W},\tilde{W}\right)\\&=g(V,\tilde{W})-\frac{g(V,\tilde{W})}{|\tilde{W}|^2}g(\tilde{W},\tilde{W})\\&=g(V,\tilde{W})-g(V,\tilde{W})\\&=0,\end{aligned}\tag{3.2.11}$$

meaning that $\frac{\partial}{\partial s}+\tilde{V}$ and $\tilde{W}$ are $g(t)$-perpendicular to each other, finishing the proof. □

The next proposition will provide us a convenient formula of the Riemannian curvature of any tangent plane of S^4, provided that we assume that the normal Riemannian curvatures are zero.

Proposition 3.2.2 (Multilinearity of the Riemannian curvature tensor)**.** *Let* $\{\frac{\partial}{\partial s},X,Y,Z\}$ *be basis of the vector field V for* (S^4,g)*, where g takes the form given by* (1.0.8). *Also assume the zero normal sectional curvature condition* (2.0.5). *Denote by* $\lambda_i:=|X_i|$ *to be* *the lengths of the Killing fields X_i for $i = 1, 2, 3$, where $X_1 := X, X_2 := Y, X_3 := Z$. Then the unnormalized sectional curvature of $\sigma = \mathrm{span}(\frac{\partial}{\partial s} + V, W)$ on (S^4, g) is*

$$\begin{aligned} R\left(\frac{\partial}{\partial s} + V, W, W, \frac{\partial}{\partial s} + V\right) = \sum_{i=1}^{3} \frac{(W^i)^2}{\lambda_i^2} R\left(\frac{\partial}{\partial s}, X_i, X_i, \frac{\partial}{\partial s}\right) \\ + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{(V^i)^2 (W^j)^2}{\lambda_i^2 \lambda_j^2} R(X_i, X_j, X_j, W). \end{aligned} \tag{3.2.12}$$

Proof. According to Proposition 3.1.1 in [17], the $(0, 4)$-tensor is skew-symmetric in the first two and last two entries and symmetric between the first two and last two entries. Applying said properties, we obtain

$$\begin{aligned} R\left(V, W, W, \frac{\partial}{\partial s}\right) &= R\left(W, \frac{\partial}{\partial s}, V, W\right) \\ &= -R\left(\frac{\partial}{\partial s}, W, V, W\right) \\ &= R\left(\frac{\partial}{\partial s}, W, W, V\right). \end{aligned} \tag{3.2.13}$$

Consequently, we can use the multilinearity of the $(0,4)$-tensor to obtain

$$\begin{aligned}
R\left(\frac{\partial}{\partial s}+V, W, W, \frac{\partial}{\partial s}+V\right) &= R\left(\frac{\partial}{\partial s}+V, W, W, \frac{\partial}{\partial s}\right) + R\left(\frac{\partial}{\partial s}+V, W, W, V\right) \\
&= R\left(\frac{\partial}{\partial s}, W, W, \frac{\partial}{\partial s}\right) + R\left(V, W, W, \frac{\partial}{\partial s}\right) \\
&\quad + R\left(\frac{\partial}{\partial s}, W, W, V\right) + R(V, W, W, V) \\
&= R\left(\frac{\partial}{\partial s}, W, W, \frac{\partial}{\partial s}\right) + R\left(V, W, W, \frac{\partial}{\partial s}\right) \\
&\quad + R\left(V, W, W, \frac{\partial}{\partial s}\right) + R(V, W, W, V) \\
&= R\left(\frac{\partial}{\partial s}, W, W, \frac{\partial}{\partial s}\right) + 2R\left(V, W, W, \frac{\partial}{\partial s}\right) \\
&\quad + R(V, W, W, V).
\end{aligned} \tag{3.2.14}$$

Consider the indices $i, j, k, l = 1, 2, 3$, which are not necessarily distinct. For all scalars $V^1, V^2, V^3, W^1, W^2, W^3 \in \mathbb{R}$ such that at least one of V^i and at least one of W^i are nonzero for some $i = 1, 2, 3$, we can write

$$V(\gamma(r,t)) = \sum_{i=1}^{3} V^i \frac{X_i}{\lambda_i}, \tag{3.2.15}$$

$$W(\gamma(r,t)) = \sum_{i=1}^{3} W^i \frac{X_i}{\lambda_i}. \tag{3.2.16}$$

We apply Proposition 3.1.2 to obtain

$$
\begin{aligned}
R\left(\frac{\partial}{\partial s}, W, W, \frac{\partial}{\partial s}\right) &= R\left(\frac{\partial}{\partial s}, \sum_{i=1}^{3} W^i \frac{X_i}{\lambda_i}, \sum_{j=1}^{3} W^j \frac{X_j}{\lambda_j}, \frac{\partial}{\partial s}\right) \\
&= \sum_{i=1}^{3} \frac{W^i W^j}{\lambda_i \lambda_j} R\left(\frac{\partial}{\partial s}, X_i, X_j, \frac{\partial}{\partial s}\right) \\
&= \sum_{i=1}^{3} \frac{(W^i)^2}{\lambda_i^2} R\left(\frac{\partial}{\partial s}, X_i, X_i, \frac{\partial}{\partial s}\right) \\
&\quad + \sum_{\substack{i=0 \\ i \neq j}}^{3} \frac{W^i W^j}{\lambda_i \lambda_j} R\left(\frac{\partial}{\partial s}, X_i, X_j, \frac{\partial}{\partial s}\right) \\
&= \sum_{i=1}^{3} \frac{(W^i)^2}{\lambda_i^2} R\left(\frac{\partial}{\partial s}, X_i, X_i, \frac{\partial}{\partial s}\right) + \sum_{\substack{i=0 \\ i \neq j}}^{3} \frac{W^i W^j}{\lambda_i \lambda_j} \cdot 0 \\
&= \sum_{i=1}^{3} \frac{(W^i)^2}{\lambda_i^2} R\left(\frac{\partial}{\partial s}, X_i, X_i, \frac{\partial}{\partial s}\right).
\end{aligned}
\tag{3.2.17}
$$

We apply Proposition 3.1.4 to obtain

$$\begin{aligned} R(V,W,W,V) &= R\left(\sum_{i=1}^{3} V^i \frac{X_i}{\lambda_i}, \sum_{j=1}^{3} W^j \frac{X_j}{\lambda_j}, \sum_{k=1}^{3} W^k \frac{X_k}{\lambda_k}, \sum_{l=1}^{3} V^l \frac{X_l}{\lambda_l}\right) \\ &= \sum_{i,j,k,l=1}^{3} \frac{V^i W^j W^k V^l}{\lambda_i \lambda_j \lambda_k \lambda_l} R(X_i, X_j, X_k, X_l) \\ &= \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \frac{(V^i)^2 (W^j)^2}{\lambda_i^2 \lambda_j^2} R(X_i, X_j, X_j, W) \\ &\qquad + \sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} \frac{V^i (W^j)^2 W^k}{\lambda_i \lambda_j^2 \lambda_k} R(X_i, X_j, X_k, W) \\ &= \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \frac{(V^i)^2 (W^j)^2}{\lambda_i^2 \lambda_j^2} R(X_i, X_j, X_j, W) \\ &\qquad + \sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} \frac{V^i (W^j)^2 W^k}{\lambda_i \lambda_j^2 \lambda_k} \cdot 0 \\ &= \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \frac{(V^i)^2 (W^j)^2}{\lambda_i^2 \lambda_j^2} R(X_i, X_j, X_j, W). \end{aligned} \tag{3.2.18}$$

We apply Proposition 3.1.5 to obtain

$$
\begin{aligned}
R\left(V, W, W, \frac{\partial}{\partial s}\right) &= R\left(\sum_{i=1}^{3} V^i \frac{X_i}{\lambda_i}, \sum_{j=1}^{2} W^j \frac{X_j}{\lambda_j}, \sum_{j=1}^{2} W^k \frac{X_k}{\lambda_k}, \frac{\partial}{\partial s}\right) \\
&= \sum_{i,j,k=1}^{2} \frac{V^i W^j W^k}{\lambda_i \lambda_j \lambda_k} R\left(X_i, X_j, X_k, \frac{\partial}{\partial s}\right) \\
&= \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{(V^i)^2 W^j}{\lambda_i^2 \lambda_j} R\left(X_i, X_j, X_i, \frac{\partial}{\partial s}\right) \\
&\qquad + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{V^i (W^j)^2}{\lambda_i \lambda_j^2} R\left(X_i, X_j, X_j, \frac{\partial}{\partial s}\right) \\
&\qquad + \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} \frac{V^i W^j W^k}{\lambda_i \lambda_j \lambda_k} R\left(X_i, X_j, X_k, \frac{\partial}{\partial s}\right) \\
&= \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{(V^i)^2 W^j}{\lambda_i^2 \lambda_j} \cdot 0 + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{V^i (W^j)^2}{\lambda_i \lambda_j^2} \cdot 0 \\
&\qquad + \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} \frac{V^i W^j W^k}{\lambda_i \lambda_j \lambda_k} \cdot 0 \\
&= 0.
\end{aligned}
\tag{3.2.19}
$$

So we conclude

$$
\begin{aligned}
R\left(\frac{\partial}{\partial s}+V, W, W, \frac{\partial}{\partial s}+V\right) &= R\left(\frac{\partial}{\partial s}, W, W, \frac{\partial}{\partial s}\right)+2R\left(V, W, W, \frac{\partial}{\partial s}\right) \\
&\quad + R(V, W, W, V) \\
&= \sum_{i=1}^{3} \frac{(W^i)^2}{\lambda_i^2} R\left(\frac{\partial}{\partial s}, X_i, X_i, \frac{\partial}{\partial s}\right) + 2(0) \\
&\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \frac{(V^i)^2 (W^j)^2}{\lambda_i^2 \lambda_j^2} R(X_i, X_j, X_j, W) \qquad (3.2.20) \\
&= \sum_{i=1}^{3} \frac{(W^i)^2}{\lambda_i^2} R\left(\frac{\partial}{\partial s}, X_i, X_i, \frac{\partial}{\partial s}\right) \\
&\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \frac{(V^i)^2 (W^j)^2}{\lambda_i^2 \lambda_j^2} R(X_i, X_j, X_j, W),
\end{aligned}
$$

which is (3.2.12). □

We will eventually apply Lemma 3.2.1 and Proposition 3.2.2 to our proof of Proposition 5.2.3 in Chapter 5.

3.3 Ricci curvatures of tangent planes generated by basis vector fields

We will use the formulas associated with the Riemannian curvature tensor in the previous sections of this chapter in order to obtain formulas associated with the Ricci tensor.

Proposition 3.3.1. *The Ricci curvatures of* S^4 *with any* SO(3)*-invariant metric* g *are*

$$\mathrm{Ric}\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = -\left(\frac{\varphi''}{\varphi} + \frac{\psi''}{\psi} + \frac{\xi''}{\xi}\right), \tag{3.3.1}$$

$$\mathrm{Ric}(X, X) = -\varphi\left(\varphi'' + \frac{2\varphi + \varphi'(\psi\xi)'}{\psi\xi}\right). \tag{3.3.2}$$

Proof. First, we will prove (3.3.1) and (3.3.9). We have

$$\begin{aligned}
\mathrm{Ric}\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) &= R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) + R\left(\frac{\partial}{\partial s}, \frac{X}{\varphi}, \frac{X}{\varphi}, \frac{\partial}{\partial s}\right) \\
&\quad + R\left(\frac{\partial}{\partial s}, \frac{Y}{\psi}, \frac{Y}{\psi}, \frac{\partial}{\partial s}\right) + R\left(\frac{\partial}{\partial s}, \frac{Z}{\xi}, \frac{Z}{\xi}, \frac{\partial}{\partial s}\right) \\
&= R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) + \frac{1}{\varphi^2}R\left(\frac{\partial}{\partial s}, X, X, \frac{\partial}{\partial s}\right) \\
&\quad + \frac{1}{\psi^2}R\left(\frac{\partial}{\partial s}, Y, Y, \frac{\partial}{\partial s}\right) + \frac{1}{\xi^2}R\left(\frac{\partial}{\partial s}, Z, Z, \frac{\partial}{\partial s}\right) \\
&= 0 - \frac{\varphi\varphi''}{\varphi^2} - \frac{\psi\psi''}{\psi^2} - \frac{\xi\xi''}{\xi^2} \\
&= -\frac{\varphi''}{\varphi} - \frac{\psi''}{\psi} - \frac{\xi''}{\xi} \\
&= -\left(\frac{\varphi''}{\varphi} + \frac{\psi''}{\psi} + \frac{\xi''}{\xi}\right),
\end{aligned} \tag{3.3.3}$$

which is (3.3.1).

Next, we will prove (3.3.2). We have

$$\begin{aligned} R\left(X, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}, X\right) &= R\left(\frac{\partial}{\partial s}, X, X, \frac{\partial}{\partial s}\right) \\ &= -\varphi\varphi''. \end{aligned} \tag{3.3.4}$$

By the antisymmetry of the Riemannian curvature tensor, we obtain

$$\begin{aligned} R\left(X, \frac{X}{\psi}, \frac{X}{\psi}, X\right) &= \frac{R(X, X, X, X)}{\psi^2} \\ &= \frac{0}{\psi^2} \\ &= 0 \end{aligned} \tag{3.3.5}$$

and

$$\begin{aligned} R\left(X, \frac{Y}{\psi}, \frac{Y}{\psi}, X\right) &= \frac{R(X, Y, Y, X)}{\psi^2} \\ &= \frac{\varphi\psi(2 - \varphi'\psi')}{\psi^2} \\ &= \frac{\varphi(2 - \varphi'\psi')}{\psi} \end{aligned} \tag{3.3.6}$$

and

$$\begin{aligned} R\left(X, \frac{Z}{\xi}, \frac{Z}{\xi}, X\right) &= \frac{R(X, Z, Z, X)}{\xi^2} \\ &= \frac{-\varphi\xi(2 + \varphi'\xi')}{\xi^2} \\ &= -\frac{\varphi(2 + \varphi'\xi')}{\xi}. \end{aligned} \tag{3.3.7}$$

So the Ricci curvature is

$$
\begin{aligned}
\mathrm{Ric}(X,X) &= R\left(X, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}, X\right) + R\left(X, \frac{X}{\varphi}, \frac{X}{\varphi}, X\right) \\
&\quad + R\left(X, \frac{Y}{\psi}, \frac{Y}{\psi}, X\right) + R\left(X, \frac{Z}{\xi}, \frac{Z}{\xi}, X\right) \\
&= -\varphi\varphi'' + 0 + \frac{\varphi(2-\varphi'\psi')}{\psi} - \frac{\varphi(2+\varphi'\xi')}{\xi} \\
&= -\varphi\left(\varphi'' - \frac{2-\varphi'\psi'}{\psi} + \frac{2+\varphi'\xi'}{\xi}\right) \\
&= -\varphi\left(\varphi'' - \frac{2}{\psi} + \frac{\varphi'\psi'}{\psi} + \frac{2}{\xi} + \frac{\varphi'\xi'}{\xi}\right) \\
&= -\varphi\left(\varphi'' + \frac{2}{\xi} - \frac{2}{\psi} + \frac{\varphi'\psi'}{\psi} + \frac{\varphi'\xi'}{\xi}\right) \qquad (3.3.8) \\
&= -\varphi\left(\varphi'' + 2\left(\frac{1}{\xi} - \frac{1}{\psi}\right) + \varphi'\left(\frac{\psi'}{\psi} + \frac{\xi'}{\xi}\right)\right) \\
&= -\varphi\left(\varphi'' + \frac{2(\psi-\xi)}{\psi\xi} + \frac{\varphi'(\psi'\xi + \psi\xi')}{\psi\xi}\right) \\
&= -\varphi\left(\varphi'' + \frac{2((\varphi+\xi)-\xi)}{\psi\xi} + \frac{\varphi'(\psi\xi)'}{\psi\xi}\right) \\
&= -\varphi\left(\varphi'' + \frac{2\varphi}{\psi\xi} + \frac{\varphi'(\psi\xi)'}{\psi\xi}\right) \\
&= -\varphi\left(\varphi'' + \frac{2\varphi + \varphi'(\psi\xi)'}{\psi\xi}\right),
\end{aligned}
$$

which is (3.3.2). □

It is possible to obtain the expressions of the Ricci curvatures $\mathrm{Ric}(Y,Y)$ and $\mathrm{Ric}(Z,Z)$ as well, but it turns out that they are not necessary for this dissertation. So we did not include them in the statement of Proposition 3.3.1.

Proposition 3.3.2. *The mixed Ricci curvatures of S^4 with any* SO(3)*-invariant metric g are*

$$\operatorname{Ric}\left(\frac{\partial}{\partial s}, X\right) = 0, \tag{3.3.9}$$

$$\operatorname{Ric}\left(\frac{\partial}{\partial s}, Y\right) = 0, \tag{3.3.10}$$

$$\operatorname{Ric}\left(\frac{\partial}{\partial s}, Z\right) = 0, \tag{3.3.11}$$

$$\operatorname{Ric}(X, Y) = 0, \tag{3.3.12}$$

$$\operatorname{Ric}(X, Z) = 0, \tag{3.3.13}$$

$$\operatorname{Ric}(Y, Z) = 0. \tag{3.3.14}$$

Proof. The proofs of (3.3.9), (3.3.10), (3.3.11) are all analogous to each other. Without loss of generality, we choose to only prove (3.3.9). We have

$$\begin{aligned}
\operatorname{Ric}\left(\frac{\partial}{\partial s}, X\right) &= R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}, X\right) + R\left(\frac{\partial}{\partial s}, \frac{X}{\varphi}, \frac{X}{\varphi}, X\right) \\
&\quad + R\left(\frac{\partial}{\partial s}, \frac{Y}{\psi}, \frac{Y}{\psi}, X_i\right) + R\left(\frac{\partial}{\partial s}, \frac{Z}{\xi}, \frac{Z}{\xi}, X\right) \\
&= R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}, X\right) - \frac{1}{\varphi^2} R\left(X, X, X, \frac{\partial}{\partial s}\right) \\
&\quad - \frac{1}{\psi^2} R\left(Y, X, Y, \frac{\partial}{\partial s}\right) - \frac{1}{\xi^2} R\left(Z, X, Z, \frac{\partial}{\partial s}\right) \\
&= 0 + \frac{0}{\varphi^2} + \frac{0}{\psi^2} + \frac{0}{\xi^2} \\
&= 0,
\end{aligned} \tag{3.3.15}$$

which is (3.3.9).

The proofs of (3.3.12), (3.3.13), (3.3.14) are all analogous to each other. Without loss of generality, we choose to only prove (3.3.12). We have

$$\begin{aligned}
\operatorname{Ric}(X,Y) &= R\left(X, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}, Y\right) + R\left(X, \frac{X}{\varphi}, \frac{X}{\varphi}, Y\right) \\
&\quad + R\left(X, \frac{Y}{\psi}, \frac{Y}{\psi}, Y\right) + R\left(X, \frac{Z}{\xi}, \frac{Z}{\xi}, Y\right) \\
&= R\left(\frac{\partial}{\partial s}, X, Y, \frac{\partial}{\partial s}\right) + \frac{R(X,X,X,Y)}{\varphi^2} \\
&\quad + \frac{R(X,Y,Y,Y)}{\psi^2} + \frac{R(X,Z,Z,Y)}{\xi^2} \\
&= 0 + \frac{0}{\varphi^2} + \frac{0}{\psi^2} + \frac{0}{\xi^2} \\
&= 0,
\end{aligned} \tag{3.3.16}$$

which is (3.3.12). □

We acknowledge that there are nonzero expressions for $\operatorname{Ric}(Y,Y)$ and $\operatorname{Ric}(Z,Z)$ as well, but we do not need to print them in this dissertation because we do not need to use them anywhere in our entire argument for Theorem 1.0.3. Nonetheless, we remark that the expressions of $\zeta_t, \varphi_t, \psi_t, \xi_t$ together form a nonlinear system of four partial differential equations. Furthermore, due to the symmetry of the Ricci tensor, (3.3.12), (3.3.13), (3.3.14) imply

$$\operatorname{Ric}(Y,X) = 0, \tag{3.3.17}$$

$$\operatorname{Ric}(Z,X) = 0, \tag{3.3.18}$$

$$\operatorname{Ric}(Z,Y) = 0, \tag{3.3.19}$$

respectively. In any case, we have enough information from Proposition 3.3.1 and Proposition 3.3.2 and our remark to conclude that the Ricci tensor is $g(t)$-orthogonal.

Chapter 4

Applications of Ricci flow

We will discuss Ricci flow in this chapter. As usual, let M be a Riemannian manifold, and recall that Ric is the $(0,2)$-Ricci tensor.

Definition 4.0.1. *Let $I \subseteq \mathbb{R}$ be a time interval. For any one-parameter family of metrics $g(t)$ on M, the* Ricci flow equation *is*

$$\frac{\partial}{\partial t} g(t) = -2\operatorname{Ric}(\cdot, t) \tag{4.0.1}$$

for all $t \in I$. Furthermore, $g(t)$ is called the Ricci flow *if it satisfies the Ricci flow equation, or* (4.0.1).

We state a proposition concerning Ricci flow and isometries.

Proposition 4.0.2. *The Ricci flow $g(t)$ preserves isometries.*

Proof. Let M_1, M_2 be two Riemannian manifolds with their respective families of metrics $g_1(t), g_2(t)$ that evolve by Ricci flow on M_1, M_2 respectively. Let $\phi : M_1 \to M_2$ is an isometry for the initial metric, and let ϕ^* be the pullback of ϕ. As a consequence of (4.0.1), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\phi^* g_2(t) &= \phi^*\left(\frac{\partial}{\partial t} g_2(t)\right) \\
&= \phi^*(-2\,\mathrm{Ric}_{g_2(t)}) \\
&= -2\phi^*\,\mathrm{Ric}_{g_2(t)} \\
&= -2\,\mathrm{Ric}_{\phi^* g_2(t)},
\end{aligned} \tag{4.0.2}$$

which signifies that $\phi^* g_2(t)$ also evolves by Ricci flow on M_1. But the uniqueness of Ricci flow implies

$$g_1(t) = \phi^* g_2(t). \tag{4.0.3}$$

So we conclude that ϕ is an isometry for $g_1(t)$. □

For further information about the Ricci flow equation and Ricci flow in general, the interested reader can consult, for instance, Chapter 2 of [6]. In this chapter, we will focus particularly on the Ricci flow whose initial metric is the usual SO(3)-invariant metric $g(0) = g$.

Karsten Grove and Wolfgang Ziller established in [9] that there exist nonnegatively curved $SO(3)$-invariant metrics on S^4 and $\mathbb{CP}^2$. Furthermore, Renato Bettiol and Anusha Krishnan showed in [2] that these metrics are diagonal; that is, along the geodesic γ, we can write

$$g(0) = \zeta(r,0)^2 dr^2 + \varphi(r,0)^2 dx^2 + \psi(r,0)^2 dy^2 + \xi(r,0)^2 dz^2, \tag{4.0.4}$$

where $\zeta, \varphi, \psi, \xi : [0, L] \times [0, \infty) \to \mathbb{R}$ are defined by

$$\zeta(r,t) := \left| \frac{\partial}{\partial r}(\gamma(r,t)) \right|, \tag{4.0.5}$$

$$\varphi(r,t) := |X(\gamma(r,t))|, \tag{4.0.6}$$

$$\psi(r,t) := |Y(\gamma(r,t))|, \tag{4.0.7}$$

$$\xi(r,t) := |Z(\gamma(r,t))|. \tag{4.0.8}$$

As $g(0)$ is diagonal, the frame $\{\frac{\partial}{\partial r}, \frac{X}{\varphi}, \frac{Y}{\psi}, \frac{Z}{\xi}\}$ is $g(0)$-orthonormal. Since we established in Proposition 4.0.2 that Ricci flow preserves isometries, it follows by the above lemma of Bettiol and Krishnan that the frame $\{\frac{\partial}{\partial r}, \frac{X}{\varphi}, \frac{Y}{\psi}, \frac{Z}{\xi}\}$ is also $g(t)$-orthonormal for all $t \geq 0$. In other words, the metric takes the form

$$g(t) = \zeta(r,t)^2 dr^2 + \varphi(r,t)^2 dx^2 + \psi(r,t)^2 dy^2 + \xi(r,t)^2 dz^2 \tag{4.0.9}$$

along the geodesic γ. This diagonal form allows us to obtain further results in this chapter.

4.1 Relationship between distance functions and general functions

Consider any family of SO(3)-invariant metrics $g(t)$ on S^4 that satisfies (4.0.1). The partial differential equations

$$(\zeta^2)_t = -2\,\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right), \tag{4.1.1}$$

$$(\varphi^2)_t = -2\,\mathrm{Ric}(X, X), \tag{4.1.2}$$

$$(\psi^2)_t = -2\,\mathrm{Ric}(Y, Y), \tag{4.1.3}$$

$$(\xi^2)_t = -2\,\mathrm{Ric}(Z, Z) \tag{4.1.4}$$

are a direct consequence of (4.0.9).

We recall that s denotes the distance from the singular orbits $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(0)}$ and $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(\sqrt{3})}$. Although Ricci flow preserves symmetries, it does not preserve distance functions unless their gradients are directions of zero Ricci curvature. Following [2], we will set $r = s$ for the distance to the singular orbits for the initial metric g. As a result, we have

$$\left|\frac{\partial}{\partial s}\right|_{g(0)} = 1, \tag{4.1.5}$$

$$\left|\frac{\partial}{\partial r}\right|_{g(t)} = \zeta(r, t), \tag{4.1.6}$$

the latter of which we typically reserve for

$$\left|\frac{\partial}{\partial r}\right|_{g(t)} \neq 1. \tag{4.1.7}$$

Lemma 4.1.1. *Let $\zeta : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be a smooth function, and we define the radial vector field*

$$\frac{\partial}{\partial r} := \zeta(r,t)\frac{\partial}{\partial s}. \tag{4.1.8}$$

Let $\phi : M \to \mathbb{R}$ be a smooth function, and we define its derivatives

$$\phi_r(r,t) := D_{\frac{\partial}{\partial r}}\phi(r,t), \tag{4.1.9}$$

$$\phi_{rr}(r,t) := D_{\frac{\partial}{\partial r}}D_{\frac{\partial}{\partial r}}\phi(r,t), \tag{4.1.10}$$

$$\phi'(s) := D_{\frac{\partial}{\partial s}}\phi(s), \tag{4.1.11}$$

$$\phi''(s) := D_{\frac{\partial}{\partial s}}D_{\frac{\partial}{\partial s}}\phi(s). \tag{4.1.12}$$

Then the first and second derivatives of ϕ with respect to s are

$$\phi'(s) = \frac{\phi_r(r,t)}{\zeta(r,t)}, \tag{4.1.13}$$

$$\phi''(s) = \frac{\zeta(r,t)\phi_{rr}(r,t) - \zeta_r(r,t)\phi_r(r,t)}{\zeta(r,t)^3}. \tag{4.1.14}$$

Proof. First, we have, for all $(r,t) \in \mathbb{R} \times [0,\infty)$ and $s \in \mathbb{R}$,

$$\begin{aligned}\left|\frac{\partial}{\partial r}\right| &= \left|\zeta(r,t)\frac{\partial}{\partial s}\right| \\ &= |\zeta(r,t)|\left|\frac{\partial}{\partial s}\right| \\ &= \zeta(r,t)\cdot 1 \\ &= \zeta(r,t),\end{aligned} \tag{4.1.15}$$

thereby establishing (4.1.6). We also obtain, for all $(r, t) \in \mathbb{R} \times [0, \infty)$ and $s \in \mathbb{R}$,

$$\begin{aligned} \phi'(s) &= D_{\frac{\partial}{\partial s}} \phi(s) \\ &= D_{\frac{1}{\zeta(r,t)} \frac{\partial}{\partial r}} \phi(r, t) \\ &= \frac{D_{\frac{\partial}{\partial r}} \phi(r, t)}{\zeta(r, t)} \\ &= \frac{\phi_r(r, t)}{\zeta(r, t)}, \end{aligned} \tag{4.1.16}$$

which is (4.1.13), and

$$\begin{aligned} \phi'(s) &= D_{\frac{\partial}{\partial s}} D_{\frac{\partial}{\partial s}} \phi(s) \\ &= D_{\frac{1}{\zeta(r,t)} \frac{\partial}{\partial r}} \phi'(s) \\ &= \frac{1}{\zeta(r, t)} D_{\frac{\partial}{\partial r}} \phi'(s) \\ &= \frac{1}{\zeta(r, t)} D_{\frac{\partial}{\partial r}} \left(\frac{\phi_r(r, t)}{\zeta(r, t)} \right) \\ &= \frac{1}{\zeta(r, t)} \frac{\zeta(r, t) D_{\frac{\partial}{\partial r}} \phi_r(r, t) - \phi_r(r, t) D_{\frac{\partial}{\partial r}} \zeta(r, t)}{\zeta(r, t)^2} \\ &= \frac{\zeta(r, t)\phi_{rr}(r, t) - \zeta_r(r, t)\phi_r(r, t)}{\zeta(r, t)^3}, \end{aligned} \tag{4.1.17}$$

which is (4.1.14). ◻

By applying Lemma 4.1.1 to our expressions of the Ricci tensor for any SO(3)-invariant cohomogeneity one metric g from Section 3.3, we can obtain expressions of the Ricci tensor for any family of SO(3)-invariant metrics $g(t)$ that is a Ricci flow.

Proposition 4.1.2. *The Ricci curvatures of S^4 with any family of* SO(3)*-invariant metrics*

$g(t)$ on S^4 that satisfies (4.0.1) *are*

$$\mathrm{Ric}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) = -\left(\frac{\varphi_{rr}}{\varphi}+\frac{\psi_{rr}}{\psi}+\frac{\xi_{rr}}{\xi}\right)+\frac{\zeta_r}{\zeta}\left(\frac{\varphi_r}{\varphi}+\frac{\psi_r}{\psi}+\frac{\xi_r}{\xi}\right), \tag{4.1.18}$$

$$\mathrm{Ric}(X,X) = -\varphi\left(\frac{\zeta\varphi_{rr}-\zeta_r\varphi_r}{\zeta^3}+\frac{2\zeta^2\varphi+\varphi_r(\psi\xi)_r}{\zeta^2\psi\xi}\right). \tag{4.1.19}$$

Proof. We apply (3.3.1) and (4.1.8) in order to obtain

$$\begin{aligned}
\mathrm{Ric}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) &= \mathrm{Ric}\left(\zeta\frac{\partial}{\partial s},\zeta\frac{\partial}{\partial s}\right)\\
&= \zeta^2\mathrm{Ric}\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right)\\
&= -\zeta^2\left(\frac{\varphi'}{\varphi}+\frac{\psi'}{\psi}+\frac{\xi'}{\xi}\right)\\
&= -\zeta^2\left(\frac{\zeta\varphi_{rr}-\zeta_r\varphi_r}{\zeta^3\varphi}-\frac{\zeta\psi_{rr}-\zeta_r\psi_r}{\zeta^3\psi}-\frac{\zeta\xi_{rr}-\zeta_r\xi_r}{\zeta^3\xi}\right)\\
&= -\left(\frac{\zeta\varphi_{rr}-\zeta_r\varphi_r}{\zeta\varphi}-\frac{\zeta\psi_{rr}-\zeta_r\psi_r}{\zeta\psi}-\frac{\zeta\xi_{rr}-\zeta_r\xi_r}{\zeta\xi}\right)\\
&= -\left(\frac{\varphi_{rr}}{\varphi}+\frac{\psi_{rr}}{\psi}+\frac{\xi_{rr}}{\xi}\right)+\frac{\zeta_r}{\zeta}\left(\frac{\varphi_r}{\varphi}+\frac{\psi_r}{\psi}+\frac{\xi_r}{\xi}\right),
\end{aligned} \tag{4.1.20}$$

which is (4.1.18). Likewise, we apply (3.3.2) and (4.1.8) in order to obtain

$$\begin{aligned}
\mathrm{Ric}(X,X) &= -\varphi\left(\varphi'+\frac{2\varphi+\varphi'(\psi\xi)'}{\psi\xi}\right)\\
&= -\varphi\left(\frac{\zeta\varphi_{rr}-\zeta_r\varphi_r}{\zeta^3}+\frac{2\varphi+\frac{\varphi_r}{\zeta}\frac{(\psi\xi)_r}{\zeta}}{\psi\xi}\right)\\
&= -\varphi\left(\frac{\zeta\varphi_{rr}-\zeta_r\varphi_r}{\zeta^3}+\frac{2\zeta^2\varphi+\varphi_r(\psi\xi)_r}{\zeta^2\psi\xi}\right),
\end{aligned} \tag{4.1.21}$$

which is (4.1.19). □

Proposition 4.1.3. *The mixed Ricci curvatures of S^4 with any family of* SO(3)*-invariant* *metrics $g(t)$ on S^4 that satisfies* (4.0.1) *are*

$$\mathrm{Ric}\left(\frac{\partial}{\partial r}, X\right) = 0, \tag{4.1.22}$$

$$\mathrm{Ric}\left(\frac{\partial}{\partial r}, Y\right) = 0, \tag{4.1.23}$$

$$\mathrm{Ric}\left(\frac{\partial}{\partial r}, Z\right) = 0. \tag{4.1.24}$$

Proof. The proofs of (4.1.22), (4.1.23), (4.1.24) are all analogous to each other. Without loss of generality, we choose to only prove (4.1.22). We have

$$\begin{aligned} \mathrm{Ric}\left(\frac{\partial}{\partial r}, X\right) &= \mathrm{Ric}\left(\zeta\frac{\partial}{\partial s}, X\right) \\ &= \zeta\mathrm{Ric}\left(\frac{\partial}{\partial s}, X\right) \\ &= \zeta \cdot 0 \\ &= 0, \end{aligned} \tag{4.1.25}$$

which is (4.1.22). □

Proposition 4.1.4. *The tangential sectional curvatures of S^4 with any family of* SO(3)*-invariant metrics $g(t)$ on S^4 that satisfies* (4.0.1) *are*

$$R(X, Y, Y, X) = \varphi\psi\left(2 - \frac{\varphi_r\psi_r}{\zeta^2}\right), \tag{4.1.26}$$

$$R(X, Z, Z, X) = -\varphi\xi\left(2 + \frac{\varphi_r\xi_r}{\zeta^2}\right), \tag{4.1.27}$$

$$R(Y, Z, Z, Y) = \psi\xi\left(2 - \frac{\psi_r\xi_r}{\zeta^2}\right). \tag{4.1.28}$$

Proof. We apply (3.1.14), in order to obtain

$$\begin{aligned} R(X,Y,Y,X) &= \varphi\psi(2-\varphi'\psi') \\ &= \varphi\psi\left(2-\frac{\varphi_r}{\zeta}\frac{\psi_r}{\zeta}\right) \\ &= \varphi\psi\left(2-\frac{\varphi_r\psi_r}{\zeta^2}\right) \end{aligned} \tag{4.1.29}$$

and

$$\begin{aligned} R(X,Z,Z,X) &= -\varphi\xi(2+\varphi'\xi') \\ &= -\varphi\xi\left(2+\frac{\varphi_r}{\zeta}\frac{\xi_r}{\zeta}\right) \\ &= -\varphi\xi\left(2+\frac{\varphi_r\xi_r}{\zeta^2}\right) \end{aligned} \tag{4.1.30}$$

and

$$\begin{aligned} R(Y,Z,Z,Y) &= \psi\xi(2-\psi'\xi') \\ &= \psi\xi\left(2-\frac{\psi_r}{\zeta}\frac{\xi_r}{\zeta}\right) \\ &= \psi\xi\left(2-\frac{\psi_r\xi_r}{\zeta^2}\right), \end{aligned} \tag{4.1.31}$$

which are (4.1.26), (4.1.27), (4.1.28), respectively. □

4.2 Evolution of sectional curvature through a Ricci flow

By applying our results for the Ricci tensor in the previous section to (4.0.1), we obtain a nonlinear system of four partial differential equations, which are (4.2.1), (4.2.2), (4.2.5), (4.2.6) below.

Proposition 4.2.1. *For any family of* $\mathrm{SO}(3)$*-invariant metrics* $g(t)$ *on* S^4 *that satisfies* (4.0.1)*, the lengths of the Killing fields* X, Y, Z *satisfy*

$$\zeta_t = \frac{1}{\zeta}\left(\frac{\varphi_{rr}}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi}\right) - \frac{\zeta_r}{\zeta^2}\left(\frac{\varphi_r}{\varphi} + \frac{\psi_r}{\psi} + \frac{\xi_r}{\xi}\right), \tag{4.2.1}$$

$$\varphi_t = \frac{\zeta\varphi_{rr} - \zeta_r\varphi_r}{\zeta^3} + \frac{2\zeta^2\varphi + \varphi_r(\psi\xi)_r}{\zeta^2\psi\xi}. \tag{4.2.2}$$

Proof. First, we will prove (4.2.1). Using Ricci flow equation (4.0.1) and (3.3.1), we obtain

$$\begin{aligned}
\zeta_t &= \frac{2\zeta\zeta_t}{2\zeta} \\
&= \frac{(\zeta^2)_t}{2\zeta} \\
&= \frac{1}{2\zeta}\left(-2\,\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)\right) \\
&= -\frac{1}{\zeta}\,\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \\
&= -\frac{1}{\zeta}\left(-\left(\frac{\varphi_{rr}}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi}\right) + \frac{\zeta_r}{\zeta}\left(\frac{\varphi_r}{\varphi} + \frac{\psi_r}{\psi} + \frac{\xi_r}{\xi}\right)\right) \\
&= \frac{1}{\zeta}\left(\frac{\varphi_{rr}}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi}\right) - \frac{\zeta_r}{\zeta^2}\left(\frac{\varphi_r}{\varphi} + \frac{\psi_r}{\psi} + \frac{\xi_r}{\xi}\right),
\end{aligned} \tag{4.2.3}$$

which is (4.2.1).

Next, we will prove (4.2.2). Using Ricci flow equation (4.0.1) and (3.3.2), we obtain

$$\begin{aligned}
\varphi_t &= \frac{2\varphi_t}{2\varphi} \\
&= \frac{(\varphi^2)_t}{2\varphi} \\
&= \frac{-2\operatorname{Ric}(X,X)}{2\varphi} \\
&= -\frac{1}{\varphi}\operatorname{Ric}(X,X) \\
&= -\frac{1}{\varphi}\left(-\varphi\left(\frac{\zeta\varphi_{rr}-\zeta_r\varphi_r}{\zeta^3}+\frac{2\zeta^2\varphi+\varphi_r(\psi\xi)_r}{\zeta^2\psi\xi}\right)\right) \\
&= \frac{\zeta\varphi_{rr}-\zeta_r\varphi_r}{\zeta^3}+\frac{2\zeta^2\varphi+\varphi_r(\psi\xi)_r}{\zeta^2\psi\xi},
\end{aligned} \tag{4.2.4}$$

which is (4.2.2). □

We remark that, by cyclically permuting the functions φ, ψ, ξ in (4.2.2), we also obtain

$$\psi_t = \frac{\zeta\psi_{rr}-\zeta_r\psi_r}{\zeta^3}+\frac{2\zeta^2\psi-\psi_r(\varphi\xi)_r}{\zeta^2\varphi\xi}, \tag{4.2.5}$$

$$\xi_t = \frac{\zeta\xi_{rr}-\zeta_r\xi_r}{\zeta^3}+\frac{2\zeta^2\xi-\xi_r(\varphi\psi)_r}{\zeta^2\varphi\psi}. \tag{4.2.6}$$

Nonetheless, we will only need to use (4.2.2) in the argument of our dissertation, which is why we did not also list (4.2.5) and (4.2.6) as assertions in the statement of Proposition 4.2.1.

We will print the following corollary of the preceding proposition.

Corollary 4.2.2. *If we also assume $\zeta = 1$, then* (4.2.1) *and* (4.2.2) *reduce to*

$$\zeta_t = \frac{\varphi_{rr}}{\varphi}+\frac{\psi_{rr}}{\psi}+\frac{\xi_{rr}}{\xi}, \tag{4.2.7}$$

$$\varphi_t = \varphi_{rr}+\frac{2\varphi+\varphi_r(\psi\xi)_r}{\psi\xi}. \tag{4.2.8}$$

In particular, if φ, ψ, ξ have zero second deriavtives (that is, $\varphi_{rr} = 0, \psi_{rr} = 0, \xi_{rr} = 0$), then (4.2.7) *and* (4.2.8) *further reduce to*

$$\zeta_t = 0, \tag{4.2.9}$$

$$\varphi_t = \frac{2\varphi + \varphi_r(\psi\xi)_r}{\psi\xi}. \tag{4.2.10}$$

Proof. We substitute $\zeta = 1$ into (4.2.1) to obtain

$$\begin{aligned}
\zeta_t &= \frac{1}{\zeta}\left(\frac{\varphi_{rr}}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi}\right) - \frac{\zeta_r}{\zeta^2}\left(\frac{\varphi_r}{\varphi} + \frac{\psi_r}{\psi} + \frac{\xi_r}{\xi}\right) \\
&= \frac{1}{1}\left(\frac{\varphi_{rr}}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi}\right) - \frac{1_r}{1^2}\left(\frac{\varphi_r}{\varphi} + \frac{\psi_r}{\psi} + \frac{\xi_r}{\xi}\right) \\
&= \frac{\varphi_{rr}}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi} - \frac{0}{1^2}\left(\frac{\varphi_r}{\varphi} + \frac{\psi_r}{\psi} + \frac{\xi_r}{\xi}\right) \\
&= \frac{\varphi_{rr}}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi},
\end{aligned} \tag{4.2.11}$$

which is (4.2.7). Furthermore, if we assume $\varphi_{rr} = 0, \psi_{rr} = 0, \xi_{rr} = 0$, then we have

$$\zeta_t = \frac{\varphi_{rr}}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi} \tag{4.2.12}$$

$$= \frac{0}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi} \tag{4.2.13}$$

$$= \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi}, \tag{4.2.14}$$

which is (4.2.9).

Likewise, we substitute $\zeta = 1$ into (4.2.2) to obtain

$$\begin{aligned}\varphi_t &= \frac{\zeta\varphi_{rr} - \zeta_r\varphi_r}{\zeta^3} + \frac{2\zeta^2\varphi + \varphi_r(\psi\xi)_r}{\zeta^2\psi\xi} \\ &= \frac{\varphi_{rr}1 - 0\varphi_r}{1^3} + \frac{2\cdot 1^2\varphi + \varphi_r(\psi\xi)_r}{1^2\psi\xi} \\ &= \varphi_{rr} + \frac{2\varphi + \varphi_r(\psi\xi)_r}{\psi\xi},\end{aligned} \tag{4.2.15}$$

which is (4.2.8). Furthermore, if we assume $\varphi_{rr} = 0$, then we have

$$\begin{aligned}\varphi_t &= \frac{\varphi_{rr}}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi} \\ &= \frac{0}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi} \\ &= \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi},\end{aligned} \tag{4.2.16}$$

which is (4.2.10). □

This corollary is particularly useful for some neighborhood of S^4 with the linearized metric $\bar{g}$ given by

$$\bar{g} = ds^2 + \bar{\varphi}^2\, d\bar{x}^2 + \bar{\psi}^2\, d\bar{y}^2 + \bar{\xi}^2\, d\bar{z}^2, \tag{4.2.17}$$

where we define $\bar{\varphi}, \bar{\psi}, \bar{\xi} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\bar{\varphi}(s) := 2s, \tag{4.2.18}$$

$$\bar{\psi}(s) := \sqrt{3} + s, \tag{4.2.19}$$

$$\bar{\xi}(s) := \sqrt{3} - s. \tag{4.2.20}$$

Indeed, $\bar{\varphi}, \bar{\psi}, \bar{\xi}$ all have zero second derivatives, and $\bar{\zeta} = 1$ satisfies $\zeta = 1$. So, according to the corollary, we may use (4.2.9) and (4.2.10) for the metrics $\mathring{g}_0, \bar{g}_0, \mathring{g}_{\frac{\pi}{3}}, \bar{g}_{\frac{\pi}{3}}$ on S^4.

4.3 Sectional curvatures for Ricci flow and one of their temporal derivatives

Here, we note that the expressions of Riemannian curvature now depend explicitly on r and t, but we do not write the independent variables in order to keep our notation simple.

Proposition 4.3.1. *The radial sectional curvatures associated with any family of* SO(3)*-invariant Riemannian metrics* $g(t)$ *that satisfies* (4.0.1) *are*

$$R\left(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r}\right) = \frac{\varphi(\zeta_r\varphi_r - \zeta\varphi_{rr})}{\zeta}, \tag{4.3.1}$$

$$R\left(\frac{\partial}{\partial r}, Y, Y, \frac{\partial}{\partial r}\right) = \frac{\psi(\zeta_r\psi_r - \zeta\psi_{rr})}{\zeta}, \tag{4.3.2}$$

$$R\left(\frac{\partial}{\partial r}, Z, Z, \frac{\partial}{\partial r}\right) = \frac{\xi(\zeta_r\xi_r - \zeta\xi_{rr})}{\zeta}. \tag{4.3.3}$$

Also, the first temporal partial derivative of (4.3.1) *is*

$$\begin{aligned}\frac{\partial}{\partial t}\left(R\left(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r}\right)\right) = {} & \frac{\varphi_t(\zeta_r\varphi_r - \zeta\varphi_{rr})}{\zeta} \\ & + \frac{\varphi(\zeta_{rt}\varphi_r + \zeta_r\varphi_{rt} - \zeta_t\varphi_{rr} - \zeta\varphi_{rrt})}{\zeta} \\ & - \frac{\varphi\zeta_t(\zeta_r\varphi_r - \zeta\varphi_{rr})}{\zeta^2},\end{aligned} \tag{4.3.4}$$

where $\varphi_{rt}, \varphi_{rrt}$ *are respectively the first and second spatial partial derivatives of* φ_t*, whose expression is given by* (4.2.2).

Proof. The proofs of (4.3.1), (4.3.2), (4.3.3) are all analogous to each other. Without loss of generality, we choose to only prove (4.3.1). Indeed, we have

$$\begin{aligned} R\left(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r}\right) &= R\left(\zeta\frac{\partial}{\partial s}, X, X, \zeta\frac{\partial}{\partial s}\right) \\ &= \zeta^2 R\left(\frac{\partial}{\partial s}, X, X, \frac{\partial}{\partial s}\right) \\ &= \zeta^2(-\varphi\varphi'') \\ &= \zeta^2\left(-\varphi\frac{\zeta\varphi_{rr} - \zeta_r\varphi_r}{\zeta^3}\right) \\ &= \frac{\varphi(\zeta_r\varphi_r - \zeta\varphi_{rr})}{\zeta}, \end{aligned} \tag{4.3.5}$$

which is (4.3.1).

We have

$$\begin{aligned} \frac{\partial}{\partial t}\left(R\left(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r}\right)\right) &= \frac{\partial}{\partial t}\left(\frac{\varphi(\zeta_r\varphi_r - \zeta\varphi_{rr})}{\zeta}\right) \\ &= \frac{\varphi_t(\zeta_r\varphi_r - \zeta\varphi_{rr})}{\zeta} + \varphi\frac{\partial}{\partial t}\left(\frac{\zeta_r\varphi_r - \zeta\varphi_{rr}}{\zeta}\right) \\ &= \frac{\varphi_t(\zeta_r\varphi_r - \zeta\varphi_{rr})}{\zeta} \\ &\quad + \varphi\frac{(\zeta_r\varphi_r - \zeta\varphi_{rr})_t\zeta - (\zeta_r\varphi_r - \zeta\varphi_{rr})\zeta_t}{\zeta^2} \\ &= \frac{\varphi_t(\zeta_r\varphi_r - \zeta\varphi_{rr})}{\zeta} \\ &\quad + \varphi\left(\frac{(\zeta_r\varphi_r)_t - (\zeta\varphi_{rr})_t}{\zeta} - \frac{(\zeta_r\varphi_r - \zeta\varphi_{rr})\zeta_t}{\zeta^2}\right) \\ &= \frac{\varphi_t(\zeta_r\varphi_r - \zeta\varphi_{rr})}{\zeta} \\ &\quad + \frac{\varphi(\zeta_{rt}\varphi_r + \zeta_r\varphi_{rt} - \zeta_t\varphi_{rr} - \zeta\varphi_{rrt})}{\zeta} \\ &\quad - \frac{\varphi\zeta_t(\zeta_r\varphi_r - \zeta\varphi_{rr})}{\zeta^2}, \end{aligned} \tag{4.3.6}$$

which is (4.3.4). □

We remark that we will, for the most part, assume $\zeta = 1$, which implies $\zeta_r = 0$. With this in mind, if we regard t to be a fixed variable, then the reader can verify as a quick exercise that (4.3.1), (4.3.2), (4.3.3) reduce to (3.1.1), (3.1.2), (3.1.3), respectively.

The next corollary will be useful for the linearized metric at initial time. In particular, we will apply Corollary 4.3.2 below to our proof of Lemma 5.3.1 in Chapter 5.

Corollary 4.3.2. *If we assume $\zeta = 1$, then, at $t = 0$,* (4.3.4) *simplifies to*

$$\frac{\partial}{\partial t}\left(R\left(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r}\right)\right)\bigg|_{t=0} = \varphi|_{t=0}(\zeta_{rt}|_{t=0}\varphi_r|_{t=0} - \zeta_t|_{t=0}\varphi_{rr}|_{t=0} - \varphi_{rrt}|_{t=0}) - \varphi_t|_{t=0}\varphi_{rr}|_{t=0}. \tag{4.3.7}$$

Furthermore, at $t = 0$, the temporal derivative of the normalized sectional curvature of the tangent plane $\mathrm{span}(\frac{\partial}{\partial r}, X)$ *is*

$$\frac{\partial}{\partial t}\left(\sec\left(\frac{\partial}{\partial r}, X\right)\right)\bigg|_{t=0} = \frac{\zeta_{rt}|_{t=0}\varphi_r|_{t=0} - \zeta_t|_{t=0}\varphi_{rr}|_{t=0} - \varphi_{rrt}|_{t=0} + \varphi_{rr}|_{t=0}}{\varphi|_{t=0}}. \tag{4.3.8}$$

Proof. We notice that $\zeta|_{t=0} = 1$ implies $\zeta_r|_{t=0} = 0$. So the first temporal derivative at $t = 0$

is

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(R\left(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r}\right)\right)\Big|_{t=0} \\
&\quad= \frac{\varphi_t|_{t=0}(\zeta_r|_{t=0}\varphi_r|_{t=0} - \zeta|_{t=0}\varphi_{rr}|_{t=0})}{\zeta|_{t=0}} \\
&\qquad+ \frac{\varphi|_{t=0}(\zeta_{rt}|_{t=0}\varphi_r|_{t=0} + \zeta_r|_{t=0}\varphi_{rt}|_{t=0} - \zeta_t|_{t=0}\varphi_{rr}|_{t=0} - \zeta|_{t=0}\varphi_{rrt}|_{t=0})}{\zeta|_{t=0}} \\
&\qquad- \frac{\varphi|_{t=0}\zeta_t|_{t=0}(\zeta_r|_{t=0}\varphi_r|_{t=0} - \zeta|_{t=0}\varphi_{rr}|_{t=0})}{(\zeta|_{t=0})^2} \\
&\quad= \frac{\varphi_t|_{t=0}(0\varphi_r|_{t=0} - 1\varphi_{rr}|_{t=0})}{1} \\
&\qquad+ \frac{\varphi|_{t=0}(\zeta_{rt}|_{t=0}\varphi_r|_{t=0} + 0\varphi_{rt}|_{t=0} - \zeta_t|_{t=0}\varphi_{rr}|_{t=0} - 1\varphi_{rrt}|_{t=0})}{1} \\
&\qquad- \frac{\varphi|_{t=0}\cdot 0\cdot(0\varphi_r|_{t=0} - 1\varphi_{rr}|_{t=0})}{1^2} \\
&\quad= -\varphi_t|_{t=0}\varphi_{rr}|_{t=0} + \varphi|_{t=0}(\zeta_{rt}|_{t=0}\varphi_r|_{t=0} - \zeta_t|_{t=0}\varphi_{rr}|_{t=0} - \varphi_{rrt}|_{t=0}) \\
&\quad= \varphi|_{t=0}(\zeta_{rt}|_{t=0}\varphi_r|_{t=0} - \zeta_t|_{t=0}\varphi_{rr}|_{t=0} - \varphi_{rrt}|_{t=0}) - \varphi_t|_{t=0}\varphi_{rr}|_{t=0},
\end{aligned} \tag{4.3.9}$$

which is (4.3.7).

The definition of the normalized sectional curvature of the tangent plane span$(\frac{\partial}{\partial r}, X)$ is

$$\begin{aligned}
\sec\left(\frac{\partial}{\partial r}, X\right) &:= \frac{R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r})}{|\frac{\partial}{\partial r}|^2|X|^2 - g(\frac{\partial}{\partial r}, X)} \\
&= \frac{R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r})}{1^2\varphi^2 - 0^2} \\
&= \frac{R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r})}{\varphi^2}.
\end{aligned} \tag{4.3.10}$$

Because we have assumed $\zeta = 1$, we can invoke (3.1.1) to find that the temporal derivative of this radial sectional curvature is

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\sec\left(\frac{\partial}{\partial r}, X\right)\right) &= \frac{\partial}{\partial t}\left(\frac{R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r})}{\varphi^2}\right) \\
&= \frac{\frac{\partial}{\partial t}(R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r}))\varphi^2 - R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r})(\varphi^2)_t}{(\varphi^2)^2} \\
&= \frac{\frac{\partial}{\partial t}(R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r}))\varphi^2 - (-\varphi\varphi_{rr})(2\varphi\varphi_t)}{\varphi^4} \\
&= \frac{\varphi^2\frac{\partial}{\partial t}(R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r})) + 2\varphi^2\varphi_t\varphi_{rr}}{\varphi^4} \\
&= \frac{\varphi^2(\frac{\partial}{\partial t}(R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r})) + 2\varphi_t\varphi_{rr})}{\varphi^4} \\
&= \frac{\frac{\partial}{\partial t}(R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r})) + 2\varphi_t\varphi_{rr}}{\varphi^2}.
\end{aligned} \tag{4.3.11}$$

At $t = 0$, we can apply (4.3.7) into (4.3.11) in order to obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\sec\left(\frac{\partial}{\partial r}, X\right)\right)\Bigg|_{t=0} \\
&= \frac{\frac{\partial}{\partial t}(R(\frac{\partial}{\partial r}, X, X, \frac{\partial}{\partial r}))|_{t=0} + 2\varphi_t|_{t=0}\varphi_{rr}|_{t=0}}{(\varphi|_{t=0})^2} \\
&= \frac{(\varphi|_{t=0}(\zeta_{rt}|_{t=0}\varphi_r|_{t=0} - \zeta_t|_{t=0}\varphi_{rr}|_{t=0} - \varphi_{rrt}|_{t=0}) - \varphi_t|_{t=0}\varphi_{rr}|_{t=0})}{(\varphi|_{t=0})^2} \\
&\quad + \frac{2\varphi_t|_{t=0}\varphi_{rr}|_{t=0}}{(\varphi|_{t=0})^2} \\
&= \frac{\varphi|_{t=0}(\zeta_{rt}|_{t=0}\varphi_r|_{t=0} - \zeta_t|_{t=0}\varphi_{rr}|_{t=0} - \varphi_{rrt}|_{t=0}) + \varphi_t|_{t=0}\varphi_{rr}|_{t=0}}{(\varphi|_{t=0})^2} \\
&= \frac{\varphi|_{t=0}(\zeta_{rt}|_{t=0}\varphi_r|_{t=0} - \zeta_t|_{t=0}\varphi_{rr}|_{t=0} - \varphi_{rrt}|_{t=0} + \varphi_{rr}|_{t=0})}{(\varphi|_{t=0})^2} \\
&= \frac{\zeta_{rt}|_{t=0}\varphi_r|_{t=0} - \zeta_t|_{t=0}\varphi_{rr}|_{t=0} - \varphi_{rrt}|_{t=0} + \varphi_{rr}|_{t=0}}{\varphi|_{t=0}},
\end{aligned} \tag{4.3.12}$$

which is (4.3.8). □

Chapter 5

The round and linearized metrics and their deviations

To begin the construction of our desired metric, we will investigate the *round metric* on S^4 at $s = 0$:

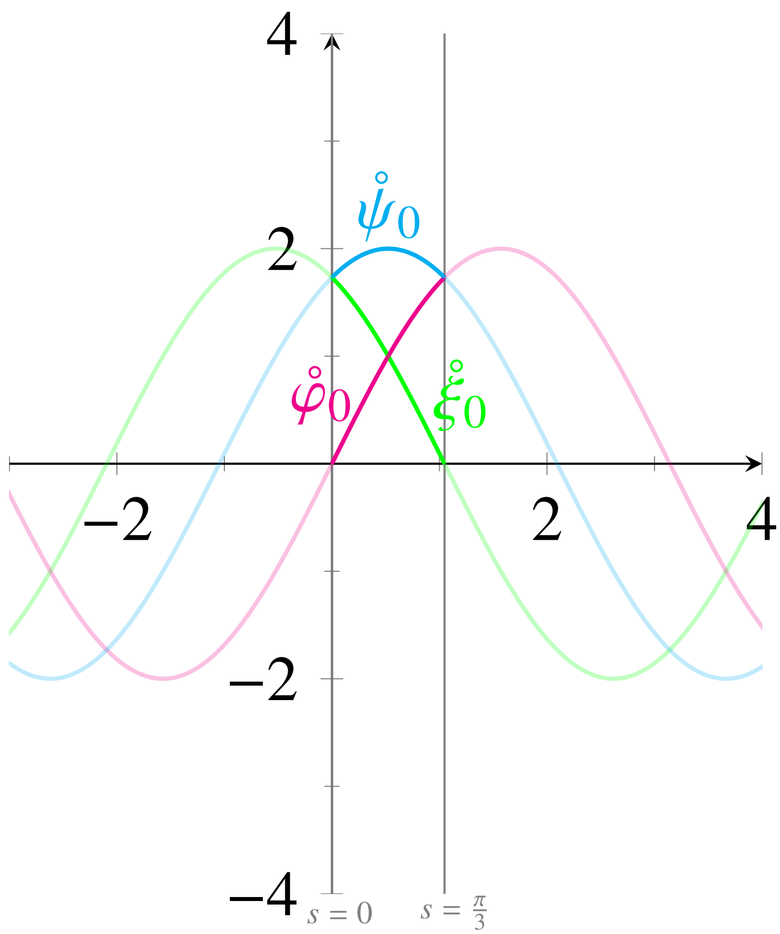
$$\mathring{g}_0 = ds^2 + \mathring{\varphi}_0^2\, d\mathring{x}^2 + \mathring{\psi}_0^2\, d\mathring{y}^2 + \mathring{\xi}_0^2\, d\mathring{z}^2, \tag{5.0.1}$$

where we define $\mathring{\varphi}_0, \mathring{\psi}_0, \mathring{\xi}_0 : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\mathring{\varphi}_0(s) := 2\sin(s), \tag{5.0.2}$$

$$\mathring{\psi}_0(s) := \sqrt{3}\cos(s) + \sin(s), \tag{5.0.3}$$

$$\mathring{\xi}_0(s) := \sqrt{3}\cos(s) - \sin(s). \tag{5.0.4}$$



We also introduce the *linearized metric* on a neighborhood of S^4 at $s = 0$:

$$\bar{g}_0 := \lim_{\alpha \to 0} \mathring{g}_0^{\alpha}. \tag{5.0.5}$$

This is equivalent to writing

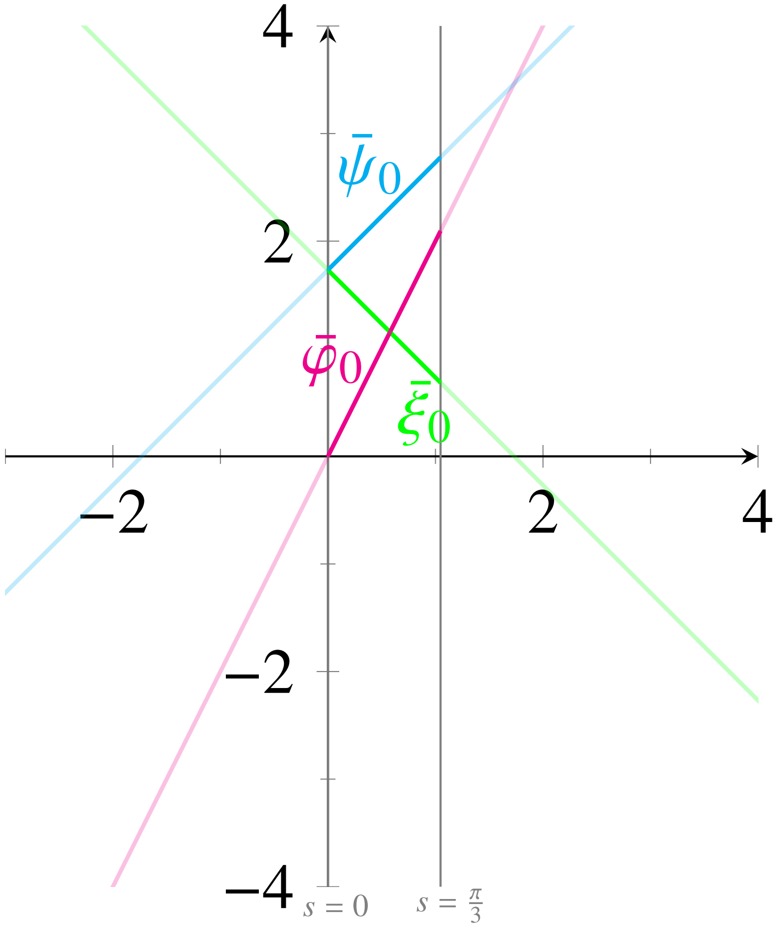
$$\bar{g}_0 = ds^2 + \bar{\varphi}_0^2\, d\bar{x}^2 + \bar{\psi}_0^2\, d\bar{y}^2 + \bar{\xi}_0^2\, d\bar{z}^2, \tag{5.0.6}$$

where we define $\bar{\varphi}_0, \bar{\psi}_0, \bar{\xi}_0 : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\bar{\varphi}_0(s) := 2s, \tag{5.0.7}$$

$$\bar{\psi}_0(s) := \sqrt{3} + s, \tag{5.0.8}$$

$$\bar{\xi}_0(s) := \sqrt{3} - s. \tag{5.0.9}$$



Analogously, we also mention the *round metric* on S^4 at $s = \frac{\pi}{3}$:

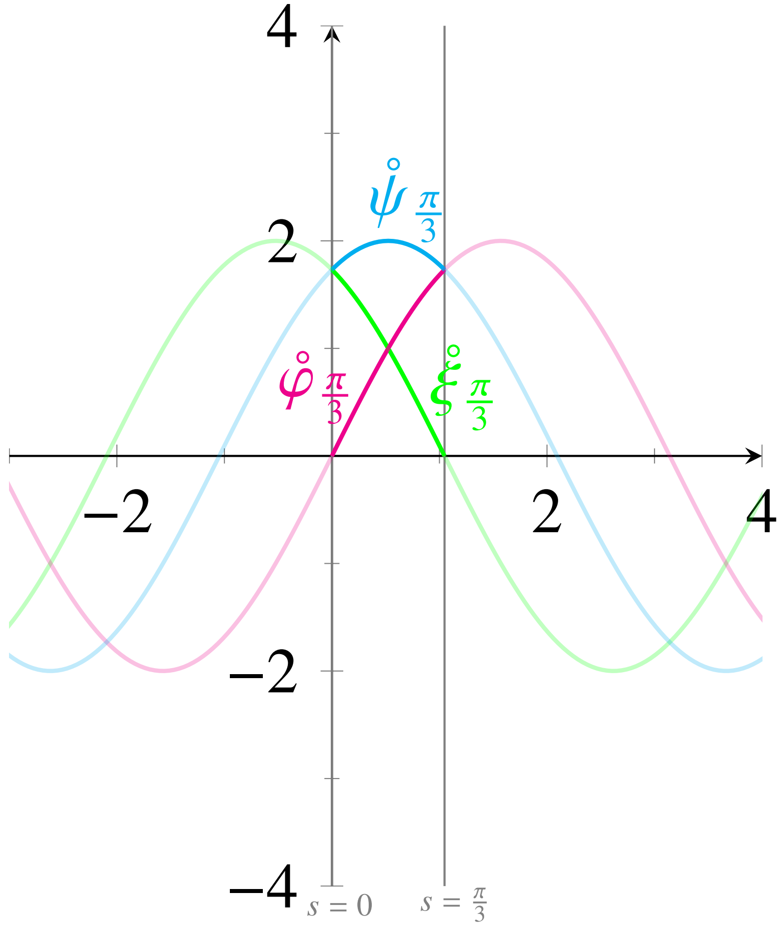
$$\mathring{g}_{\frac{\pi}{3}} = ds^2 + \mathring{\varphi}^2_{\frac{\pi}{3}}\, d\mathring{x}^2 + \mathring{\psi}^2_{\frac{\pi}{3}}\, d\mathring{y}^2 + \mathring{\xi}^2_{\frac{\pi}{3}}\, d\mathring{z}^2, \tag{5.0.10}$$

where we define $\mathring{\varphi}_{\frac{\pi}{3}}, \mathring{\psi}_{\frac{\pi}{3}}, \mathring{\xi}_{\frac{\pi}{3}} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\mathring{\varphi}_{\frac{\pi}{3}}(s) := \mathring{\psi}_0\left(s - \frac{\pi}{3}\right), \tag{5.0.11}$$

$$\mathring{\psi}_{\frac{\pi}{3}}(s) := \mathring{\xi}_0\left(s - \frac{\pi}{3}\right), \tag{5.0.12}$$

$$\mathring{\xi}_{\frac{\pi}{3}}(s) := -\mathring{\varphi}_0\left(s - \frac{\pi}{3}\right). \tag{5.0.13}$$



We also introduce *linearized metric* on a neighborhood of S^4 at $s = \frac{\pi}{3}$:

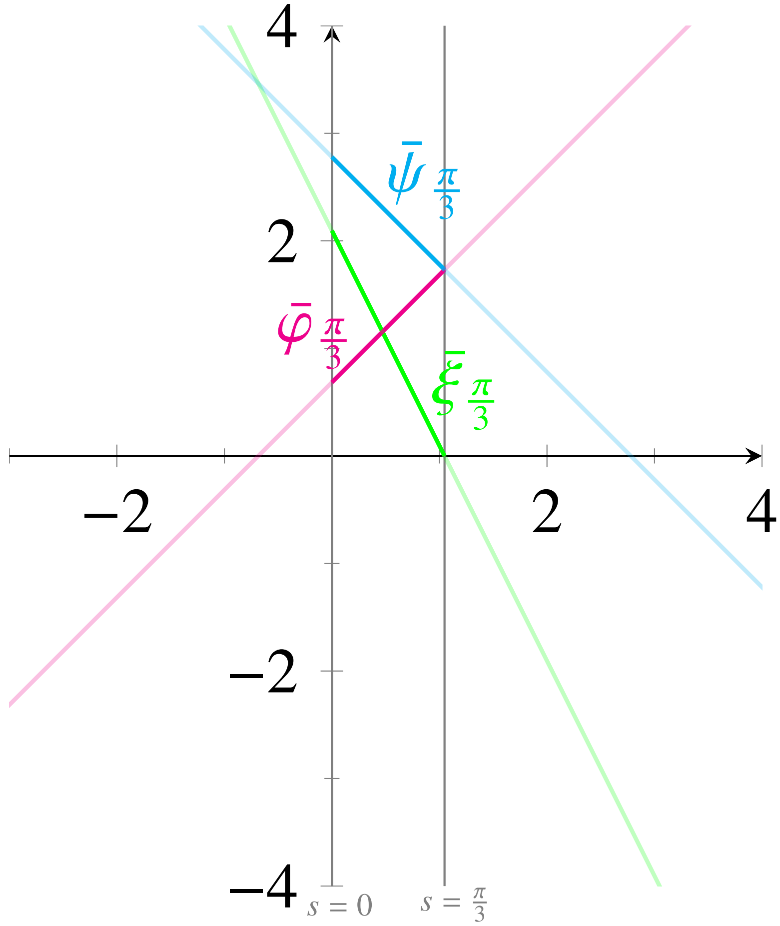
$$\bar{g}_{\frac{\pi}{3}} = ds^2 + \bar{\varphi}^2_{\frac{\pi}{3}}\, d\bar{x}^2 + \bar{\psi}^2_{\frac{\pi}{3}}\, d\bar{y}^2 + \bar{\xi}^2_{\frac{\pi}{3}}\, d\bar{z}^2, \tag{5.0.14}$$

where we define $\bar{\varphi}_{\frac{\pi}{3}}, \bar{\psi}_{\frac{\pi}{3}}, \bar{\xi}_{\frac{\pi}{3}} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\bar{\varphi}_{\frac{\pi}{3}}(s) := \bar{\psi}_0 \left(s - \frac{\pi}{3}\right), \tag{5.0.15}$$

$$\bar{\psi}_{\frac{\pi}{3}}(s) := \bar{\xi}_0 \left(s - \frac{\pi}{3}\right), \tag{5.0.16}$$

$$\bar{\xi}_{\frac{\pi}{3}}(s) := -\bar{\varphi}_0 \left(s - \frac{\pi}{3}\right). \tag{5.0.17}$$



With (5.0.2), (5.0.3), (5.0.4) in mind, we can rewrite (5.0.11), (5.0.12), (5.0.13) explicitly as

$$\mathring{\varphi}_{\frac{\pi}{3}}(s) = \sqrt{3}\cos\left(s - \frac{\pi}{3}\right) + \sin\left(s - \frac{\pi}{3}\right), \tag{5.0.18}$$

$$\mathring{\psi}_{\frac{\pi}{3}}(s) = \sqrt{3}\cos\left(s - \frac{\pi}{3}\right) - \sin\left(s - \frac{\pi}{3}\right), \tag{5.0.19}$$

$$\mathring{\xi}_{\frac{\pi}{3}}(s) = -2\sin\left(s - \frac{\pi}{3}\right), \tag{5.0.20}$$

respectively. Likewise, with (5.0.7), (5.0.8), (5.0.9) in mind, we can rewrite (5.0.15), (5.0.16), (5.0.17) explicitly as

$$\bar{\varphi}_{\frac{\pi}{3}}(s) = \sqrt{3} + \left(s - \frac{\pi}{3}\right), \tag{5.0.21}$$

$$\bar{\psi}_{\frac{\pi}{3}}(s) = \sqrt{3} - \left(s - \frac{\pi}{3}\right), \tag{5.0.22}$$

$$\bar{\xi}_{\frac{\pi}{3}}(s) = -2\left(s - \frac{\pi}{3}\right), \tag{5.0.23}$$

respectively. In the next section, we will show that the metrics $\mathring{g}_0$ and $\bar{g}_0$ are homotopic and that the metrics $\mathring{g}_{\frac{\pi}{3}}$ and $\bar{g}_{\frac{\pi}{3}}$ are homotopic.

5.1 Homotopies between the round metrics and the linearized metrics

We will consider for any $\alpha \in \mathbb{R} \setminus \{0\}$ the metric on S^4 at $s = 0$:

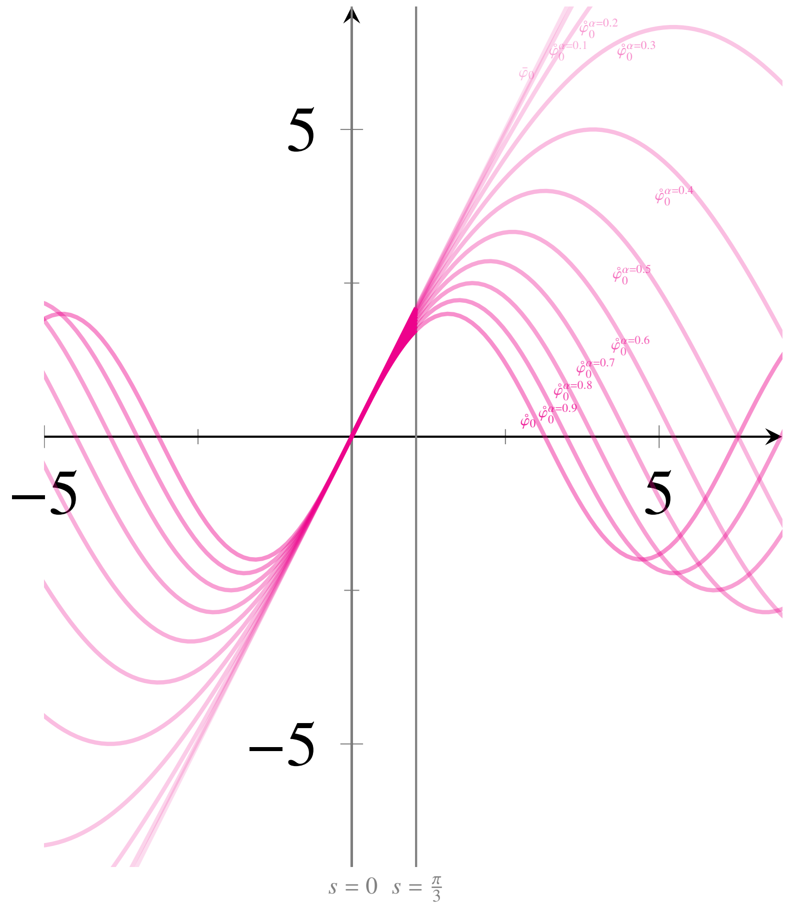
$$\mathring{g}_0^\alpha = ds^2 + (\mathring{\varphi}_0^\alpha)^2\, d\mathring{x}^2 + (\mathring{\psi}_0^\alpha)^2\, d\mathring{y}^2 + (\mathring{\xi}_0^\alpha)^2\, d\mathring{z}^2, \tag{5.1.1}$$

where we define $\mathring{\varphi}_0^\alpha, \mathring{\psi}_0^\alpha, \mathring{\xi}_0^\alpha : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

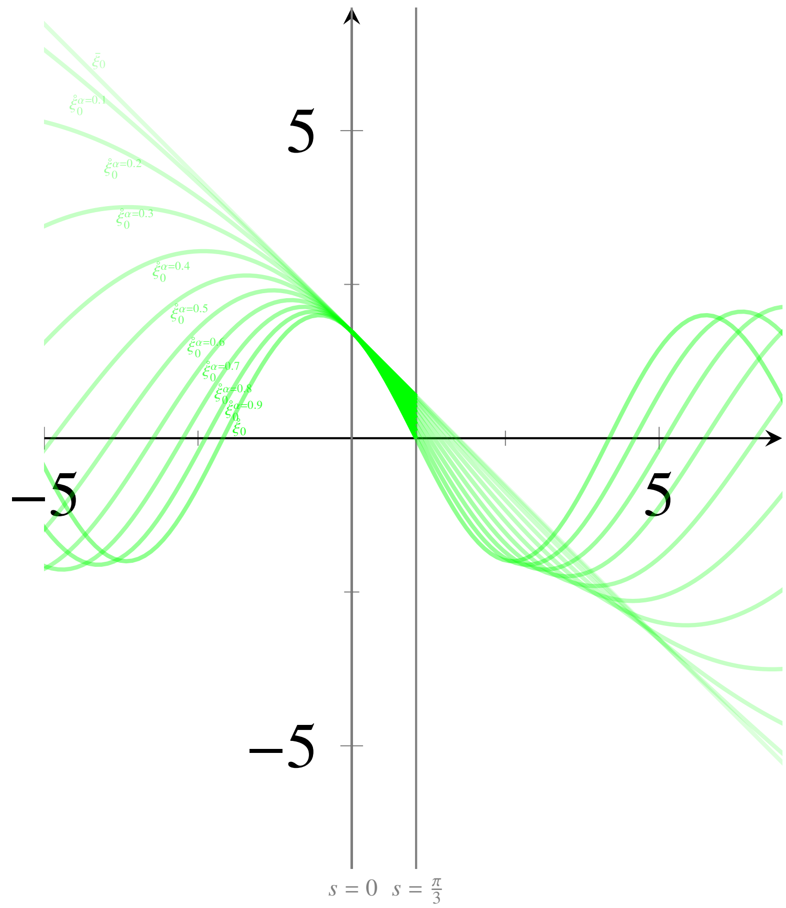
$$\mathring{\varphi}_0^\alpha(s) := \frac{2}{\alpha}\sin(\alpha s), \tag{5.1.2}$$

$$\mathring{\psi}_0^\alpha(s) := \sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s), \tag{5.1.3}$$

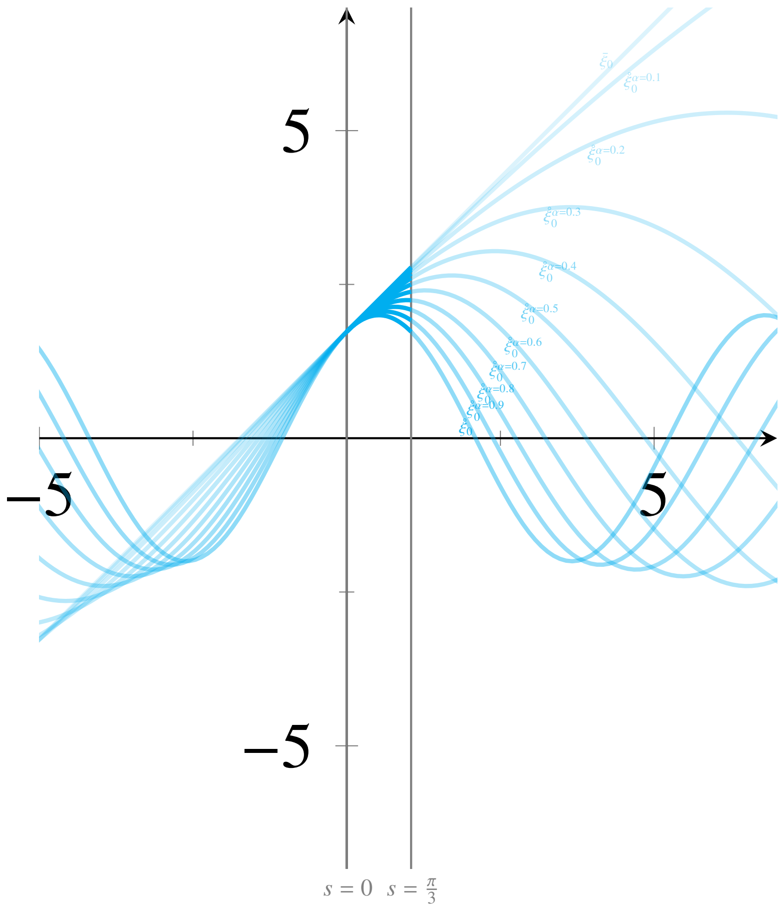
$$\mathring{\xi}_0^\alpha(s) := \sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s), \tag{5.1.4}$$



5
−5
−5
5
s = 0
s = π/3



5
−5
−5
5
$s = 0$
$s = \frac{\pi}{3}$



Likewise, we will also consider for any $\alpha > 0$ the metric on S^4 at $s = \frac{\pi}{3}$:

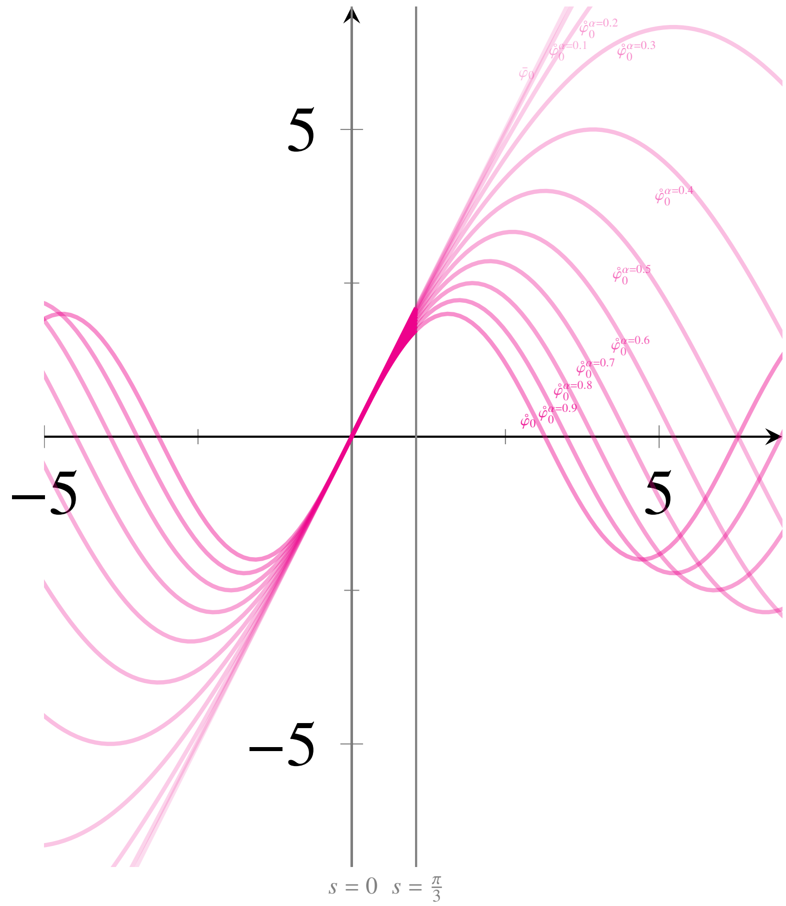
$$\mathring{g}^{\alpha}_{\frac{\pi}{3}} = ds^2 + (\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}})^2\, d\mathring{x}^2 + (\mathring{\psi}^{\alpha}_{\frac{\pi}{3}})^2\, d\mathring{y}^2 + (\mathring{\xi}^{\alpha}_{\frac{\pi}{3}})^2\, d\mathring{z}^2, \tag{5.1.5}$$

where we define $\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}, \mathring{\psi}^{\alpha}_{\frac{\pi}{3}}, \mathring{\xi}^{\alpha}_{\frac{\pi}{3}} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

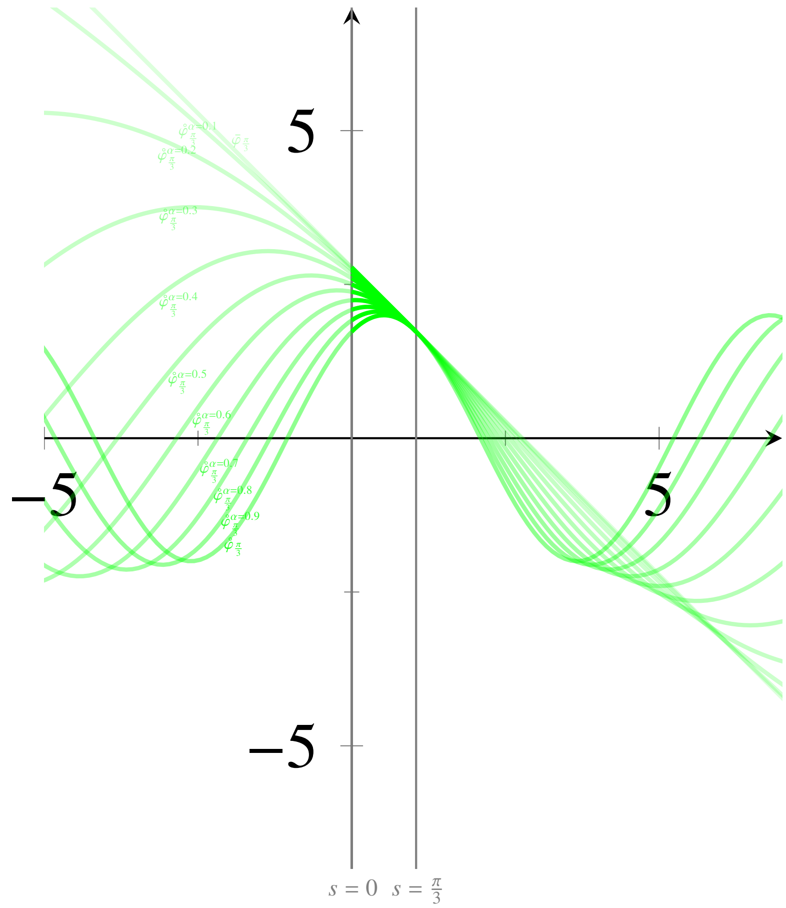
$$\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}(s) := \mathring{\psi}^{\alpha}_{0}\left(s - \frac{\pi}{3}\right), \tag{5.1.6}$$

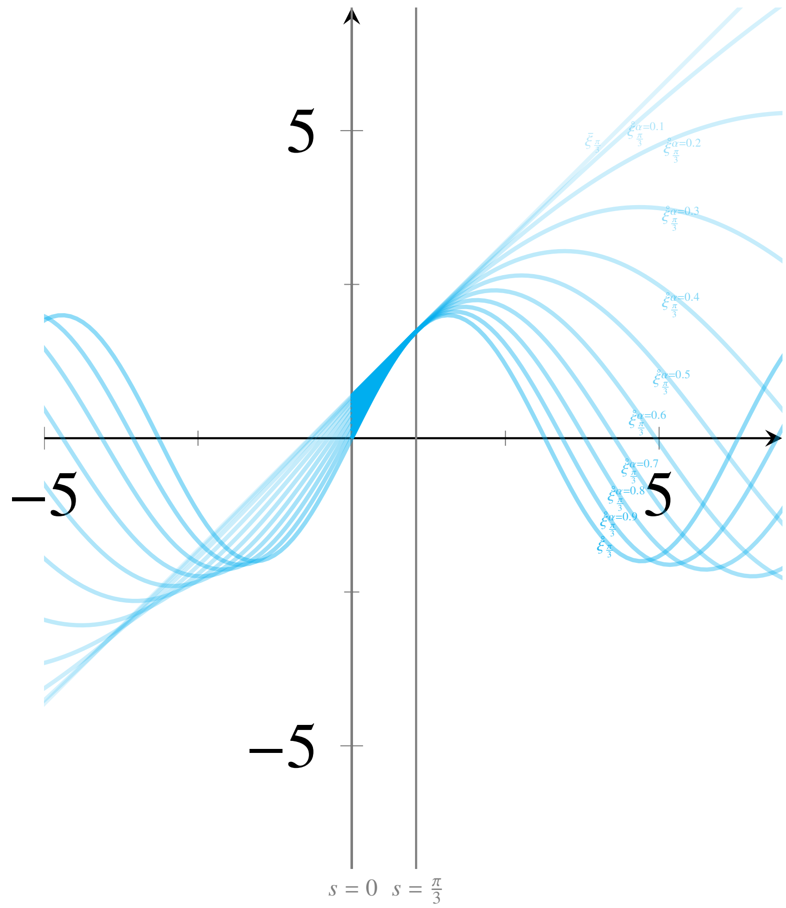
$$\mathring{\psi}^{\alpha}_{\frac{\pi}{3}}(s) := \mathring{\xi}^{\alpha}_{0}\left(s - \frac{\pi}{3}\right), \tag{5.1.7}$$

$$\mathring{\xi}^{\alpha}_{\frac{\pi}{3}}(s) := -\mathring{\varphi}^{\alpha}_{0}\left(s - \frac{\pi}{3}\right). \tag{5.1.8}$$



5
−5
5
−5
$s = 0$
$s = \frac{\pi}{3}$


5
−5
−5
5
$\mathring{\varphi}^{\alpha=0.1}_{\frac{\pi}{3}}$
$\bar{\varphi}_{\frac{\pi}{3}}$
$\mathring{\varphi}^{\alpha=0.2}_{\frac{\pi}{3}}$
$\mathring{\varphi}^{\alpha=0.3}_{\frac{\pi}{3}}$
$\mathring{\varphi}^{\alpha=0.4}_{\frac{\pi}{3}}$
$\mathring{\varphi}^{\alpha=0.5}_{\frac{\pi}{3}}$
$\mathring{\varphi}^{\alpha=0.6}_{\frac{\pi}{3}}$
$\mathring{\varphi}^{\alpha=0.7}_{\frac{\pi}{3}}$
$\mathring{\varphi}^{\alpha=0.8}_{\frac{\pi}{3}}$
$\mathring{\varphi}^{\alpha=0.9}_{\frac{\pi}{3}}$
$\mathring{\varphi}_{\frac{\pi}{3}}$
$s = 0$
$s = \frac{\pi}{3}$



With (5.1.2), (5.1.3), (5.1.4) in mind, we can rewrite (5.1.6), (5.1.7), (5.1.8) explicitly as

$$\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}(s) = \sqrt{3}\cos\left(\alpha\left(s-\frac{\pi}{3}\right)\right) + \frac{1}{\alpha}\sin\left(\alpha\left(s-\frac{\pi}{3}\right)\right), \tag{5.1.9}$$

$$\mathring{\psi}^{\alpha}_{\frac{\pi}{3}}(s) = \sqrt{3}\cos\left(\alpha\left(s-\frac{\pi}{3}\right)\right) - \frac{1}{\alpha}\sin\left(\alpha\left(s-\frac{\pi}{3}\right)\right), \tag{5.1.10}$$

$$\mathring{\xi}^{\alpha}_{\frac{\pi}{3}}(s) = -\frac{2}{\alpha}\sin\left(\alpha\left(s-\frac{\pi}{3}\right)\right), \tag{5.1.11}$$

respectively.

We will state a lemma concerning the smoothness of $\mathring{g}^{\alpha}_{0}$ on a neighborhood of the singular orbit $(\mathrm{SO}(3)\times\mathbb{D}^2)/\mathrm{SO}(3)_{\gamma(0)}$. This lemma will be useful towards the end of Chapter 6 when we construct our final one-parameter family of metrics that will facilitate the proof of Theorem 1.0.3.

Lemma 5.1.1. *For all $\alpha \in \mathbb{R} \setminus \{0\}$, the* SO(3)*-invariant metric $\mathring{g}_0^\alpha$ satisfies the following:*

(1) *It is smooth on a neighborhood of the singular orbit* $(\mathrm{SO}(3) \times \mathbb{D}^2)/\mathrm{SO}(3)_{\gamma(0)}$.

(2) *It is smooth on a neighborhood of the singular orbit* $(\mathrm{SO}(3) \times \mathbb{D}^2)/\mathrm{SO}(3)_{\gamma(\frac{\pi}{3})}$.

Proof. To prove (1), we need to show that

$$\mathring{g}_0^\alpha|_{\mathbb{D}^2} := ds^2 + (\mathring{\varphi}_0^\alpha)^2\, dx^2, \tag{5.1.12}$$

the restriction of $\mathring{g}_0^\alpha$ to $\mathbb{D}^2$, is smooth and that the extended functions $(\mathring{\psi}_0^\alpha)_{\mathrm{ext}}, (\mathring{\xi}_0^\alpha)_{\mathrm{ext}} : [-\frac{\pi}{3}, \frac{\pi}{3}] \to \mathbb{R}$ defined by

$$(\mathring{\psi}_0^\alpha)_{\mathrm{ext}}(s) := \begin{cases} \mathring{\psi}_0^\alpha(s) & \text{for } s \geq 0, \\ \mathring{\xi}_0^\alpha(-s) & \text{for } s < 0 \end{cases} \tag{5.1.13}$$

$$(\mathring{\xi}_0^\alpha)_{\mathrm{ext}}(s) := \begin{cases} \mathring{\xi}_0^\alpha(s) & \text{for } s \geq 0, \\ \mathring{\psi}_0^\alpha(-s) & \text{for } s < 0 \end{cases} \tag{5.1.14}$$

are smooth on $[-\frac{\pi}{3}, \frac{\pi}{3}]$. The metric $\mathring{g}_0^\alpha|_{\mathbb{D}^2}$ given by (5.1.12) is

$$\mathring{g}_0^\alpha|_{\mathbb{D}^2} := ds^2 + (\mathring{\varphi}_0^\alpha)^2\, dx^2, \tag{5.1.15}$$

a rescaling of

$$\mathring{g}_0|_{\mathbb{D}^2} := ds^2 + \mathring{\varphi}_0^2\, dx^2, \tag{5.1.16}$$

the restriction of the round metric $\mathring{g}_0$ to $\mathbb{D}^2$, which is smooth. This implies that $\mathring{g}_0^\alpha|_{\mathbb{D}^2}$ given by (5.1.12) is also smooth. Also, for all $s \in [-\frac{\pi}{3}, \frac{\pi}{3}]$, we have

$$
\begin{aligned}
(\mathring{\psi}_0^\alpha)_{\text{ext}}(s) &= \begin{cases} \mathring{\psi}_0^\alpha(s) & \text{for } s \geq 0, \\ \mathring{\xi}_0^\alpha(-s) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha(-s)) - \frac{1}{\alpha}\sin(\alpha(-s)) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s) & \text{for } s \geq 0, \\ \sqrt{3}\cos(-\alpha s) - \frac{1}{\alpha}\sin(-\alpha s) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}(-\sin(\alpha s)) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s) & \text{for } s < 0 \end{cases} \\
&= \sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s),
\end{aligned}
\tag{5.1.17}
$$

which means $(\mathring{\psi}_0^\alpha)_{\text{ext}}$ is a smooth sinusoidal function on $[-\frac{\pi}{3}, \frac{\pi}{3}]$ for all $\alpha \in \mathbb{R} \setminus \{0\}$.

Similarly, for all $s \in [-\frac{\pi}{3}, \frac{\pi}{3}]$, we have

$$
\begin{aligned}
(\mathring{\xi}_0^\alpha)_{\text{ext}}(s) &= \begin{cases} \mathring{\xi}_0^\alpha(s) & \text{for } s \geq 0, \\ \mathring{\psi}_0^\alpha(-s) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha(-s)) + \frac{1}{\alpha}\sin(\alpha(-s)) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s) & \text{for } s \geq 0, \\ \sqrt{3}\cos(-\alpha s) + \frac{1}{\alpha}\sin(-\alpha s) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}(-\sin(\alpha s)) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s) & \text{for } s < 0 \end{cases} \\
&= \sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s),
\end{aligned}
\tag{5.1.18}
$$

which means $(\mathring{\xi}_0^\alpha)_{\text{ext}}$ is a smooth sinusoidal function on $[-\frac{\pi}{3}, \frac{\pi}{3}]$ for all $\alpha \in \mathbb{R} \setminus \{0\}$. By Theorem 1.3.2, we conclude that $\mathring{g}_0^\alpha$ is smooth.

To prove (2), we need to show that

$$
\mathring{g}_{\frac{\pi}{3}}^\alpha|_{\mathbb{D}^2} := ds^2 + (\mathring{\varphi}_{\frac{\pi}{3}}^\alpha)^2\, dx^2, \tag{5.1.19}
$$

the restriction of $\mathring{g}_{\frac{\pi}{3}}^\alpha$ to $\mathbb{D}^2$, is smooth and that the extended functions $(\mathring{\psi}_{\frac{\pi}{3}}^\alpha)_{\text{ext}}, (\mathring{\xi}_{\frac{\pi}{3}}^\alpha)_{\text{ext}}$:

$[-\frac{\pi}{3}, \frac{\pi}{3}] \to \mathbb{R}$ defined by

$$(\mathring{\psi}^{\alpha}_{\frac{\pi}{3}})_{\text{ext}}(s) := (\mathring{\psi}^{\alpha}_{0})_{\text{ext}}\left(s - \frac{\pi}{3}\right) \\ = \begin{cases} \mathring{\psi}^{\alpha}_{0}(s - \frac{\pi}{3}) & \text{for } s \geq 0, \\ \mathring{\xi}^{\alpha}_{0}(-(s - \frac{\pi}{3})) & \text{for } s < 0 \end{cases} \tag{5.1.20}$$

and

$$(\mathring{\xi}^{\alpha}_{\frac{\pi}{3}})_{\text{ext}}(s) := (\mathring{\xi}^{\alpha}_{0})_{\text{ext}}\left(s - \frac{\pi}{3}\right) \\ = \begin{cases} \mathring{\xi}^{\alpha}_{0}(s - \frac{\pi}{3}) & \text{for } s \geq 0, \\ \mathring{\psi}^{\alpha}_{0}(-(s - \frac{\pi}{3})) & \text{for } s < 0 \end{cases} \tag{5.1.21}$$

are also smooth on $[-\frac{\pi}{3}, \frac{\pi}{3}]$. The metric $\mathring{g}^{\alpha}_{\frac{\pi}{3}}|_{\mathbb{D}^2}$ given by (5.1.19) is

$$\mathring{g}^{\alpha}_{\frac{\pi}{3}}|_{\mathbb{D}^2} := ds^2 + (\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}})^2\, dx^2, \tag{5.1.22}$$

a rescaling of

$$\mathring{g}_{\frac{\pi}{3}}|_{\mathbb{D}^2} := ds^2 + \mathring{\varphi}^2_{\frac{\pi}{3}}\, dx^2, \tag{5.1.23}$$

the restriction of the round metric $\mathring{g}_{\frac{\pi}{3}}$ to $\mathbb{D}^2$, which is smooth. This implies that $\mathring{g}^{\alpha}_{\frac{\pi}{3}}|_{\mathbb{D}^2}$ given by (5.1.12) is also smooth. Also, for all $s \in [-\frac{\pi}{3}, \frac{\pi}{3}]$, we have

$$
\begin{aligned}
(\mathring{\psi}^{\alpha}_{\frac{\pi}{3}})_{\text{ext}}(s) &= \begin{cases} \mathring{\psi}^{\alpha}_{0}(s - \frac{\pi}{3}) & \text{for } s \geq 0, \\ \mathring{\xi}^{\alpha}_{0}(-(s - \frac{\pi}{3})) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha(s - \frac{\pi}{3})) + \frac{1}{\alpha}\sin(\alpha(s - \frac{\pi}{3})) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha(-(s - \frac{\pi}{3}))) - \frac{1}{\alpha}\sin(\alpha(-(s - \frac{\pi}{3}))) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha(s - \frac{\pi}{3})) + \frac{1}{\alpha}\sin(\alpha(s - \frac{\pi}{3})) & \text{for } s \geq 0, \\ \sqrt{3}\cos(-\alpha(s - \frac{\pi}{3})) - \frac{1}{\alpha}\sin(-\alpha(s - \frac{\pi}{3})) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha(s - \frac{\pi}{3})) + \frac{1}{\alpha}\sin(\alpha(s - \frac{\pi}{3})) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha(s - \frac{\pi}{3})) - \frac{1}{\alpha}(-\sin(\alpha(s - \frac{\pi}{3}))) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha(s - \frac{\pi}{3})) + \frac{1}{\alpha}\sin(\alpha(s - \frac{\pi}{3})) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha(s - \frac{\pi}{3})) + \frac{1}{\alpha}\sin(\alpha(s - \frac{\pi}{3})) & \text{for } s < 0 \end{cases} \\
&= \sqrt{3}\cos\left(\alpha\left(s - \frac{\pi}{3}\right)\right) + \frac{1}{\alpha}\sin\left(\alpha\left(s - \frac{\pi}{3}\right)\right),
\end{aligned}
\tag{5.1.24}
$$

which means $(\mathring{\psi}^{\alpha}_{\frac{\pi}{3}})_{\text{ext}}$ is a smooth sinusoidal function on $[-\frac{\pi}{3}, \frac{\pi}{3}]$ for all $\alpha \in \mathbb{R} \setminus \{0\}$.

Similarly, for all $s \in [-\frac{\pi}{3}, \frac{\pi}{3}]$, we have

$$
\begin{aligned}
(\mathring{\xi}^{\alpha}_{\frac{\pi}{3}})_{\text{ext}}(s) &= \begin{cases} \mathring{\xi}^{\alpha}_0(s-\frac{\pi}{3}) & \text{for } s \geq 0, \\ \mathring{\psi}^{\alpha}_0(-(s-\frac{\pi}{3})) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha(s-\frac{\pi}{3})) - \frac{1}{\alpha}\sin(\alpha(s-\frac{\pi}{3})) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha(-(s-\frac{\pi}{3}))) + \frac{1}{\alpha}\sin(\alpha(-(s-\frac{\pi}{3}))) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha(s-\frac{\pi}{3})) - \frac{1}{\alpha}\sin(\alpha(s-\frac{\pi}{3})) & \text{for } s \geq 0, \\ \sqrt{3}\cos(-\alpha(s-\frac{\pi}{3})) + \frac{1}{\alpha}\sin(-\alpha(s-\frac{\pi}{3})) & \text{for } s < 0 \end{cases} \qquad (5.1.25) \\
&= \begin{cases} \sqrt{3}\cos(\alpha(s-\frac{\pi}{3})) - \frac{1}{\alpha}\sin(\alpha(s-\frac{\pi}{3})) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha(s-\frac{\pi}{3})) + \frac{1}{\alpha}(-\sin(\alpha(s-\frac{\pi}{3}))) & \text{for } s < 0 \end{cases} \\
&= \begin{cases} \sqrt{3}\cos(\alpha(s-\frac{\pi}{3})) - \frac{1}{\alpha}\sin(\alpha(s-\frac{\pi}{3})) & \text{for } s \geq 0, \\ \sqrt{3}\cos(\alpha(s-\frac{\pi}{3})) - \frac{1}{\alpha}\sin(\alpha(s-\frac{\pi}{3})) & \text{for } s < 0 \end{cases} \\
&= \sqrt{3}\cos\left(\alpha\left(s-\frac{\pi}{3}\right)\right) - \frac{1}{\alpha}\sin\left(\alpha\left(s-\frac{\pi}{3}\right)\right),
\end{aligned}
$$

which means $(\mathring{\xi}^{\alpha}_{\frac{\pi}{3}})_{\text{ext}}$ is a smooth sinusoidal function on $[-\frac{\pi}{3}, \frac{\pi}{3}]$ for all $\alpha \in \mathbb{R} \setminus \{0\}$. By Theorem 1.3.2, we conclude that $\mathring{g}^{\alpha}_{\frac{\pi}{3}}$ is smooth. □

Now, we turn our attention to showing that $\mathring{g}^{\alpha}_0$ establishes a relationship between the round metric $\mathring{g}_0$ and the linearized metric $\bar{g}_0$. We can also use the modified round metric $\mathring{g}^{\alpha}_{\frac{\pi}{3}}$ to do the same between the round metric $\mathring{g}_{\frac{\pi}{3}}$ and the linearized metric $\bar{g}_{\frac{\pi}{3}}$. The following definition, which we take from Chapter 9 of [16], will state this relationship precisely.

Definition 5.1.2. *Let $f, h : \mathbb{R} \to \mathbb{R}$ be two continuous functions. We say that f is* homotopic *to h if, for any $a \in (0, 1)$, there exists a continuous function $G^a : \mathbb{R} \to \mathbb{R}$ that* *satisfies*

$$\lim_{a\to 0^+} G^a(x) = f(x), \tag{5.1.26}$$

$$\lim_{a\to 1^-} G^a(x) = h(x). \tag{5.1.27}$$

We also call G^a a homotopy *between f and h.*

Proposition 5.1.3. *Consider the round metrics $\mathring{g}_0, \mathring{g}_{\frac{\pi}{3}}$ and the linearized metrics $\bar{g}_0, \bar{g}_{\frac{\pi}{3}}$.*

(1) *The functions of the round metric $\mathring{\varphi}_0, \mathring{\psi}_0, \mathring{\xi}_0 : [0, \frac{\pi}{3}] \to \mathbb{R}$ are homotopic to those of the linearized metric $\bar{\varphi}_0, \bar{\psi}_0, \bar{\xi}_0 : [0, \frac{\pi}{3}] \to \mathbb{R}$, respectively.*

(2) *The functions of the round metric $\mathring{\varphi}_{\frac{\pi}{3}}, \mathring{\psi}_{\frac{\pi}{3}}, \mathring{\xi}_{\frac{\pi}{3}} : [0, \frac{\pi}{3}] \to \mathbb{R}$ are homotopic to those of the linearized metric $\bar{\varphi}_{\frac{\pi}{3}}, \bar{\psi}_{\frac{\pi}{3}}, \bar{\xi}_{\frac{\pi}{3}} : [0, \frac{\pi}{3}] \to \mathbb{R}$, respectively.*

Proof. To prove (1), it suffices to show that $\mathring{\varphi}_0^\alpha, \mathring{\psi}_0^\alpha, \mathring{\xi}_0^\alpha$ given by (5.1.2), (5.1.3), (5.1.4) are homotopies between $\mathring{\varphi}_0, \mathring{\psi}_0, \mathring{\xi}_0$ and $\bar{\varphi}_0, \bar{\psi}_0, \bar{\xi}_0$, respectively. Similarly, to prove (2), it suffices to show that $\mathring{\varphi}_{\frac{\pi}{3}}^\alpha, \mathring{\psi}_{\frac{\pi}{3}}^\alpha, \mathring{\xi}_{\frac{\pi}{3}}^\alpha$ given by (5.1.2), (5.1.3), (5.1.4) are homotopies between $\mathring{\varphi}_{\frac{\pi}{3}}, \mathring{\psi}_{\frac{\pi}{3}}, \mathring{\xi}_{\frac{\pi}{3}}$ and $\bar{\varphi}_{\frac{\pi}{3}}, \bar{\psi}_{\frac{\pi}{3}}, \bar{\xi}_{\frac{\pi}{3}}$, respectively. Since the proofs of (1) and (2) are analogous, we choose to only prove (1). The interested reader can prove (2) as an exercise.

First, we will show that $\mathring{\varphi}_0^\alpha$ given by (5.1.2) is a homotopy of $\mathring{\varphi}_0$ and $\bar{\varphi}_0$. We have

$$\begin{aligned}
\lim_{\alpha\to 0^+} \mathring{\varphi}_0^\alpha(s) &= \lim_{\alpha\to 0} \left(\frac{2}{\alpha}\sin(\alpha s)\right) \\
&= \lim_{\alpha\to 0} \left(\frac{2}{\alpha}\sin(\alpha s)\right)\frac{s}{s} \\
&= 2s \lim_{\alpha\to 0} \frac{\sin(\alpha s)}{\alpha s} \\
&= 2s \cdot 1 \\
&= 2s \\
&= \bar{\varphi}_0(s)
\end{aligned} \tag{5.1.28}$$

and

$$\begin{aligned}\lim_{\alpha\to 1^-}\mathring{\varphi}_0^{\alpha}(s) &= \lim_{\alpha\to 1^-}\left(\frac{2}{\alpha}\sin(\alpha s)\right)\\ &= \frac{2}{1}\sin(1s)\\ &= 2\sin(s)\\ &= \mathring{\varphi}_0(s).\end{aligned}\tag{5.1.29}$$

So we conclude that $\mathring{\varphi}_0^{\alpha}$ is a homotopy of $\mathring{\varphi}_0$ and $\bar{\varphi}_0$.

Next, we will show that $\mathring{\psi}_0^{\alpha}$ given by (5.1.3) is a homotopy of $\mathring{\psi}_0$ and $\bar{\psi}_0$. We have

$$\begin{aligned}\lim_{\alpha\to 0^+}\mathring{\psi}_0^{\alpha} &= \lim_{\alpha\to 0^+}\left(\sqrt{3}\cos(\alpha s)+\frac{1}{\alpha}\sin(\alpha s)\right)\\ &= \lim_{\alpha\to 0^+}\sqrt{3}\cos(\alpha s)+\lim_{\alpha\to 0^+}\left(\frac{1}{\alpha}\sin(\alpha s)\right)\\ &= \sqrt{3}\cos(0s)+\lim_{\alpha\to 0^+}\left(\frac{1}{\alpha}\sin(\alpha s)\right)\frac{s}{s}\\ &= \sqrt{3}\cos(0)+s\lim_{\alpha\to 0^+}\frac{\sin(\alpha s)}{\alpha s}\\ &= \sqrt{3}\cdot 1+s\cdot 1\\ &= \sqrt{3}+s\\ &= \bar{\psi}_0(s)\end{aligned}\tag{5.1.30}$$

and

$$\begin{aligned}\lim_{\alpha\to 1^-}\mathring{\psi}_0^{\alpha}(s) &= \lim_{\alpha\to 1^-}\left(\sqrt{3}\cos(\alpha s)+\frac{1}{\alpha}\sin(\alpha s)\right)\\ &= \sqrt{3}\cos(1s)+\frac{1}{1}\sin(1s)\\ &= \sqrt{3}\cos(s)+\sin(s)\\ &= \mathring{\psi}_0(s).\end{aligned}\tag{5.1.31}$$

So we conclude that $\mathring{\psi}_0^\alpha$ is a homotopy of $\mathring{\psi}_0$ and $\bar{\psi}_0$.

Finally, we will show that $\mathring{\xi}_0^\alpha$ given by (5.1.4) is a homotopy of $\mathring{\xi}_0$ and $\bar{\xi}_0$. We have

$$\begin{aligned}
\lim_{\alpha\to 0^+} \mathring{\xi}_0^\alpha &= \lim_{\alpha\to 0^+}\left(\sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s)\right) \\
&= \lim_{\alpha\to 0^+}\sqrt{3}\cos(\alpha s) - \lim_{\alpha\to 0^+}\left(\frac{1}{\alpha}\sin(\alpha s)\right) \\
&= \sqrt{3}\cos(0s) - \lim_{\alpha\to 0^+}\left(\frac{1}{\alpha}\sin(\alpha s)\right)\frac{s}{s} \\
&= \sqrt{3}\cos(0) - s\lim_{\alpha\to 0^+}\frac{\sin(\alpha s)}{\alpha s} \\
&= \sqrt{3}\cdot 1 - s\cdot 1 \\
&= \sqrt{3} - s \\
&= \bar{\xi}_0(s)
\end{aligned} \tag{5.1.32}$$

and

$$\begin{aligned}
\lim_{\alpha\to 1^-} \mathring{\xi}_0^\alpha(s) &= \lim_{\alpha\to 1^-}\left(\sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s)\right) \\
&= \sqrt{3}\cos(1s) - \frac{1}{1}\sin(1s) \\
&= \sqrt{3}\cos(s) - \sin(s) \\
&= \mathring{\xi}_0(s).
\end{aligned} \tag{5.1.33}$$

So we conclude that $\mathring{\xi}_0^\alpha$ is a homotopy of $\mathring{\psi}_0$ and $\bar{\xi}_0$. □

For the rest of this chapter, we will only focus on the modified round metric $\mathring{g}_0^\alpha$, the round metric $\mathring{g}_0$, and the linearized metric $\bar{g}_0$. Our results for $\mathring{g}_0^\alpha, \mathring{g}_0, \bar{g}_0$ are analogous to those for $\mathring{g}_{\frac{\pi}{3}}^\alpha, \mathring{g}_{\frac{\pi}{3}}, \bar{g}_{\frac{\pi}{3}}$. As part of our efforts to streamline the exposition of this chapter, we will not repeat our results and arguments for $\mathring{g}_{\frac{\pi}{3}}^\alpha, \mathring{g}_{\frac{\pi}{3}}, \bar{g}_{\frac{\pi}{3}}$ in this dissertation. The interested reader can prove the same results of this chapter for $\mathring{g}_{\frac{\pi}{3}}^\alpha, \mathring{g}_{\frac{\pi}{3}}, \bar{g}_{\frac{\pi}{3}}$ as exercises.

5.2 Sectional curvatures for the modified round metric

In this section, we will compute the sectional curvature of S^4 with the modified round metric. In particular, we will verify that the sectional curvature of S^4 with the round metric g_0^α is constant and, in fact, the unit sectional curvature. We will also verify that the normal Riemannian curvatures of S^4 with g_0^α are zero.

Proposition 5.2.1. *Consider S^4 with the modified round metric $\mathring{g}_0^\alpha$ for any $\alpha \in \mathbb{R} \setminus \{0\}$. Then we have the following unnormalized sectional curvatures:*

$$\mathring{R}_0^\alpha\left(\frac{\partial}{\partial s}, \mathring{X}, \mathring{X}, \frac{\partial}{\partial s}\right) = \alpha^2(\mathring{\varphi}_0^\alpha)^2, \tag{5.2.1}$$

$$\mathring{R}_0^\alpha\left(\frac{\partial}{\partial s}, \mathring{Y}, \mathring{Y}, \frac{\partial}{\partial s}\right) = \alpha^2(\mathring{\psi}_0^\alpha)^2, \tag{5.2.2}$$

$$\mathring{R}_0^\alpha\left(\frac{\partial}{\partial s}, \mathring{Z}, \mathring{Z}, \frac{\partial}{\partial s}\right) = \alpha^2(\mathring{\xi}_0^\alpha)^2, \tag{5.2.3}$$

$$\mathring{R}_0^\alpha(\mathring{X}, \mathring{Y}, \mathring{Y}, \mathring{X}) = \alpha^2(\mathring{\varphi}_0^\alpha)^2(\mathring{\psi}_0^\alpha)^2, \tag{5.2.4}$$

$$\mathring{R}_0^\alpha(\mathring{X}, \mathring{Z}, \mathring{Z}, \mathring{X}) = \alpha^2(\mathring{\varphi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2, \tag{5.2.5}$$

$$\mathring{R}_0^\alpha(\mathring{Y}, \mathring{Z}, \mathring{Z}, \mathring{Y}) = \alpha^2(\mathring{\psi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2 + 3(1-\alpha^2)\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha. \tag{5.2.6}$$

Proof. First, we obtain, for all $s \in \mathbb{R}$ and for any $\alpha > 0$,

$$\begin{aligned}
(\mathring{\varphi}_0^\alpha)''(s) &= \left(\frac{2}{\alpha}\sin(\alpha s)\right)'' \\
&= (2\cos(\alpha s))' \\
&= -2\alpha\sin(\alpha s) \\
&= -\alpha^2\left(\frac{2}{\alpha}\sin(\alpha s)\right) \\
&= -\alpha^2\mathring{\varphi}_0^\alpha(s)
\end{aligned} \tag{5.2.7}$$

and

$$
\begin{aligned}
(\mathring{\psi}_0^\alpha)''(s) &= \left(\sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s)\right)'' \\
&= (-\sqrt{3}\alpha\sin(\alpha s) + \cos(\alpha s))' \\
&= -\sqrt{3}\alpha^2\cos(\alpha s) - \alpha\sin(\alpha s) \\
&= -\alpha^2\left(\sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s)\right) \\
&= -\alpha^2\mathring{\psi}_0^\alpha(s)
\end{aligned}
\tag{5.2.8}
$$

and

$$
\begin{aligned}
(\mathring{\xi}_0^\alpha)''(s) &= \left(\sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s)\right)'' \\
&= (-\sqrt{3}\alpha\sin(\alpha s) - \cos(\alpha s))' \\
&= -\sqrt{3}\alpha^2\cos(\alpha s) + \alpha\sin(\alpha s) \\
&= -\alpha^2\left(\sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s)\right) \\
&= -\alpha^2\mathring{\xi}_0^\alpha(s).
\end{aligned}
\tag{5.2.9}
$$

In other words, for any $\alpha > 0$, we have

$$(\mathring{\varphi}_0^\alpha)'' = -\alpha^2\mathring{\varphi}_0^\alpha, \tag{5.2.10}$$

$$(\mathring{\psi}_0^\alpha)'' = -\alpha^2\mathring{\psi}_0^\alpha, \tag{5.2.11}$$

$$(\mathring{\xi}_0^\alpha)'' = -\alpha^2\mathring{\xi}_0^\alpha. \tag{5.2.12}$$

So we conclude

$$\begin{aligned}
\mathring{R}_0^\alpha\left(\frac{\partial}{\partial s}, \mathring{X}, \mathring{X}, \frac{\partial}{\partial s}\right) &= -\mathring{\varphi}_0^\alpha(\mathring{\varphi}_0^\alpha)'' \\
&= -\mathring{\varphi}_0^\alpha(-\alpha^2\mathring{\varphi}_0^\alpha) \\
&= \alpha^2(\mathring{\varphi}_0^\alpha)^2
\end{aligned} \tag{5.2.13}$$

and

$$\begin{aligned}
\mathring{R}_0^\alpha\left(\frac{\partial}{\partial s}, \mathring{Y}, \mathring{Y}, \frac{\partial}{\partial s}\right) &= -\mathring{\psi}_0^\alpha(\mathring{\psi}_0^\alpha)'' \\
&= -\mathring{\psi}_0^\alpha(-\alpha^2\mathring{\psi}_0^\alpha) \\
&= \alpha^2(\mathring{\psi}_0^\alpha)^2
\end{aligned} \tag{5.2.14}$$

and

$$\begin{aligned}
\mathring{R}_0^\alpha\left(\frac{\partial}{\partial s}, \mathring{Z}, \mathring{Z}, \frac{\partial}{\partial s}\right) &= -\mathring{\xi}_0^\alpha(\mathring{\xi}_0^\alpha)'' \\
&= -\mathring{\xi}_0^\alpha(-\alpha^2\mathring{\xi}_0^\alpha) \\
&= \alpha^2(\mathring{\xi}_0^\alpha)^2,
\end{aligned} \tag{5.2.15}$$

which are (5.2.1), (5.2.2), (5.2.3), respectively.

Next, we obtain, for all $s \in \mathbb{R}$,

$$
\begin{aligned}
(\mathring{\varphi}_0^\alpha)'(s)(\mathring{\psi}_0^\alpha)'(s) &= \left(\frac{2}{\alpha}\sin(\alpha s)\right)' \left(\sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s)\right)' \\
&= (2\cos(\alpha s))(-\sqrt{3}\alpha\sin(\alpha s) + \cos(\alpha s)) \\
&= -2\sqrt{3}\alpha\sin(\alpha s)\cos(\alpha s) + 2\cos^2(\alpha s) \\
&= -2\sqrt{3}\alpha\sin(\alpha s)\cos(\alpha s) + 2(1 - \sin^2(\alpha s)) \\
&= -2\sqrt{3}\alpha\sin(\alpha s)\cos(\alpha s) + 2 - 2\sin^2(\alpha s)) \\
&= 2 - 2\sqrt{3}\alpha\sin(\alpha s)\cos(\alpha s) - 2\sin^2(\alpha s)) \\
&= 2 - \alpha^2\left(\frac{2}{\alpha}\sin(\alpha s)\right)\left(\sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s)\right) \\
&= 2 - \alpha^2\mathring{\varphi}_0^\alpha(s)\mathring{\psi}_0^\alpha(s)
\end{aligned}
\tag{5.2.16}
$$

and

$$
\begin{aligned}
(\mathring{\varphi}_0^\alpha)'(s)(\mathring{\xi}_0^\alpha)'(s) &= \left(\frac{2}{\alpha}\sin(\alpha s)\right)' \left(\sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s)\right)' \\
&= (2\cos(\alpha s))(-\sqrt{3}\sin(\alpha s) - \cos(\alpha s)) \\
&= -2\sqrt{3}\sin(\alpha s)\cos(\alpha s) - 2\cos^2(\alpha s) \\
&= -2\sqrt{3}\sin(\alpha s)\cos(\alpha s) - 2(1 - \sin^2(\alpha s)) \\
&= -2\sqrt{3}\sin(\alpha s)\cos(\alpha s) - 2 + 2\sin^2(\alpha s)) \\
&= -2 - 2\sqrt{3}\sin(\alpha s)\cos(\alpha s) + 2\sin^2(\alpha s)) \\
&= -2 - \alpha^2\left(\frac{2}{\alpha}\sin(\alpha s)\right)\left(\sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s)\right) \\
&= -2 - \alpha^2\mathring{\varphi}_0^\alpha(s)\mathring{\xi}_0^\alpha(s)
\end{aligned}
\tag{5.2.17}
$$

and

$$
\begin{aligned}
(\mathring{\psi}_0^\alpha)'(s)(\mathring{\xi}_0^\alpha)'(s) &= \left(\sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s)\right)' \left(\sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s)\right)' \\
&= (-\sqrt{3}\alpha\sin(\alpha s) + \cos(\alpha s))(-\sqrt{3}\alpha\sin(\alpha s) - \cos(\alpha s)) \\
&= 3\alpha^2\sin^2(\alpha s) - \cos^2(\alpha s) \\
&= 3\alpha^2(1-\cos^2(\alpha s)) - (1-\sin^2(\alpha s)) \\
&= 3\alpha^2 - 3\alpha^2\cos^2(\alpha s) - 1 + \sin^2(\alpha s) \\
&= 3\alpha^2 - 1 - 3\alpha^2\cos^2(\alpha s) + \sin^2(\alpha s) \\
&= 3\alpha^2 - 1 - \alpha^2\left(3\cos^2(\alpha s) - \frac{1}{\alpha^2}\sin^2(\alpha s)\right) \\
&= 3\alpha^2 - 1 - \alpha^2\left((\sqrt{3}\cos(\alpha s))^2 - \left(\frac{1}{\alpha}\sin(\alpha s)\right)^2\right) \\
&= 3\alpha^2 - 1 - \alpha^2\left(\sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s)\right) \\
&\qquad\qquad \cdot \left(\sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s)\right) \\
&= 3\alpha^2 - 1 - \alpha^2\mathring{\psi}_0^\alpha(s)\mathring{\xi}_0^\alpha(s).
\end{aligned}
\tag{5.2.18}
$$

In other words, we have

$$(\mathring{\varphi}_0^\alpha)'(\mathring{\psi}_0^\alpha)' = 2 - \alpha^2\mathring{\varphi}_0^\alpha\mathring{\psi}_0^\alpha, \tag{5.2.19}$$

$$(\mathring{\varphi}_0^\alpha)'(\mathring{\xi}_0^\alpha)' = -2 - \alpha^2\mathring{\varphi}_0^\alpha\mathring{\xi}_0^\alpha, \tag{5.2.20}$$

$$(\mathring{\psi}_0^\alpha)'(\mathring{\xi}_0^\alpha)' = 3\alpha^2 - 1 - \alpha^2\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha, \tag{5.2.21}$$

which imply, respectively,

$$\begin{aligned}\mathring{R}(\mathring{X},\mathring{Y},\mathring{Y},\mathring{X}) &= \mathring{\varphi}_0^\alpha\mathring{\psi}_0^\alpha(2-(\mathring{\varphi}_0^\alpha)'(\mathring{\psi}_0^\alpha)')\\ &= \mathring{\varphi}_0^\alpha\mathring{\psi}_0^\alpha(2-(2-\alpha^2\mathring{\varphi}_0^\alpha\mathring{\psi}_0^\alpha))\\ &= \mathring{\varphi}_0^\alpha\mathring{\psi}_0^\alpha(\alpha^2\mathring{\varphi}_0^\alpha\mathring{\psi}_0^\alpha)\\ &= \alpha^2(\mathring{\varphi}_0^\alpha)^2(\mathring{\psi}_0^\alpha)^2\end{aligned} \tag{5.2.22}$$

and

$$\begin{aligned}\mathring{R}(\mathring{X},\mathring{Z},\mathring{Z},\mathring{X}) &= -\mathring{\varphi}_0^\alpha\mathring{\xi}_0^\alpha(2+(\mathring{\varphi}_0^\alpha)'(\mathring{\xi}_0^\alpha)')\\ &= -\mathring{\varphi}_0^\alpha\mathring{\xi}_0^\alpha(2+(-2-\alpha^2\mathring{\varphi}_0^\alpha\mathring{\xi}_0^\alpha))\\ &= -\mathring{\varphi}_0^\alpha\mathring{\xi}_0^\alpha(-\alpha^2\mathring{\varphi}_0^\alpha\mathring{\xi}_0^\alpha)\\ &= \alpha^2(\mathring{\varphi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2\end{aligned} \tag{5.2.23}$$

and

$$\begin{aligned}\mathring{R}(\mathring{Y},\mathring{Z},\mathring{Z},\mathring{Y}) &= \mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha(2-(\mathring{\psi}_0^\alpha)'(\mathring{\xi}_0^\alpha)')\\ &= \mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha(2-(3\alpha^2-1-\alpha^2\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha))\\ &= \mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha(2-3\alpha^2+1+\alpha^2\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha)\\ &= \mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha(3-3\alpha^2+\alpha^2\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha)\\ &= (3-3\alpha^2)\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha+\alpha^2(\mathring{\psi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2\\ &= \alpha^2(\mathring{\psi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2+(3-3\alpha^2)\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha\\ &= \alpha^2(\mathring{\psi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2+3(1-\alpha^2)\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha,\end{aligned} \tag{5.2.24}$$

which are (5.2.4), (5.2.5), (5.2.6), respectively. □

Proposition 5.2.2. *The normal sectional curvature tensors satisfy*

$$\mathring{R}_0^\alpha\left(\mathring{X}, \mathring{Y}, \mathring{Z}, \frac{\partial}{\partial s}\right) = 0, \tag{5.2.25}$$

$$\mathring{R}_0^\alpha\left(\mathring{Y}, \mathring{Z}, \mathring{X}, \frac{\partial}{\partial s}\right) = 0, \tag{5.2.26}$$

$$\mathring{R}_0^\alpha\left(\mathring{Z}, \mathring{X}, \mathring{Y}, \frac{\partial}{\partial s}\right) = 0, \tag{5.2.27}$$

$$\mathring{R}_0^\alpha\left(\mathring{X}, \mathring{Z}, \mathring{Y}, \frac{\partial}{\partial s}\right) = 0, \tag{5.2.28}$$

$$\mathring{R}_0^\alpha\left(\mathring{Z}, \mathring{Y}, \mathring{X}, \frac{\partial}{\partial s}\right) = 0, \tag{5.2.29}$$

$$\mathring{R}_0^\alpha\left(\mathring{Y}, \mathring{X}, \mathring{Z}, \frac{\partial}{\partial s}\right) = 0, \tag{5.2.30}$$

Proof. First, we obtain

$$\begin{aligned}\mathring{\psi}_0^\alpha(s) &= \sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s) \\ &= \frac{2}{\alpha}\sin(\alpha s) + \left(\sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s)\right) \\ &= \mathring{\varphi}_0^\alpha(s) + \mathring{\xi}_0^\alpha(s),\end{aligned} \tag{5.2.31}$$

which is (2.0.5) for the functions associated with $\mathring{g}_0^\alpha$. Now we apply Proposition 3.1.5 to obtain (5.2.25), (5.2.26), (5.2.27). Furthermore, by applying the antisymmetry property of the Riemannian curvature tensor in the first two entries, we see that (5.2.25), (5.2.26), (5.2.27) are equivalent to (5.2.28), (5.2.29), (5.2.30). ◻

Now, we will provide estimates of the sectional curvature of S^4 with the modified round metric $\mathring{g}_0^\alpha$.

Proposition 5.2.3. *Let $\mathring{V}, \mathring{W}$ be $\mathring{g}_0^\alpha$-perpendicular vector fields that are tangent to the orbits* $\mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(s)} \subseteq M$ *for all $s \in [0, \frac{\pi}{3}]$ along the geodesic γ. Then we have the following results about the normalized sectional curvature:*

(1) *For any* $\alpha \in \mathbb{R}$ *with* $|\alpha| \leq 1$ *and* $\alpha \neq 0$*, the normalized sectional curvature of* S^4 *with* $\mathring{g}_0^\alpha$ *satisfies*

$$\alpha^2 \leq \mathring{\sec}_0^\alpha \left(\frac{\partial}{\partial s} + \mathring{V}, \mathring{W} \right) \leq \alpha^2 + \frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha}. \tag{5.2.32}$$

In particular, the normalized sectional curvature of S^4 *with* $\mathring{g}_0^\alpha$ *is positive.*

(2) *For any* $\alpha \in \mathbb{R}$ *with* $|\alpha| \geq 1$*, the normalized sectional curvature of* S^4 *with* $\mathring{g}_0^\alpha$ *satisfies*

$$\alpha^2 - \frac{3(\alpha^2-1)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \leq \mathring{\sec}_0^\alpha \left(\frac{\partial}{\partial s} + \mathring{V}, \mathring{W} \right) \leq \alpha^2. \tag{5.2.33}$$

In particular, the normalized sectional curvature of S^4 *with* $\mathring{g}_0^\alpha$ *is positive if* α *satisfies*

$$1 \leq \alpha^2 < \frac{3}{3 - \mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha}. \tag{5.2.34}$$

Proof. We can write the vector fields $\mathring{V}, \mathring{W}$ as their linear combinations

$$\mathring{V} = \mathring{V}^1 \mathring{X} + \mathring{V}^2 \mathring{Y} + \mathring{V}^3 \mathring{Z}, \tag{5.2.35}$$

$$\mathring{W} = \mathring{W}^1 \mathring{X} + \mathring{W}^2 \mathring{Y} + \mathring{W}^3 \mathring{Z} \tag{5.2.36}$$

for some scalars $\mathring{V}^1, \mathring{V}^2, \mathring{V}^3, \mathring{W}^1, \mathring{W}^2, \mathring{W}^3 \in \mathbb{R}$. By applying (3.2.13) to the modified round metric $\mathring{g}_0^\alpha$, we conclude

$$\begin{aligned}
&\mathring{R}\left(\frac{\partial}{\partial s}+\mathring{V},\mathring{W},\mathring{W},\frac{\partial}{\partial s}+\mathring{V}\right)\\
&\quad=\frac{(\mathring{W}^1)^2}{(\mathring{\varphi}_0^\alpha)^2}\mathring{R}_0^\alpha\left(\frac{\partial}{\partial s},\mathring{X},\mathring{X},\frac{\partial}{\partial s}\right)+\frac{(\mathring{W}^2)^2}{(\mathring{\psi}_0^\alpha)^2}\mathring{R}_0^\alpha\left(\frac{\partial}{\partial s},\mathring{Y},\mathring{Y},\frac{\partial}{\partial s}\right)\\
&\qquad+\frac{(\mathring{W}^1)^2}{(\mathring{\xi}_0^\alpha)^2}\mathring{R}_0^\alpha\left(\frac{\partial}{\partial s},\mathring{Z},\mathring{Z},\frac{\partial}{\partial s}\right)+\frac{(\mathring{V}^1)^2(\mathring{W}^2)^2}{(\mathring{\varphi}_0^\alpha)^2(\mathring{\psi}_0^\alpha)^2}R(\mathring{X},\mathring{Y},\mathring{Y},\mathring{X})\\
&\qquad+\frac{(\mathring{V}^1)^2(\mathring{W}^3)^2}{(\mathring{\varphi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2}R(\mathring{X},\mathring{Z},\mathring{Z},\mathring{X})+\frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{(\mathring{\psi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2}R(\mathring{Y},\mathring{Z},\mathring{Z},\mathring{Y})\\
&\quad=\frac{(\mathring{W}^1)^2}{(\mathring{\varphi}_0^\alpha)^2}\alpha^2(\mathring{\varphi}_0^\alpha)^2+\frac{(\mathring{W}^2)^2}{(\mathring{\psi}_0^\alpha)^2}\alpha^2(\mathring{\psi}_0^\alpha)^2+\frac{(\mathring{W}^1)^2}{(\mathring{\xi}_0^\alpha)^2}\alpha^2(\mathring{\xi}_0^\alpha)^2\\
&\qquad+\frac{(\mathring{V}^1)^2(\mathring{W}^2)^2}{(\mathring{\varphi}_0^\alpha)^2(\mathring{\psi}_0^\alpha)^2}\alpha^2(\mathring{\varphi}_0^\alpha)^2(\mathring{\psi}_0^\alpha)^2+\frac{(\mathring{V}^1)^2(\mathring{W}^3)^2}{(\mathring{\varphi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2}\alpha^2(\mathring{\varphi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2\\
&\qquad+\frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{(\mathring{\psi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2}(\alpha^2(\mathring{\psi}_0^\alpha)^2(\mathring{\xi}_0^\alpha)^2+3(1-\alpha^2)\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha) \qquad (5.2.37)\\
&\quad=\alpha^2((\mathring{W}^1)^2+(\mathring{W}^2)^2+(\mathring{W}^3)^2)\\
&\qquad+\alpha^2((\mathring{V}^1)^2(\mathring{W}^2)^2+(\mathring{V}^1)^2(\mathring{W}^3)^2+(\mathring{V}^2)^2(\mathring{W}^3)^2)\\
&\qquad+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2\\
&\quad=\alpha^2\sum_{i=1}^3(\mathring{W}^i)^2+\alpha^2\sum_{\substack{i,j=1\\i\neq j}}^3(\mathring{V}^i)^2(\mathring{W}^j)^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2\\
&\quad=\alpha^2|\mathring{W}|^2+\alpha^2|\mathring{V}|^2|\mathring{W}|^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2\\
&\quad=\alpha^2(1+|\mathring{V}|^2)|\mathring{W}|^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2.
\end{aligned}$$

We can further rewrite our final expression of (5.2.37) as

$$
\begin{aligned}
&\mathring{R}\left(\frac{\partial}{\partial s}+\mathring{V},\mathring{W},\mathring{W},\frac{\partial}{\partial s}+\mathring{V}\right)\\
&\quad=\alpha^2(1+|\mathring{V}|^2)|\mathring{W}|^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2\\
&\quad=\alpha^2(1^2+0+0+|\mathring{V}|^2)|\mathring{W}|^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2\\
&\quad=\alpha^2\left(\left|\frac{\partial}{\partial s}\right|^2+\mathring{g}\left(\frac{\partial}{\partial s},\mathring{V}\right)+\mathring{g}\left(\mathring{V},\frac{\partial}{\partial s}\right)+|\mathring{V}|^2\right)|\mathring{W}|^2\\
&\qquad+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2 \qquad\qquad (5.2.38)\\
&\quad=\alpha^2\left(\mathring{g}\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right)+\mathring{g}\left(\frac{\partial}{\partial s},\mathring{V}\right)+\mathring{g}\left(\mathring{V},\frac{\partial}{\partial s}\right)+\mathring{g}(\mathring{V},\mathring{V})\right)|\mathring{W}|^2\\
&\qquad+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2\\
&\quad=\alpha^2\mathring{g}\left(\frac{\partial}{\partial s}+\mathring{V},\frac{\partial}{\partial s}+\mathring{V}\right)|\mathring{W}|^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2\\
&\quad=\alpha^2\left|\frac{\partial}{\partial s}+\mathring{V}\right|^2|\mathring{W}|^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2.
\end{aligned}
$$

We claim that $\mathring{W}(\gamma)$ must be a nonzero vector field and either $\frac{\partial}{\partial s}(\gamma)$ or $\mathring{V}(\gamma)$ must also be a nonzero vector field. If, instead, $\mathring{W}(\gamma)$ were zero or both $\frac{\partial}{\partial s}(\gamma)$ and $\mathring{V}(\gamma)$ were zero, then we would have

$$
\begin{aligned}
\operatorname{span}\left\{\left(\frac{\partial}{\partial s}+\mathring{V}\right)(\gamma),W(\gamma)\right\}&=\{0\}\\
&\neq M,
\end{aligned}
\tag{5.2.39}
$$

which contradicts Lemma 3.2.1. As $(\frac{\partial}{\partial s}+\mathring{V})(\gamma)$ and $\mathring{W}(\gamma)$ are nonzero vector fields along γ that are $\mathring{g}_0$-perpendicular to each other, the normalized sectional curvature is well-defined, and its expression is

$$\begin{aligned}\mathrm{s\mathring{e}c}_0^\alpha\left(\frac{\partial}{\partial s}+\mathring{V},\mathring{W}\right) &= \frac{\mathring{R}(\frac{\partial}{\partial s}+\mathring{V},\mathring{W},\frac{\partial}{\partial s}+\mathring{V})}{\mathring{g}(\frac{\partial}{\partial s}+\mathring{V},\frac{\partial}{\partial s}+\mathring{V})\mathring{g}(\mathring{W},\mathring{W})-\mathring{g}(\frac{\partial}{\partial s}+\mathring{V},\mathring{W})^2} \\ &= \frac{\alpha^2|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2-0^2} \\ &= \frac{\alpha^2|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2}{|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2}+\frac{\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2} \\ &= \alpha^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}\frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2}.\end{aligned} \tag{5.2.40}$$

We will use (5.2.40) to help us prove (1) and (2).

First, we will prove (1). For all $\alpha\in\mathbb{R}$ with $|\alpha|\leq 1$, we have $1-\alpha^2\geq 0$, and so we obtain

$$\begin{aligned}\mathrm{s\mathring{e}c}_0^\alpha\left(\frac{\partial}{\partial s}+\mathring{V},\mathring{W}\right) &= \alpha^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}\frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2} \\ &\geq \alpha^2+\frac{3\cdot 0}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}\frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2} \\ &= \alpha^2\end{aligned} \tag{5.2.41}$$

and

$$\begin{aligned}\mathrm{s\mathring{e}c}_0^\alpha\left(\frac{\partial}{\partial s}+\mathring{V},\mathring{W}\right) &= \alpha^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}\frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2} \\ &\leq \alpha^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}\frac{|\mathring{V}|^2|\mathring{W}|^2}{|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2} \\ &\leq \alpha^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}\frac{|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2}{|\frac{\partial}{\partial s}+\mathring{V}|^2|\mathring{W}|^2} \\ &= \alpha^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha\mathring{\xi}_0^\alpha}.\end{aligned} \tag{5.2.42}$$

We combine (5.2.41) and (5.2.42) to conclude (5.2.32), completing our proof of (1).

Next, we will prove (2). For all $\alpha \in \mathbb{R}$ with $|\alpha| \geq 1$, we have $\alpha^2 - 1 \geq 0$, and so we obtain

$$\begin{aligned} \mathring{\sec}_0^\alpha\left(\frac{\partial}{\partial s} + \mathring{V}, \mathring{W}\right) &= \alpha^2 + \frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s} + \mathring{V}|^2|\mathring{W}|^2} \\ &= \alpha^2 - \frac{3(\alpha^2-1)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s} + \mathring{V}|^2|\mathring{W}|^2} \\ &\leq \alpha^2 - \frac{3 \cdot 0}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s} + \mathring{V}|^2|\mathring{W}|^2} \\ &= \alpha^2 \end{aligned} \tag{5.2.43}$$

and

$$\begin{aligned} \mathring{\sec}_0^\alpha\left(\frac{\partial}{\partial s} + \mathring{V}, \mathring{W}\right) &= \alpha^2 + \frac{3(1-\alpha^2)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s} + \mathring{V}|^2|\mathring{W}|^2} \\ &= \alpha^2 - \frac{3(\alpha^2-1)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \frac{(\mathring{V}^2)^2(\mathring{W}^3)^2}{|\frac{\partial}{\partial s} + \mathring{V}|^2|\mathring{W}|^2} \\ &\geq \alpha^2 - \frac{3(\alpha^2-1)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \frac{|\mathring{V}|^2|\mathring{W}|^2}{|\frac{\partial}{\partial s} + \mathring{V}|^2|\mathring{W}|^2} \\ &\geq \alpha^2 - \frac{3(\alpha^2-1)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \frac{|\frac{\partial}{\partial s} + \mathring{V}|^2|\mathring{W}|^2}{|\frac{\partial}{\partial s} + \mathring{V}|^2|\mathring{W}|^2} \\ &= \alpha^2 - \frac{3(\alpha^2-1)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha}. \end{aligned} \tag{5.2.44}$$

We combine (5.2.43) and (5.2.44) to conclude (5.2.33).

We are not yet finished with our proof of (2). We need to also establish (5.2.34). Our assumption $|\alpha| \geq 1$ implies

$$\begin{aligned} \alpha^2 &= |\alpha|^2 \\ &\geq 1^2 \\ &\geq 1, \end{aligned} \tag{5.2.45}$$

which is one inequality of (5.2.34). Now we need to prove

$$\alpha^2 < \frac{3}{3 - \mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha}, \tag{5.2.46}$$

which is the other inequality of (5.2.34). The normalized sectional curvature is guaranteed to be positive if its lower bound in (5.2.34) is also positive. To achieve this, we set

$$\alpha^2 - \frac{3(\alpha^2 - 1)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} > 0. \tag{5.2.47}$$

We can rewrite the left-hand side of (5.2.47) as

$$\begin{aligned} \alpha^2 - \frac{3(\alpha^2 - 1)}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} &= \alpha^2 - \frac{3\alpha^2 - 3}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \\ &= \alpha^2 - \frac{3\alpha^2}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} + \frac{3}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \\ &= \left(1 - \frac{3}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha}\right)\alpha^2 + \frac{3}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \\ &= \left(\frac{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha - 3}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha}\right)\alpha^2 + \frac{3}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} \end{aligned} \tag{5.2.48}$$

so that (5.2.47) becomes

$$\left(\frac{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha - 3}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha}\right)\alpha^2 + \frac{3}{\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha} > 0. \tag{5.2.49}$$

Since $\mathring{\psi}_0^\alpha$ and $\mathring{\xi}_0^\alpha$ are both positive on $[0, \frac{\pi}{3}]$, we can multiply both sides by $\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha$ to obtain

$$(\mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha - 3)\alpha^2 + 3 > 0, \tag{5.2.50}$$

from which we can algebraically rearrange to obtain

$$(3 - \mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha)\alpha^2 < 3. \tag{5.2.51}$$

We claim that the quantity $3 - \mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha$ is nonnegative. Finally, by using the facts $-1 \leq \sin(x) \leq 1$ and $-1 \leq \cos(x) \leq 1$ for all $x \in \mathbb{R}$ and the triangle inequality, we have

$$\begin{aligned}
\mathring{\psi}_0^\alpha(s)\mathring{\xi}_0^\alpha(s) &= |\mathring{\psi}_0^\alpha(s)\mathring{\xi}_0^\alpha(s)| \\
&= \left|\left(\sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s)\right)\left(\sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s)\right)\right| \\
&= \left|3\cos^2(\alpha s) - \frac{1}{\alpha^2}\sin^2(\alpha s)\right| \\
&\leq \left|3\cos^2(\alpha s)\right| + \left|\frac{1}{\alpha^2}\sin^2(\alpha s)\right| \\
&= 3\cos^2(\alpha s) + \frac{1}{\alpha^2}\sin^2(\alpha s) \\
&\leq 3 \cdot 1^2 + \frac{1}{\alpha^2} \cdot 1^2 \\
&= 3 + \frac{1}{\alpha^2}.
\end{aligned} \tag{5.2.52}$$

Since (5.2.52) holds true for all α with $|\alpha| \geq 1$, we can send $\alpha \to \infty$ to conclude

$$\begin{aligned}
\mathring{\psi}_0^\alpha(s)\mathring{\xi}_0^\alpha &\leq 3 + \lim_{\alpha\to\infty}\frac{1}{\alpha^2} \\
&= 3 + 0 \\
&= 3,
\end{aligned} \tag{5.2.53}$$

which implies

$$3 - \mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha \geq 0, \tag{5.2.54}$$

and so $3 - \mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha$ is nonnegative, as we claimed. Consequently, dividing both sides of (5.2.51) by $3 - \mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha$ does not change the inequality sign in (5.2.51). So we conclude that (5.2.51) is equivalent to

$$\alpha^2 < \frac{3}{3 - \mathring{\psi}_0^\alpha \mathring{\xi}_0^\alpha}, \tag{5.2.55}$$

which is (5.2.46). This completes our proof of (2). □

Now, we will verify that the sectional curvature of S^4 with the round metric $\mathring{g}_0$ is 1. We will also investigate the sectional curvature of the linearized metric $\bar{g}_0$.

Corollary 5.2.4. *The four-dimensional sphere S^4 with the round metric $\mathring{g}_0$ has unit sectional curvature. Namely, for any tangent vector fields $\mathring{V}, \mathring{W}$ on S^4, the normalized sectional curvature is constant with*

$$\mathring{\sec}_0 \left(\frac{\partial}{\partial s} + \mathring{V}, \mathring{W} \right) = 1. \tag{5.2.56}$$

Proof. We substitute $\alpha = 1$ into (5.2.32) in order to obtain

$$\begin{aligned} \mathring{\sec}_0 \left(\frac{\partial}{\partial s} + \mathring{V}, \mathring{W} \right) &= \mathring{\sec}_0^{\alpha=1} \left(\frac{\partial}{\partial s} + \mathring{V}, \mathring{W} \right) \\ &\geq \alpha^2|_{\alpha=1} \\ &= 1^2 \\ &= 1 \end{aligned} \tag{5.2.57}$$

and

$$\begin{aligned}
\mathring{\sec}_0\left(\frac{\partial}{\partial s}+\mathring{V},\mathring{W}\right) &= \mathring{\sec}_0^{\alpha=1}\left(\frac{\partial}{\partial s}+\mathring{V},\mathring{W}\right) \\
&\leq \left.\left(\alpha^2+\frac{3(1-\alpha^2)}{\mathring{\psi}_0^{\alpha}\mathring{\xi}_0^{\alpha}}\right)\right|_{\alpha=1} \\
&\leq (1)^2+\frac{3(1-(1)^2)}{\mathring{\psi}_0^{\alpha}\mathring{\xi}_0^{\alpha}} \qquad (5.2.58) \\
&= 1+\frac{3\cdot 0}{\mathring{\psi}_0^{\alpha}\mathring{\xi}_0^{\alpha}} \\
&= 1.
\end{aligned}$$

In other words, we have

$$1 \leq \mathring{\sec}_0\left(\frac{\partial}{\partial s}+\mathring{V},\mathring{W}\right) \leq 1, \qquad (5.2.59)$$

from which we conclude

$$\mathring{\sec}_0\left(\frac{\partial}{\partial s}+\mathring{V},\mathring{W}\right) = 1, \qquad (5.2.60)$$

which is (5.2.56). □

We remark that we can prove Corollary 5.2.4 using either (5.2.32) or (5.2.33).

Corollary 5.2.5. *The four-dimensional sphere S^4 with the linearized metric*

$$\bar{g}_0 := \lim_{\alpha\to 0}\mathring{g}_0^{\alpha} \qquad (5.2.61)$$

has bounded and nonnegative sectional curvature on $[0,\frac{\pi}{3}]$. Namely, the normalized sec-

tional curvature of S^4 with $\bar{g}_0$ satisfies

$$0 \leq \bar{\sec}_0\left(\frac{\partial}{\partial s} + \mathring{V}, \mathring{W}\right) \leq \frac{3}{\bar{\psi}_0\bar{\xi}_0}. \tag{5.2.62}$$

Proof. As we approach $\alpha \to 0$ in (5.2.32), then we obtain

$$\lim_{\alpha\to 0} \alpha^2 \leq \lim_{\alpha\to 0} \mathring{\sec}_0^{\alpha}\left(\frac{\partial}{\partial s} + \mathring{V}, \mathring{W}\right) \leq \lim_{\alpha\to 0}\left(\alpha^2 + \frac{3(1-\alpha^2)}{\mathring{\psi}_0^{\alpha}\mathring{\xi}_0^{\alpha}}\right). \tag{5.2.63}$$

We evaluate the limits in the expressions of (5.2.63) to obtain

$$0^2 \leq \bar{\sec}_0\left(\frac{\partial}{\partial s} + \mathring{V}, \mathring{W}\right) \leq 0^2 + \frac{3(1-0^2)}{\bar{\psi}_0\bar{\xi}_0}, \tag{5.2.64}$$

which we can further simplify to conclude (5.2.62). □

5.3 A metric that yields a negative temporal derivative of sectional curvature

In this section, we construct a metric that has positive sectional curvature and a negative temporal derivative of sectional curvature at initial time for some tangent plane generated by the vector fields $\{\frac{\partial}{\partial s}, X, Y, Z\}$.

For any $m, c \in \mathbb{R}$, we consider the *linearized middle metric*

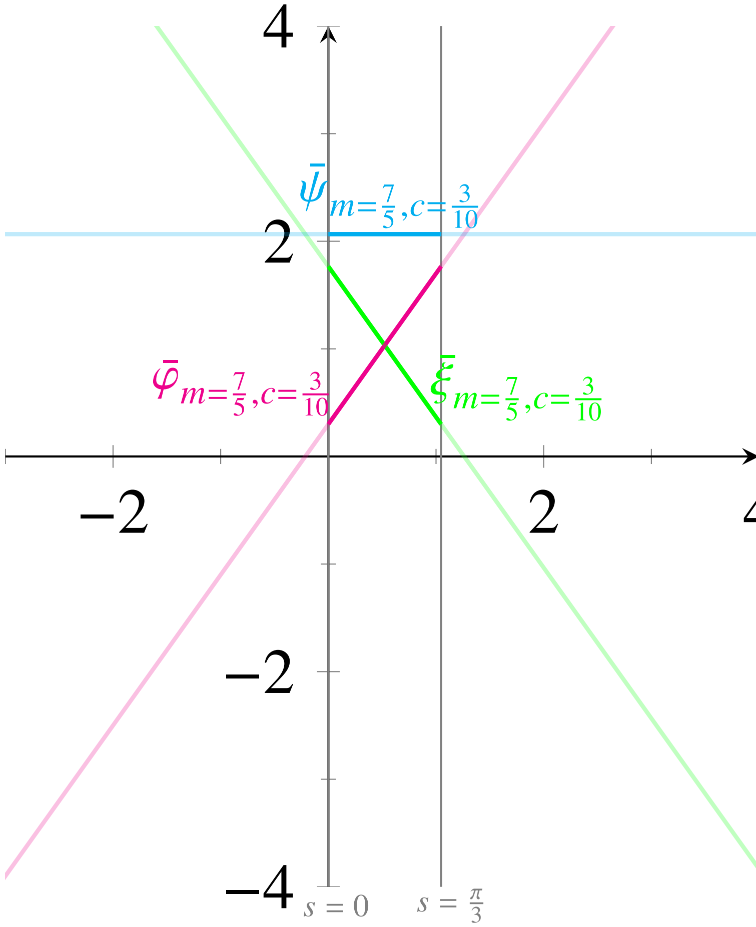
$$\bar{g}_{m,c} = ds^2 + (\bar{\varphi}_{m,c})^2\, dx^2 + (\bar{\psi}_{m,c})^2\, dy^2 + (\bar{\xi}_{m,c})^2\, dz^2, \tag{5.3.1}$$

where we define $\bar{\varphi}_{m,c}, \bar{\psi}_{m,c}, \bar{\xi}_{m,c} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\bar{\varphi}_{m,c}(s) := c + ms \tag{5.3.2}$$

$$\bar{\psi}_{m,c}(s) := \frac{m\pi}{3} + 2c, \tag{5.3.3}$$

$$\bar{\xi}_{m,c}(s) := \frac{m\pi}{3} + c - ms. \tag{5.3.4}$$



By definition, we have

$$\begin{aligned} \bar{\varphi}_{m,c}(s) + \bar{\xi}_{m,c}(s) &= (c + ms) + \left(\frac{m\pi}{3} + c - ms\right) \\ &= \frac{m\pi}{3} + 2c \\ &= \bar{\psi}_{m,c}(s) \end{aligned} \tag{5.3.5}$$

for all $s \in [0, \frac{\pi}{3}]$.

The next lemma will show that we can apply Lemma 1.0.4 for the middle metric $\bar{g}_{m,c}$ and the tangent plane $\sigma := \mathrm{span}(\frac{\partial}{\partial s}, X)$.

Lemma 5.3.1. *Let* $g_{m,c}(t)$ *be a family of* SO(3)*-invariant metrics that satisfies* (4.0.1) *and* *passes through the linearized middle metric $\bar{g}_{m,c}$ at initial time. In other words, suppose the Ricci flow $g_{m,c}(t)$ satisfies*

$$g_{m,c}(0) = \bar{g}_{m,c}. \tag{5.3.6}$$

Then there exist $m \in (-\sqrt{2}, \sqrt{2})$, $c \in \mathbb{R}$, and a sufficiently small $\alpha > 0$ such that, for all $\frac{\pi}{6} - \alpha < s < \frac{\pi}{6} + \alpha$, the middle metric $\bar{g}_{m,c}$ given by (5.3.1) *has zero mixed curvatures and satisfies*

$$\sec_{\bar{g}_{m,c}}\left(\frac{\partial}{\partial s}, X\right) = 0, \tag{5.3.7}$$

$$\frac{\partial}{\partial t}\left(\sec_{g_{m,c}(t)}\left(\frac{\partial}{\partial s}, X\right)\right)\Bigg|_{t=0} < 0. \tag{5.3.8}$$

Proof. First, we will prove (5.3.7). We have

$$\begin{aligned}\left(R_{\bar{g}_{m,c}}\left(\frac{\partial}{\partial s}, X, X, \frac{\partial}{\partial s}\right)\right)(s) &= -\bar{\varphi}_{m,c}(s)\bar{\varphi}''_{m,c}(s) \\ &= -(c+ms)(c+ms)'' \\ &= -(c+ms)m' \\ &= -(c+ms)0 \\ &= 0\end{aligned} \tag{5.3.9}$$

for all $s \in [0, \frac{\pi}{3}]$. In other words,

$$R_{\bar{g}_{m,c}}\left(\frac{\partial}{\partial s}, X, X, \frac{\partial}{\partial s}\right) = 0. \tag{5.3.10}$$

So we conclude

$$\begin{aligned}
\sec_{\bar{g}_{m,c}}\left(\frac{\partial}{\partial s}, X\right) &= \frac{R_{\bar{g}_{m,c}}(\frac{\partial}{\partial s}, X, X, \frac{\partial}{\partial s})}{|\frac{\partial}{\partial s}|^2|X|^2 - g(\frac{\partial}{\partial s}, X)^2} \\
&= \frac{0}{1 \cdot \varphi^2 - 0^2} \\
&= 0,
\end{aligned} \tag{5.3.11}$$

which is (5.3.7).

Next, we will prove (5.3.8). We apply (4.2.10) to (5.3.2), (5.3.3), (5.3.4) in order to obtain

$$\begin{aligned}
(\varphi_{m,c})_t|_{t=0}(s) &= (\varphi_{m,c}|_{t=0})''(s) \\
&\quad + \frac{2\varphi_{m,c}|_{t=0}(s) + (\varphi_{m,c}|_{t=0})'(s)(\psi_{m,c}|_{t=0}(s)\xi_{m,c}|_{t=0}(s))'}{\psi_{m,c}|_{t=0}(s)\xi_{m,c}|_{t=0}(s)} \\
&= (\bar{\varphi}_{m,c})''(s) + \frac{2\bar{\varphi}_{m,c}(s) + (\bar{\varphi}_{m,c})'(s)(\bar{\psi}_{m,c}(s)\bar{\xi}_{m,c}(s))'}{\bar{\psi}_{m,c}(s)\bar{\xi}_{m,c}(s)} \\
&= (c+ms)'' \\
&\quad + \frac{2(c+ms) + (c+ms)'((\frac{m\pi}{3}+2c)(\frac{m\pi}{3}+c-ms))'}{(\frac{m\pi}{3}+2c)(\frac{m\pi}{3}+c-ms)} \\
&= 0 + \frac{2(c+ms) + m(-m(\frac{m\pi}{3}+2c))}{(\frac{m\pi}{3}+2c)(\frac{m\pi}{3}+c-ms)} \\
&= \frac{2c+2ms-m^2(\frac{m\pi}{3}+2c)}{(\frac{m\pi}{3}+2c)(\frac{m\pi}{3}+c-ms)}
\end{aligned} \tag{5.3.12}$$

for all $s \in [0, \frac{\pi}{3}]$. Then the first spatial partial derivative is given by the expression

$$\begin{aligned}
(\varphi_{m,c})_{st}|_{t=0}(s) &= \left(\frac{2c + 2ms - m^2(\frac{m\pi}{3} + 2c)}{(\frac{m\pi}{3} + 2c)(\frac{m\pi}{3} + c - ms)}\right)' \\
&= \frac{1}{\frac{m\pi}{3} + 2c}\left(\frac{2c + 2ms - m^2(\frac{m\pi}{3} + 2c)}{(\frac{m\pi}{3} + c - ms)}\right)' \\
&= \frac{1}{\frac{m\pi}{3} + 2c}\left(\frac{(2c + 2ms - m^2(\frac{m\pi}{3} + 2c))'(\frac{m\pi}{3} + c - ms)}{(\frac{m\pi}{3} + c - ms)^2}\right. \\
&\qquad\left. - \frac{(2c + 2ms - m^2(\frac{m\pi}{3} + 2c))(\frac{m\pi}{3} + c - ms)'}{(\frac{m\pi}{3} + c - ms)^2}\right) \\
&= \frac{1}{\frac{m\pi}{3} + 2c}\left(\frac{(2m)(\frac{m\pi}{3} + c - ms)}{(\frac{m\pi}{3} + c - ms)^2}\right. \\
&\qquad\left. - \frac{(2c + 2ms - m^2(\frac{m\pi}{3} + 2c))(-m)}{(\frac{m\pi}{3} + c - ms)^2}\right) \\
&= \frac{1}{\frac{m\pi}{3} + 2c}\frac{\frac{2m^2\pi}{3} + 2mc - 2m^2s + 2mc + 2m^2s - m^3(\frac{m\pi}{3} + 2c)}{(\frac{m\pi}{3} + c - ms)^2} \\
&= \frac{1}{\frac{m\pi}{3} + 2c}\frac{\frac{2m^2\pi}{3} + 4mc - m^3(\frac{m\pi}{3} + 2c)}{(\frac{m\pi}{3} + c - ms)^2} \\
&= \frac{1}{\frac{m\pi}{3} + 2c}\frac{2m(\frac{m\pi}{3} + 2c) - m^3(\frac{m\pi}{3} + 2c)}{(\frac{m\pi}{3} + c - ms)^2} \\
&= \frac{1}{\frac{m\pi}{3} + 2c}\frac{(2m - m^3)(\frac{m\pi}{3} + 2c)}{(\frac{m\pi}{3} + c - ms)^2} \\
&= \frac{2m - m^3}{(\frac{m\pi}{3} + c - ms)^2} \\
&= \frac{2m - m^3}{\bar{\xi}_{m,c}(s)^2}
\end{aligned} \tag{5.3.13}$$

for all $s \in [0, \frac{\pi}{3}]$. The second spatial partial derivative is given by the expression

$$
\begin{aligned}
(\varphi_{m,c})_{sst}|_{t=0}(s) &= \left(\frac{2m - m^3}{\bar{\xi}_{m,c}(s)^2}\right)' \\
&= -\frac{2(2m - m^3)\bar{\xi}'_{m,c}(s)}{\bar{\xi}_{m,c}(s)^3} \\
&= -\frac{(4m - 2m^3)(-m)}{\bar{\xi}_{m,c}(s)^3} \\
&= \frac{4m^2 - 2m^4}{\bar{\xi}_{m,c}(s)^3}
\end{aligned}
\tag{5.3.14}
$$

for all $s \in [0, \frac{\pi}{3}]$. In other words, we obtain

$$
(\varphi_{m,c})_{sst}|_{t=0} = \frac{4m^2 - 2m^4}{\bar{\xi}^3_{m,c}}. \tag{5.3.15}
$$

At $t = 0$, we also have, according to (4.2.7),

$$
\begin{aligned}
\zeta_t|_{t=0} &= \frac{\varphi_{ss}|_{t=0}}{\varphi|_{t=0}} + \frac{\psi_{ss}|_{t=0}}{\psi|_{t=0}} + \frac{\xi_{ss}|_{t=0}}{\xi|_{t=0}} \\
&= \frac{\bar{\varphi}''_{m,c}}{\bar{\varphi}_{m,c}} + \frac{\bar{\psi}''_{m,c}}{\bar{\psi}_{m,c}} + \frac{\bar{\xi}''_{m,c}}{\bar{\xi}_{m,c}} \\
&= \frac{0}{\bar{\varphi}_{m,c}} + \frac{0}{\bar{\psi}_{m,c}} + \frac{0}{\bar{\xi}_{m,c}} \\
&= 0.
\end{aligned}
\tag{5.3.16}
$$

According to (4.3.8), we conclude

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\sec\left(\frac{\partial}{\partial s}, X\right)\right)\Big|_{t=0} &= \frac{\zeta_{st}|_{t=0}\varphi_s|_{t=0} - \zeta_t|_{t=0}\varphi_{ss}|_{t=0} - \varphi_{sst}|_{t=0} + \varphi_{ss}|_{t=0}}{\varphi|_{t=0}} \\
&= \frac{0_s\bar{\varphi}'_{m,c} - 0\bar{\varphi}''_{m,c} - (\varphi_{m,c})_{sst}|_{t=0} + \bar{\varphi}''_{m,c}}{\bar{\varphi}_{m,c}} \\
&= \frac{-(\varphi_{m,c})_{sst}|_{t=0} + \bar{\varphi}''_{m,c}}{\bar{\varphi}_{m,c}} \\
&= \frac{-\frac{4m^2-2m^4}{\bar{\xi}^3_{m,c}} + 0}{\bar{\varphi}_{m,c}} \\
&= \frac{2m^2(m^2-2)}{\bar{\varphi}_{m,c}\bar{\xi}^3_{m,c}} \\
&< 0,
\end{aligned}
\tag{5.3.17}
$$

which is (5.3.8), provided that we assume $m \in (-\sqrt{2}, \sqrt{2})$. In fact, because of the continuity of $\frac{\partial}{\partial t}(\sec(\frac{\partial}{\partial s}, X))|_{t=0}$ in s, for any $0 < k < \frac{\sqrt{2}}{2}$ and for all $m \in (-\sqrt{2} + k, \sqrt{2} - k)$, there exists a constant $C^k_{m,c} < 0$ that satisfies

$$
\frac{\partial}{\partial t}\left(\sec_{g_{m,c}(t)}\left(\frac{\partial}{\partial s}, X\right)\right)\Big|_{t=0} \leq C^k_{m,c}. \tag{5.3.18}
$$

Therefore, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\sec_{g_{m,c}(t)}\left(\frac{\partial}{\partial s}, X\right)\right)\Big|_{t=0} &\leq C^k_{m,c} \\
&< 0,
\end{aligned}
\tag{5.3.19}
$$

which is (5.3.8). □

We will also deform the metric $\bar{g}_{m,c}$, so that the new metric has positive sectional curvature. For any $m, c \in \mathbb{R}$, we consider the metric

$$\bar{g}^{\epsilon}_{m,c} = ds^2 + (\bar{\varphi}^{\epsilon}_{m,c})^2\, dx^2 + (\bar{\psi}^{\epsilon}_{m,c})^2\, dy^2 + (\bar{\xi}^{\epsilon}_{m,c})^2\, dz^2, \tag{5.3.20}$$

where we define $\bar{\varphi}^{\epsilon}_{m,c}, \bar{\psi}^{\epsilon}_{m,c}, \bar{\xi}^{\epsilon}_{m,c} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\bar{\varphi}^{\epsilon}_{m,c}(s) := c + ms - \epsilon s^2, \tag{5.3.21}$$

$$\bar{\psi}^{\epsilon}_{m,c}(s) := \frac{m\pi}{3} + 2c - 2\epsilon s^2, \tag{5.3.22}$$

$$\bar{\xi}^{\epsilon}_{m,c}(s) := \frac{m\pi}{3} + c - ms - \epsilon s^2. \tag{5.3.23}$$

For any sufficiently small $\epsilon > 0$, the graphs of $\bar{\varphi}^{\epsilon}_{m,c}, \bar{\psi}^{\epsilon}_{m,c}, \bar{\xi}^{\epsilon}_{m,c}$ are almost identical to those of $\bar{\varphi}_{m,c}, \bar{\psi}_{m,c}, \bar{\xi}_{m,c}$, respectively. So we will not print the graphs of $\bar{\varphi}^{\epsilon}_{m,c}, \bar{\psi}^{\epsilon}_{m,c}, \bar{\xi}^{\epsilon}_{m,c}$ here.

In particular, we have

$$\begin{aligned}\bar{\varphi}^{\epsilon}_{m,c}(s) + \bar{\xi}^{\epsilon}_{m,c}(s) &= (c + ms - \epsilon s^2) + \left(\frac{m\pi}{3} + c - ms - \epsilon s^2\right) \\ &= \frac{m\pi}{3} + 2c - 2\epsilon s^2 \\ &= \bar{\psi}^{\epsilon}_{m,c}(s)\end{aligned} \tag{5.3.24}$$

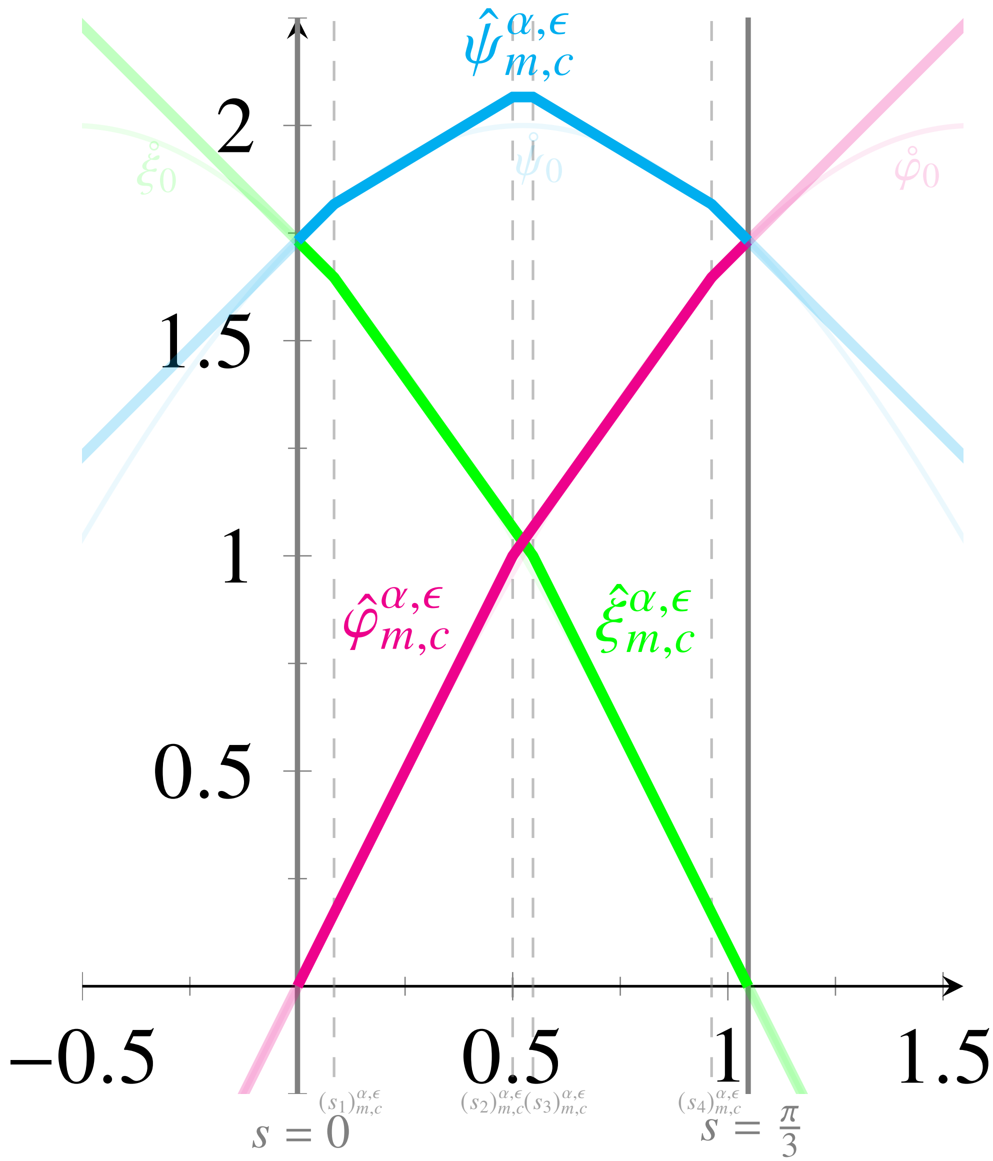
for all $s \in [0, \frac{\pi}{3}]$. That means, according to Proposition 3.1.5, all normal sectional curvatures associated with $\bar{g}^{\epsilon}_{m,c}$ are zero.

Chapter 6

Construction of a family of smooth metrics

The goal of this final chapter is to construct a one-parameter familly of metrics that will allow us to complete our proof of Theorem 1.0.3. We will also need to prove Lemma 1.0.5, which will fulfill Step 1 of our procedure that we outlined in Chapter 1 between Lemma 1.0.4 and Lemma 1.0.5. This will show in particular that the temporal derivative of a radial sectional curvature of some tangent plane associated with our proposed metric is negative.

To remind the reader, we will show again the graphs of the functions that we will construct in this chapter.



The next theorem is an elaboration of Lemma 1.0.5. In other words, Lemma 1.0.5 follows from Theorem 6.0.1.

Theorem 6.0.1. *There exists g on S^4 of the form given by* (1.0.8) *with the following properties:*

(1) *The functions φ, ψ, ξ are piecewise linear and concave down.*

(2) *The boundary values and their first derivatives at those values are equal to those of the round metric $\mathring{g}_0$ defined by* (5.0.1)*. In other words, for $k = 0, 1$, the functions φ, ψ, ξ satisfy*

$$\varphi^{(k)}(0) = (\mathring{\varphi}_0^{\alpha})^{(k)}(0), \tag{6.0.1}$$

$$\psi^{(k)}(0) = (\mathring{\psi}_0^{\alpha})^{(k)}(0), \tag{6.0.2}$$

$$\xi^{(k)}(0) = (\mathring{\xi}_0^{\alpha})^{(k)}(0), \tag{6.0.3}$$

$$\varphi^{(k)}\left(\frac{\pi}{3}\right) = (\mathring{\varphi}_0^{\alpha})^{(k)}\left(\frac{\pi}{3}\right), \tag{6.0.4}$$

$$\psi^{(k)}\left(\frac{\pi}{3}\right) = (\mathring{\psi}_0^{\alpha})^{(k)}\left(\frac{\pi}{3}\right), \tag{6.0.5}$$

$$\xi^{(k)}\left(\frac{\pi}{3}\right) = (\mathring{\xi}_0^{\alpha})^{(k)}\left(\frac{\pi}{3}\right). \tag{6.0.6}$$

(3) *There exists $t_0 \in (0, \frac{\pi}{3})$ such that, for any point $p \in \mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(t_0)}$, the functions φ, ψ, ξ are smooth on a sufficiently small neighborhood of t_0, and there exists a tangent plane $\sigma \in T_pS^4$ that satisfies*

$$\sec_{g(0)}(\sigma) = 0, \tag{6.0.7}$$

$$(\sec_{g(t)}(\sigma))_t|_{t=0} < 0, \tag{6.0.8}$$

where $g(t)$ solves Ricci flow near (t_0, p) whose initial metric is $g(0) = g$.

6.1 Set of admissible Riemannian metrics

We introduce for S^4 the deformed continuous Riemannian metric

$$\hat{g}^{\alpha,\epsilon}_{m,c} := ds^2 + (\hat{\varphi}^{\alpha,\epsilon}_{m,c})^2\, dx^2 + (\hat{\psi}^{\alpha,\epsilon}_{m,c})^2\, dy^2 + (\hat{\xi}^{\alpha,\epsilon}_{m,c})^2\, dz^2 \tag{6.1.1}$$

for any $m, c \in \mathbb{R}$ and a sufficiently small $\epsilon > 0$, where we define $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\epsilon}_{m,c} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\hat{\varphi}^{\alpha,\epsilon}_{m,c}(s) := \min\{\mathring{\varphi}^{\alpha}_{0}(s), \bar{\varphi}^{\epsilon}_{m,c}(s), \mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}(s)\}, \tag{6.1.2}$$

$$\hat{\xi}^{\alpha,\epsilon}_{m,c}(s) := \min\{\mathring{\xi}^{\alpha}_{0}(s), \bar{\xi}^{\epsilon}_{m,c}(s), \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}(s)\}, \tag{6.1.3}$$

where we have previously defined $\mathring{\varphi}^{\alpha}_{0}, \mathring{\psi}^{\alpha}_{0}, \mathring{\xi}^{\alpha}_{0}, \mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}, \mathring{\psi}^{\alpha}_{\frac{\pi}{3}}, \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}$ by (5.1.2), (5.1.3), (5.1.4), (5.1.6), (5.1.7), (5.1.8), respectively, and where we define $\hat{\psi}^{\alpha,\epsilon}_{m,c} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$, \hat{\psi}^{\alpha,\epsilon}_{m,c}(s) := \hat{\varphi}^{\alpha,\epsilon}_{m,c}(s) + \hat{\xi}^{\alpha,\epsilon}_{m,c}(s). \tag{6.1.4}$$

Let $(s_1)^{\alpha,\epsilon}_{m,c}, (s_2)^{\alpha,\epsilon}_{m,c}, (s_3)^{\alpha,\epsilon}_{m,c}, (s_4)^{\alpha,\epsilon}_{m,c}$ be values that solve the equations

$$\mathring{\xi}^{\alpha}_{0}((s_1)^{\alpha,\epsilon}_{m,c}) = \bar{\xi}^{\epsilon}_{m,c}((s_1)^{\alpha,\epsilon}_{m,c}), \tag{6.1.5}$$

$$\mathring{\varphi}^{\alpha}_{0}((s_2)^{\alpha,\epsilon}_{m,c}) = \bar{\varphi}^{\epsilon}_{m,c}((s_2)^{\alpha,\epsilon}_{m,c}), \tag{6.1.6}$$

$$\bar{\xi}^{\epsilon}_{m,c}((s_3)^{\alpha,\epsilon}_{m,c}) = \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}((s_3)^{\alpha,\epsilon}_{m,c}), \tag{6.1.7}$$

$$\bar{\varphi}^{\epsilon}_{m,c}((s_4)^{\alpha,\epsilon}_{m,c}) = \mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}((s_4)^{\alpha,\epsilon}_{m,c}), \tag{6.1.8}$$

respectively. Note that it is possible, but not necessary for this dissertation, to numerically compute the decimal approximations of $(s_1)^{\alpha,\epsilon}_{m,c}, (s_2)^{\alpha,\epsilon}_{m,c}, (s_3)^{\alpha,\epsilon}_{m,c}, (s_4)^{\alpha,\epsilon}_{m,c}$ from (6.1.5), (6.1.6), (6.1.7), (6.1.8), respectively.

Now, we consider the admissible set

$$A^{\alpha,\epsilon}_{m,c} := \left\{\hat{g}^{\alpha,\epsilon}_{m,c} : 0 < (s_1)^{\alpha,\epsilon}_{m,c} < (s_2)^{\alpha,\epsilon}_{m,c} < (s_3)^{\alpha,\epsilon}_{m,c} < (s_4)^{\alpha,\epsilon}_{m,c} < \frac{\pi}{3}\right\}. \tag{6.1.9}$$

Observe that, if we assume $\hat{g}^{\alpha,\epsilon}_{m,c} \in A^{\alpha,\epsilon}_{m,c}$, where we have defined $A^{\alpha,\epsilon}_{m,c}$ by (6.1.9), then we can express $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\epsilon}_{m,c}$ more explicitly as

$$\hat{\varphi}^{\alpha,\epsilon}_{m,c}(s) = \begin{cases} \mathring{\varphi}^{\alpha}_{0}(s) & \text{for all } 0 \le s \le (s_2)^{\alpha,\epsilon}_{m,c}, \\ \bar{\varphi}^{\epsilon}_{m,c}(s) & \text{for all } (s_2)^{\alpha,\epsilon}_{m,c} \le s \le (s_4)^{\alpha,\epsilon}_{m,c}, \\ \mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}(s) & \text{for all } (s_4)^{\alpha,\epsilon}_{m,c} \le s \le \frac{\pi}{3}, \end{cases} \tag{6.1.10}$$

$$\hat{\psi}^{\alpha,\epsilon}_{m,c}(s) = \begin{cases} \mathring{\psi}^{\alpha}_{0}(s) & \text{for all } 0 \le s \le (s_1)^{\alpha,\epsilon}_{m,c}, \\ \mathring{\varphi}^{\alpha}_{0}(s) + \bar{\xi}^{\epsilon}_{m,c}(s) & \text{for all } (s_1)^{\alpha,\epsilon}_{m,c} \le s \le (s_2)^{\alpha,\epsilon}_{m,c}, \\ \bar{\psi}^{\epsilon}_{m,c}(s) & \text{for all } (s_2)^{\alpha,\epsilon}_{m,c} \le s \le (s_3)^{\alpha,\epsilon}_{m,c}, \\ \bar{\varphi}^{\epsilon}_{m,c}(s) + \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}(s) & \text{for all } (s_3)^{\alpha,\epsilon}_{m,c} \le s \le (s_4)^{\alpha,\epsilon}_{m,c}, \\ \mathring{\psi}^{\alpha}_{\frac{\pi}{3}}(s) & \text{for all } (s_4)^{\alpha,\epsilon}_{m,c} \le s \le \frac{\pi}{3}, \end{cases} \tag{6.1.11}$$

$$\hat{\xi}^{\alpha,\epsilon}_{m,c}(s) = \begin{cases} \mathring{\xi}^{\alpha}_{0}(s) & \text{for all } 0 \le s \le (s_1)^{\alpha,\epsilon}_{m,c}, \\ \bar{\xi}^{\epsilon}_{m,c}(s) & \text{for all } (s_1)^{\alpha,\epsilon}_{m,c} \le s \le (s_3)^{\alpha,\epsilon}_{m,c}, \\ \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}(s) & \text{for all } (s_3)^{\alpha,\epsilon}_{m,c} \le s \le \frac{\pi}{3}. \end{cases} \tag{6.1.12}$$

It would be impractical for us to print the graphs of $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\varphi}^{\alpha,\epsilon}_{m,c}$, as they appear to be very similar to those of $\hat{\varphi}_{m,c}, \hat{\varphi}_{m,c}, \hat{\varphi}_{m,c}$, respectively. We remind the reader that the graphs of $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\varphi}^{\alpha,\epsilon}_{m,c}$ are very close to those of $\hat{\varphi}_{m,c}, \hat{\varphi}_{m,c}, \hat{\varphi}_{m,c}$, except at the cusps.

6.2 Mollifiers and gluing functions

We will dedicate this section to constructing a smooth Riemannian metric on S^4. To achieve this, we will need to smooth the two cusps that occur in the graph of $\hat{\varphi}_{m,c}^{\alpha,\epsilon}$ at $s = (s_2)_{m,c}^{\alpha,\epsilon}, (s_4)_{m,c}^{\alpha,\epsilon}$ and in the graph of $\hat{\xi}_{m,c}^{\alpha,\epsilon}$ at $s = (s_1)_{m,c}^{\alpha,\epsilon}, (s_3)_{m,c}^{\alpha,\epsilon}$. Thanks to (2.0.5) for $\hat{\varphi}_{m,c}^{\alpha,\epsilon}, \hat{\xi}_{m,c}^{\alpha,\epsilon}$, these processes will also take care of smoothing the four cusps that occur in the graph of $\hat{\psi}_{m,c}^{\alpha,\epsilon}$ at $s = (s_1)_{m,c}^{\alpha,\epsilon}, (s_2)_{m,c}^{\alpha,\epsilon}, (s_3)_{m,c}^{\alpha,\epsilon}, (s_4)_{m,c}^{\alpha,\epsilon}$. One method of smoothing all the cusps is introducing mollifiers, which are smooth, compactly supported, and integrable functions that can be convolved with any non-smooth function to introduce its smooth approximation.

We define the standard mollifier $\eta : \mathbb{R} \to \mathbb{R}$ by

$$\eta(x) = \begin{cases} Ce^{\frac{1}{x^2-1}} & \text{for all } -1 < x < 1, \\ 0 & \text{for all } x \geq 1 \text{ or } x \leq -1, \end{cases} \tag{6.2.1}$$

where $C \in \mathbb{R}$ is some constant that satisfies

$$\int_{-\infty}^{\infty} \eta(x)\, dx = 1. \tag{6.2.2}$$

Consider for any $\delta > 0$ the standard mollifier $\eta^\delta : \mathbb{R} \to \mathbb{R}$ given by

$$\eta^\delta(x) := \frac{1}{\delta}\eta\left(\frac{x}{\delta}\right). \tag{6.2.3}$$

Now, we define for any $\delta > 0$ the mollification $f^\delta : \mathbb{R} \to \mathbb{R}$ by the convolution of η^δ and

f; namely, we define

$$f^\delta(x) := (\eta^\delta * f)(x) \\ = \int_{-\infty}^{\infty} \eta^\delta(y) f(x-z)\, dz \\ = \int_{-\infty}^{\infty} \eta^\delta(x-z) f(z)\, dz. \tag{6.2.4}$$

According to (6.2.1), we notice that η has compact support on $\mathbb{R}$ since we have $\eta(x) = 0$ for all $x \in \mathbb{R} \setminus (-1, 1)$. In turn, according to (6.2.2), we notice that η^δ also has compact support on $\mathbb{R}$ since we have $\eta^\delta(x) = 0$ for all $x \in \mathbb{R} \setminus (-\delta, \delta)$. As the author of this dissertation, I have borrowed η and these properties from Section C.5 in Appendix C of [8].

Proposition 6.2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and Lebesgue integrable function. Then the mollification $f^\delta := \eta^\delta * f$ is smooth on $\mathbb{R}$; that is, for all integers $k \geq 0$, its k^{th} derivative $(f^\delta)^{(k)}$ exists and is given by the expression*

$$(f^\delta)^{(k)}(x) = ((\eta^\delta)^{(k)} * f)(x) \tag{6.2.5}$$

for all $x \in \mathbb{R}$. Furthermore, if $f^{(k)}$ is continuous on an open interval $(a, b) \subseteq \mathbb{R}$, then $(f^\delta)^{(k)}$ converges uniformly to $f^{(k)}$ on the closed subinterval $[\tilde{a}, \tilde{b}] \subseteq (a, b)$ for $\delta \to 0^+$.

Proof. We will use a proof by induction. First, we will prove the statement for $k = 1$; that is, we will prove

$$(f^\delta)'(x) = ((\eta^\delta)' * f)(x). \tag{6.2.6}$$

For any sufficiently small $h > 0$, we obtain the difference quotient

$$\begin{aligned}
\frac{f^\delta(x+h) - f^\delta(x)}{h} &= \frac{1}{h}\left(\int_{-\infty}^{\infty} \eta^\delta(x+h-z) f(z)\, dz - \int_{-\infty}^{\infty} \eta^\delta(x-z) f(z)\, dz\right) \\
&= \frac{1}{h}\left(\int_{-\infty}^{\infty} \frac{1}{\delta}\eta\left(\frac{x+h-z}{\delta}\right) f(z)\, dz \right. \\
&\qquad \left. - \int_{-\infty}^{\infty} \frac{1}{\delta}\eta\left(\frac{x-z}{\delta}\right) f(z)\, dz\right) \\
&= \frac{1}{h}\left(\int_{-\infty}^{\infty} \frac{1}{\delta}\eta\left(\frac{x+h-z}{\delta}\right) f(z) - \frac{1}{\delta}\eta\left(\frac{x-z}{\delta}\right) f(z)\, dz\right) \\
&= \int_{-\infty}^{\infty} \frac{1}{\delta h}\left(\eta\left(\frac{x+h-z}{\delta}\right) - \eta\left(\frac{x-z}{\delta}\right)\right) f(z)\, dz.
\end{aligned} \tag{6.2.7}$$

Since η is smooth and compactly supported, it follows in particular that η is also Lipschitz continuous. So there exists a Lipschitz constant L_η that satisfies

$$|\eta(x) - \eta(y)| \leq L_\eta |x - y| \tag{6.2.8}$$

for all $x, y \in \mathbb{R}$. So, for all $z \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\left|\frac{1}{\delta h}\left(\eta\left(\frac{x+h-z}{\delta}\right) - \eta\left(\frac{x-z}{\delta}\right)\right) f(z)\right| \\
&\qquad = \frac{1}{\delta h}\left|\eta\left(\frac{x+h-z}{\delta}\right) - \eta\left(\frac{x-z}{\delta}\right)\right| |f(z)| \\
&\qquad \leq \frac{1}{\delta h} L_\eta \left|\frac{x+h-z}{\delta} - \frac{x-z}{\delta}\right| |f(z)| \\
&\qquad = \frac{1}{\delta h} L_\eta \frac{h}{\delta} |f(z)| \\
&\qquad = \frac{L_\eta}{\delta^2} |f(z)|,
\end{aligned} \tag{6.2.9}$$

meaning that the integrand appearing in the final expression of (6.2.7) is bounded by a number that does not depend on h. So we can invoke the Dominated Convergence Theorem to conclude

$$\lim_{h\to 0}\int_{-\infty}^{\infty}\frac{1}{\delta h}\left(\eta\left(\frac{x+h-z}{\delta}\right)-\eta\left(\frac{x-z}{\delta}\right)\right)f(z)\,dz$$
$$=\int_{-\infty}^{\infty}\lim_{h\to 0}\frac{1}{\delta h}\left(\eta\left(\frac{x+h-z}{\delta}\right)-\eta\left(\frac{x-z}{\delta}\right)\right)f(z)\,dz. \tag{6.2.10}$$

So we have

$$\begin{aligned}(f^{\delta})'(x) &= \lim_{h\to 0}\frac{f^{\delta}(x+h)-f^{\delta}(x)}{h}\\ &= \lim_{h\to 0}\int_{-\infty}^{\infty}\frac{1}{\delta h}\left(\eta\left(\frac{x+h-z}{\delta}\right)-\eta\left(\frac{x-z}{\delta}\right)\right)f(z)\,dz\\ &= \int_{-\infty}^{\infty}\lim_{h\to 0}\frac{1}{\delta h}\left(\eta\left(\frac{x+h-z}{\delta}\right)-\eta\left(\frac{x-z}{\delta}\right)\right)f(z)\,dz\\ &= \frac{1}{\delta^2}\int_{-\infty}^{\infty}\lim_{h\to 0}\frac{1}{\frac{h}{\delta}}\left(\eta\left(\frac{x-z}{\delta}+\frac{h}{\delta}\right)-\eta\left(\frac{x-z}{\delta}\right)\right)f(z)\,dz\\ &= \frac{1}{\delta^2}\int_{-\infty}^{\infty}\eta'\left(\frac{x-z}{\delta}\right)f(z)\,dz\\ &= \frac{1}{\delta}\int_{-\infty}^{\infty}\frac{1}{\delta}\eta'\left(\frac{x-z}{\delta}\right)f(z)\,dz\\ &= \frac{1}{\delta}\int_{-\infty}^{\infty}\left(\eta\left(\frac{x-z}{\delta}\right)\right)'f(z)\,dz\\ &= \int_{-\infty}^{\infty}\left(\frac{1}{\delta}\eta\left(\frac{x-z}{\delta}\right)\right)'f(z)\,dz\\ &= \int_{-\infty}^{\infty}(\eta^{\delta})'(x-z)f(z)\,dz\\ &= ((\eta^{\delta})'*f)(x),\end{aligned} \tag{6.2.11}$$

which is (6.2.6).

Now we will assume (6.2.5) and prove

$$(f^{\delta})^{(k+1)}(x)=((\eta^{\delta})^{(k+1)}*f)(x). \tag{6.2.12}$$

The proof of (6.2.12) is analogous to the proof of (6.2.6). More specifically, the reader can repeat the argument of the previous paragraph with $k = 1$ replaced by an arbitrary integer $k > 1$.

Next, we will show that $(f^\delta)^{(k)}$ converges uniformly to $f^{(k)}$ on a closed interval $[\tilde{a}, \tilde{b}] \subseteq \mathbb{R}$ as δ approaches 0. Let $\epsilon > 0$ be given. Since we assume that $f^{(k)}$ is continuous on (a, b), it follows that $f^{(k)}$ is uniformly continuous on $[\tilde{a}, \tilde{b}] \subseteq (a, b)$. So there exists a $\delta > 0$ such that, for all $x, y \in [\tilde{a}, \tilde{b}]$ with $|x - z| < \delta$, we have $|f(x) - f(z)| < \epsilon$. So we conclude that, for all $x \in [\tilde{a}, \tilde{b}]$ and for any integer $k \geq 0$, we have

$$
\begin{aligned}
(f^\delta)^{(k)}(x) - f^{(k)}(x) &= (f^\delta)^{(k)}(x) - f^{(k)}(x) \cdot 1 \\
&= \int_{-\infty}^{\infty} \eta^\delta(x - z) f^{(k)}(y)\, dz - f^{(k)}(x) \int_{-\infty}^{\infty} \eta^\delta(x - z)\, dz \qquad (6.2.13) \\
&= \int_{-\infty}^{\infty} \eta^\delta(x - z)(f^{(k)}(y) - f^{(k)}(x))\, dz.
\end{aligned}
$$

We also recall from (6.2.2) that η^δ has compact support on $\mathbb{R}$ since we have $\eta^\delta(x) = 0$ for all $x \in \mathbb{R} \setminus (-\delta, \delta)$, which implies in particular

$$\eta^\delta(x - z) = 0 \qquad (6.2.14)$$

for all $z \in \mathbb{R} \setminus (x - \delta, x + \delta)$. By applying the triangle inequality and using (6.2.13) and

(6.2.14), we obtain

$$\begin{aligned}|(f^{\delta})^{(k)}(x)-f^{(k)}(x)| &= \left|\int_{-\infty}^{\infty}\eta^{\delta}(x-z)(f^{(k)}(y)-f^{(k)}(x))\,dz\right| \\ &\le \int_{-\infty}^{\infty}\eta^{\delta}(x-z)|f^{(k)}(y)-f^{(k)}(x)|\,dz \\ &= \int_{-\infty}^{x-\delta}\eta^{\delta}(x-z)|f^{(k)}(y)-f^{(k)}(x)|\,dz \\ &\quad + \int_{x-\delta}^{x+\delta}\eta^{\delta}(x-z)|f^{(k)}(y)-f^{(k)}(x)|\,dz \\ &\quad + \int_{x+\delta}^{\infty}\eta^{\delta}(x-z)|f^{(k)}(y)-f^{(k)}(x)|\,dz \\ &= \int_{-\infty}^{x-\delta}0\cdot|f^{(k)}(y)-f^{(k)}(x)|\,dz \\ &\quad + \int_{x-\delta}^{x+\delta}\eta^{\delta}(x-z)|f^{(k)}(y)-f^{(k)}(x)|\,dz \\ &\quad + \int_{x+\delta}^{\infty}0\cdot|f^{(k)}(y)-f^{(k)}(x)|\,dz \\ &= \int_{x-\delta}^{x+\delta}\eta^{\delta}(x-z)|f^{(k)}(x)-f^{(k)}(y)|\,dz \\ &< \int_{x-\delta}^{x+\delta}\eta^{\delta}(x-z)\epsilon\,dz \\ &= \epsilon\int_{x-\delta}^{x+\delta}\eta^{\delta}(x-z)\,dz \\ &= \epsilon\cdot 1 \\ &= \epsilon.\end{aligned} \tag{6.2.15}$$

In other words, we have

$$|(f^{\delta})^{(k)}(x)-f^{(k)}(x)| \to 0, \tag{6.2.16}$$

for all $x \in [\tilde{a}, \tilde{b}]$, given $\delta \to 0^{+}$, and so we conclude that $(f^{\delta})^{(k)}$ converges uniformly to $f^{(k)}$ for $\delta \to 0^{+}$. □

Next, we introduce an increasing smooth function that goes from 0 to 1 over a finite interval and a decreasing smooth function that goes from 1 to 0 over a finite interval. As the author of this dissertation, I have borrowed this function and its properties from Lemma 7.1 of [5].

Proposition 6.2.2. *For any closed interval* $[a, b] \subseteq \mathbb{R}$*, define* $H_{a,b} : \mathbb{R} \to \mathbb{R}$ *by*

$$H_{a,b}(x) := \frac{F(x-a)}{F(x-a) + F(b-x)}, \tag{6.2.17}$$

where $F : \mathbb{R} \to \mathbb{R}$ *is the flat function given by*

$$F(x) := \begin{cases} 0 & \text{for } x \leq 0, \\ e^{-\frac{1}{x^2}} & \text{for } x > 0. \end{cases} \tag{6.2.18}$$

Then $H_{a,b}$ *is smooth and increasing on* $\mathbb{R}$ *and satisfies*

$$H_{a,b}|_{(-\infty,a]}(x) = 0, \tag{6.2.19}$$

$$H_{a,b}|_{[b,\infty)}(x) = 1. \tag{6.2.20}$$

Similarly, the function $1 - H_{a,b} : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\begin{aligned} (1 - H_{a,b})(x) &:= 1 - H_{a,b}(x) \\ &= 1 - \frac{F(x-a)}{F(x-a) + F(b-x)} \\ &= \frac{F(b-x)}{F(x-a) + F(b-x)} \end{aligned} \tag{6.2.21}$$

is smooth and decreasing on $\mathbb{R}$ *and satisfies*

$$H_{a,b}|_{(-\infty,a]}(x) = 1, \tag{6.2.22}$$

$$H_{a,b}|_{[b,\infty)}(x) = 0. \tag{6.2.23}$$

Proof. In order to establish the desired properties of $H_{a,b}$, it will be convenient for us to write

$$\begin{aligned} H_{a,b}(x) &= \frac{F(x-a)}{F(x-a)+F(b-x)} \\ &= \begin{cases} \frac{0}{0+e^{-(b-x)^{-2}}} & \text{for } x \leq a, \\ \frac{e^{-(x-a)^{-2}}}{e^{-(x-a)^{-2}}+e^{-(b-x)^{-2}}} & \text{for } a < x < b, \\ \frac{e^{-(x-a)^{-2}}}{e^{-(x-a)^{-2}}+0} & \text{for } x \geq b \end{cases} \\ &= \begin{cases} 0 & \text{for } x \leq a, \\ \frac{e^{-(x-a)^{-2}}}{e^{-(x-a)^{-2}}+e^{-(b-x)^{-2}}} & \text{for } a < x < b, \\ 1 & \text{for } x \geq b. \end{cases} \end{aligned} \tag{6.2.24}$$

The final expression of (6.2.24) tells us that $H_{a,b}$ is nonnegative on $\mathbb{R}$. The final expression also tells us $H_{a,b}(x) = 0$ for all $x \leq a$ and $H_{a,b}(x) = 1$ for all $x \geq b$, which are (6.2.19) and (6.2.20), respectively.

Next, we will establish that $H_{a,b}$ is smooth. Let k be a nonnegative integer. We can see from the final expression of (6.2.24) that $H_{a,b}$ consists of algebraic or transcendental smooth functions on $\mathbb{R} \setminus \{a, b\}$. In other words, the k^{th} derivative $H_{a,b}^{(k)}$ consists of continuous functions on $\mathbb{R} \setminus \{a, b\}$. Additionally, $H_{a,b}^{(k)}$ satisfies

$$\lim_{x \to a_1} H_{a,b}^{(k)}(x) = 0, \tag{6.2.25}$$

$$\lim_{x \to a_2} H_{a,b}^{(k)}(x) = \begin{cases} 1 & \text{for } k = 1, \\ 0 & \text{for } k = 1, 2, 3, \dots. \end{cases} \tag{6.2.26}$$

So we conclude that $H_{a,b}^{(k)}$ is continuous on all of $\mathbb{R}$ and for any integer $k \geq 0$, which implies that $H_{a,b}$ is smooth on $\mathbb{R}$.

Finally, we will show that $H_{a,b}$ is increasing. To simplify the presentation of our below computations, define $G_{a,b} : [a, b] \to \mathbb{R}$ by

$$\begin{aligned} G_{a,b}(x) &:= \frac{F'(x-a)}{F(x-a) + F(b-x)} \\ &= \begin{cases} \frac{0'}{0 + e^{-(b-x)^{-2}}} & \text{for } x \leq a, \\ \frac{(e^{-(x-a)^{-2}})'}{e^{-(x-a)^{-2}} + e^{-(b-x)^{-2}}} & \text{for } a < x < b, \\ \frac{(e^{-(x-a)^{-2}})'}{e^{-(x-a)^{-2}} + 0} & \text{for } x \geq b \end{cases} \\ &= \begin{cases} \frac{0}{0 + e^{-(b-x)^{-2}}} & \text{for } x \leq a, \\ \frac{2(x-a)^{-3} e^{-(x-a)^{-2}}}{e^{-(x-a)^{-2}} + e^{-(b-x)^{-2}}} & \text{for } a < x < b, \\ \frac{2(x-a)^{-3} e^{-(x-a)^{-2}}}{e^{-(x-a)^{-2}}} & \text{for } x \geq b. \end{cases} \\ &= \begin{cases} 0 & \text{for } x \leq a, \\ \frac{2e^{-(x-a)^{-2}}}{(x-a)^3 (e^{-(x-a)^{-2}} + e^{-(b-x)^{-2}})} & \text{for } a < x < b, \\ \frac{2}{(x-a)^3} & \text{for } x \geq b. \end{cases} \end{aligned} \tag{6.2.27}$$

The final expression of (6.2.27) tells us that $G_{a,b}$ is nonnegative on $\mathbb{R}$, much like $H_{a,b}$. So we can write the first derivative $(H_{a,b})'$ as

$$
\begin{aligned}
(H_{a,b})'(x) &= \left(\frac{F(x-a)}{F(x-a)+F(b-x)}\right)' \\
&= \frac{F'(x-a)F(b-x)+F(x-a)F'(b-x)}{(F(x-a)+F(b-x))^2} \\
&= \frac{F'(x-a)F(b-x)}{(F(x-a)+F(b-x))^2} + \frac{F(x-a)F'(b-x)}{(F(x-a)+F(b-x))^2} \\
&= \frac{F'(x-a)}{F(x-a)+F(b-x)}\frac{F(b-x)}{F(b-x)+F(x-a)} \\
&\quad + \frac{F'(b-x)}{F(b-x)+F(x-a)}\frac{F(x-a)}{F(x-a)+F(b-x)} \\
&= \frac{F'(x-a)}{F(x-a)+F(b-x)} \\
&\qquad \cdot \frac{F((a+b-x)-a)}{F((a+b-x)-a)+F(b-(a+b-x))} \\
&\quad + \frac{F'((a+b-x)-a)}{F((a+b-x)-a)+F(b-(a+b-x))} \\
&\qquad \cdot \frac{F(x-a)}{F(x-a)+F(b-x)} \\
&= G_{a,b}(x)H_{a,b}(a+b-x)+G_{a,b}(a+b-x)H_{a,b}(x).
\end{aligned}
\tag{6.2.28}
$$

Since we have previously stated that both $G_{a,b}$ and $H_{a,b}$ are nonnegative on $\mathbb{R}$, we conclude from our final expression of (6.2.28) that $(H_{a,b})'$ is also nonnegative on $\mathbb{R}$, and so $H_{a,b}$ is increasing on $\mathbb{R}$, as we claimed.

In order to establish the desired properties of $1 - H_{a,b}$, it will be convenient for us to write

$$
\begin{aligned}
1 - H_{a,b}(x) &= 1 - \frac{F(x-a)}{F(x-a)+F(b-x)} \\
&= \begin{cases} 1 - \frac{0}{0+e^{-(b-x)^{-2}}} & \text{for } x \le a, \\ 1 - \frac{e^{-(x-a)^{-2}}}{e^{-(x-a)^{-2}}+e^{-(b-x)^{-2}}} & \text{for } a < x < b, \\ 1 - \frac{e^{-(x-a)^{-2}}}{e^{-(x-a)^{-2}}+0} & \text{for } x \ge b. \end{cases} \qquad (6.2.29) \\
&= \begin{cases} 1 & \text{for } x \le a, \\ \frac{e^{-(b-x)^{-2}}}{e^{-(x-a)^{-2}}+e^{-(b-x)^{-2}}} & \text{for } a < x < b, \\ 0 & \text{for } x \ge b. \end{cases}
\end{aligned}
$$

The arguments to show that $1-H_{a,b}$ is smooth and decreasing on $\mathbb{R}$ and satisfies (6.2.22) and (6.2.23) are analogous to those for $H_{a,b}$. We will not repeat our entire argument again here. One would use the final expression of (6.2.29) in place of (6.2.24) in order to obtain the desired properties of $1 - H_{a,b}$. The interested reader can fill in the details of this argument as an exercise. □

We also remark for the record that, if we send $a \to b^-$ or $b \to a^+$, then $H_{a,b}$ converges to the Heaviside step function $H^{a=b} : \mathbb{R} \to \mathbb{R}$, defined by

$$
H^{a=b}(x) := \begin{cases} 0 & \text{for } x < a = b, \\ \frac{1}{2} & \text{for } x = a = b, \\ 1 & \text{for } x > a = b. \end{cases} \qquad (6.2.30)
$$

In other words, $H_{a,b}$ is a smooth approximation of the Heaviside step function at $a = b$.

Next, we consider two more smooth functions $f, h : \mathbb{R} \to \mathbb{R}$ and numbers $a, b \in \mathbb{R}$ with $a < b$, and we introduce the glue functions that smoothly connect f and h over any open interval (a, b) via our binary operation ${}_a\diamond_b$. We define the glue function $f\,{}_a\diamond_b\,h : \mathbb{R} \to \mathbb{R}$ by

$$
\begin{aligned}
&(f\,{}_a\diamond_b\,h)(x) \\
&\quad := (1 - H_{a,b})(x) f(x) + H_{a,b}(x) h(x) \\
&\quad = \begin{cases} 1 f(x) + 0 h(x) & \text{for } x \le a, \\ \frac{e^{-(x-a)^{-2}}}{e^{-(x-a)^{-2}} + e^{-(b-x)^{-2}}} f(x) + \frac{e^{-(b-x)^{-2}}}{e^{-(x-a)^{-2}} + e^{-(b-x)^{-2}}} h(x) & \text{for } a < x < b, \\ 0 f(x) + 1 h(x) & \text{for } x \ge b. \end{cases} \\
&\quad = \begin{cases} f(x) & \text{for } x \le a, \\ \frac{e^{-(x-a)^{-2}}}{e^{-(x-a)^{-2}} + e^{-(b-x)^{-2}}} f(x) + \frac{e^{-(b-x)^{-2}}}{e^{-(x-a)^{-2}} + e^{-(b-x)^{-2}}} h(x) & \text{for } a < x < b, \\ h(x) & \text{for } x \ge b. \end{cases}
\end{aligned}
\tag{6.2.31}
$$

Lemma 6.2.3. *Let $f, h : \mathbb{R} \to \mathbb{R}$ be smooth functions. Then, for any integer $k \ge 0$, the k^{th} derivative of the glue function $f\,{}_a\diamond_b\,h : \mathbb{R} \to \mathbb{R}$ is*

$$
(f\,{}_a\diamond_b\,h)^{(k)} = \sum_{l=0}^{k} \binom{k}{l} H_{a,b}^{(l)} (h^{(k-l)} - f^{(k-l)}) + f^{(k)}, \tag{6.2.32}
$$

which is continuous on $[0, \frac{\pi}{3}]$. As a result, $f\,{}_a\diamond_b\,h$ is a smooth function as well.

Proof. We are essentially going to prove a version of the binomial theorem, and we will accomplish this with a proof by induction. For the base case, we have, for $k = 0$,

$$
\begin{aligned}
(f\,{}_a\diamond_b\,h) &= (1 - H_{a,b}) f + H_{a,b} h \\
&= f - H_{a,b} f + H_{a,b} h \\
&= H_{a,b}(h - f) + f.
\end{aligned}
\tag{6.2.33}
$$

For the inductive step, assume that (6.2.32) holds true for $k = n$; that is, assume

$$(f\,{}_a\!\diamond_b\, h)^{(n)} = \sum_{l=0}^{n} \binom{n}{l} H_{a,b}^{(l)} (h^{(n-l)} - f^{(n-l)}) + f^{(n)}. \tag{6.2.34}$$

Then we have

$$\begin{aligned}
(f\,{}_a\!\diamond_b\, h)^{(n+1)} &= ((f\,{}_a\!\diamond_b\, h)^{(n)})' \\
&= \left(\sum_{l=0}^{n} \binom{n}{l} H_{a,b}^{(l)} (h^{(n-l)} - f^{(n-l)}) + f^{(n)} \right)' \\
&= \sum_{l=0}^{n} \binom{n}{l} (H_{a,b}^{(l)} (h^{(n-l)} - f^{(n-l)})' + (f^{(n)})' \\
&= \sum_{l=0}^{n} \binom{n}{l} H_{a,b}^{(l+1)} (h^{(n-l)} - f^{(n-l)}) + f^{(n+1)} \\
&\qquad + \sum_{l=0}^{n} \binom{n}{l} H_{a,b}^{(l)} (h^{(n+1-l)} - f^{(n+1-l)}) + f^{(n+1)} \\
&= \sum_{l=0}^{n+1} \left(\binom{n}{l+1} + \binom{n}{l} \right) H_{a,b}^{(l)} (h^{(n+1-l)} - f^{(n+1-l)}) + f^{(n+1)} \\
&= \sum_{l=0}^{n+1} \binom{n+1}{l+1} H_{a,b}^{(l)} (h^{(n+1-l)} - f^{(n+1-l)}) + f^{(n+1)},
\end{aligned} \tag{6.2.35}$$

which means (6.2.32) holds true for $k = n + 1$. This completes our proof by induction.

Next, we will establish that the k^{th} derivatives are all continuous. Presumably, we know that any finite sum and product of continuous functions is again a continuous function. And $(f\,{}_a\!\diamond_b\, h)^{(k)}$ is the finite sum and product of higher-order derivatives of $H_{a,b}, f, h$, all of which are continuous because we have already established that $H_{a,b}, f, h$ are all smooth. So we conclude that $(f\,{}_a\!\diamond_b\, h)^{(k)}$ is also continuous for any integer $k \geq 0$, which means $f\,{}_a\!\diamond_b\, h$ is smooth. □

For the next proposition, we will introduce the following definition of two functions that are sufficiently close to each other and whose derivatives are sufficiently close to each other as well on any given interval $[a, b] \subseteq \mathbb{R}$.

Definition 6.2.4. *Let $f, h : \mathbb{R} \to \mathbb{R}$ be k-times differentiable functions, where $k \geq 0$ is an integer. Define the C^k*-norm of *f on some interval $[a, b] \subseteq \mathbb{R}$ by*

$$\|f\|_{C^k([a,b])} := \max_{i=0,1,2,\ldots,k} \max_{x\in[a,b]} |f^{(i)}(x)|. \tag{6.2.36}$$

The definition implies, in particular,

$$\begin{aligned} |f^{(i)}(x)| &\leq \max_{x\in[a,b]} |f^{(i)}(x)| \\ &\leq \max_{i=0,1,2,\ldots,k} \max_{x\in[a,b]} |f^{(i)}(x)| \\ &= \|f\|_{C^k([a,b])} \end{aligned} \tag{6.2.37}$$

for all $x \in [a, b]$.

The next proposition will make use of Definition 6.2.4 with $k = 1$.

Proposition 6.2.5. *Consider any closed interval $[a, b] \subseteq \mathbb{R}$, and let $f, h : \mathbb{R} \to \mathbb{R}$ be twice differentiable functions that are strictly concave on $[a, b]$. Then:*

(1) *The sum function $f + h$ is strictly concave on $\mathbb{R}$.*

(2) *If $\delta > 0$ is sufficiently small, then the mollification $f^\delta := \eta^\delta * f$ is strictly concave on $[a, b]$.*

(3) *For any $\beta > 0$, there exists $\epsilon > 0$ such that, if f, h satisfy*

$$\max(\max_{x\in\mathbb{R}} f''(x), \max_{x\in\mathbb{R}} h''(x)) \leq -\beta, \tag{6.2.38}$$

$$\|f - h\|_{C^1([a,b])} < \epsilon, \tag{6.2.39}$$

then the glue function $f\,{}_a\!\diamond_b\, h$ is strictly concave on $\mathbb{R}$.

Proof. First, for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned}(f+h)(\lambda x+(1-\lambda)y) &= f(\lambda x+(1-\lambda)y)+h(\lambda x+(1-\lambda)y)\\ &> \lambda f(x)+(1-\lambda)f(y)+\lambda h(x)+(1-\lambda)h(y)\\ &= \lambda(f(x)+h(x))+(1-\lambda)(f(y)+h(y))\\ &= \lambda(f+h)(x)+(1-\lambda)(f+h)(y),\end{aligned} \tag{6.2.40}$$

which means that $f+h$ is also strictly concave on $\mathbb{R}$.

Next, we will show that the mollification $f^\delta := \eta^\delta * f$ is strictly concave on $\mathbb{R}$. Since f is strictly concave on $\mathbb{R}$, by definition we have, for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$f(\lambda x+(1-\lambda)y) > \lambda f(x)+(1-\lambda)f(y). \tag{6.2.41}$$

Consequently, we obtain, for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$\begin{aligned}&f^\delta(\lambda x+(1-\lambda)y)\\ &\quad= \int_{-\infty}^{\infty}(\eta^\delta)^{(k)}(z)f(\lambda x+(1-\lambda)y-z)\,dz\\ &\quad= \int_{-\infty}^{\infty}(\eta^\delta)^{(k)}(z)f(\lambda x+(1-\lambda)y-\lambda z-(1-\lambda)z)\,dz\\ &\quad= \int_{-\infty}^{\infty}(\eta^\delta)^{(k)}(z)f(\lambda(x-z)+(1-\lambda)(y-z))\,dz\\ &\quad> \int_{-\infty}^{\infty}(\eta^\delta)^{(k)}(z)(\lambda f(x-z)+(1-\lambda)f(y-z))\,dz\\ &\quad= \int_{-\infty}^{\infty}\lambda(\eta^\delta)^{(k)}(z)(f(x-z)+(1-\lambda)(\eta^\delta)^{(k)}(z)f(y-z)\,dz\\ &\quad= \lambda\int_{-\infty}^{\infty}(\eta^\delta)^{(k)}(z)f(x-z)\,dz+(1-\lambda)\int_{-\infty}^{\infty}(\eta^\delta)^{(k)}(z)f(y-z)\,dz\\ &\quad= \lambda f^\delta(x)+(1-\lambda)f^\delta(y),\end{aligned} \tag{6.2.42}$$

which means that f is strictly concave on $\mathbb{R}$. And this proof is analogous for h.

Finally, we will show that the glue function $f\,{}_a\!\diamond_b\, h$ is strictly concave on $\mathbb{R}$. Since f, h are strictly concave, (6.2.38) implies

$$f''(x) \leq -\beta, \tag{6.2.43}$$

$$h''(x) \leq -\beta \tag{6.2.44}$$

for some $\beta > 0$. Also, since H is smooth on $[a, b]$, it follows in particular that its first derivative H' and its second derivative H'' are bounded. In other words, there exist constants $C_1 > 0$ and $C_2 > 0$ that satisfy

$$|H'(x)| \leq C_1, \tag{6.2.45}$$

$$|H''(x)| \leq C_2 \tag{6.2.46}$$

for all $x \in [a, b]$. Therefore, according to (6.2.32) with $k = 2$, we conclude that our second derivative of $f\,{}_a\!\diamond_b\, h$ is

$$\begin{aligned}
(f\,{}_a\!\diamond_b\, h)'' &= \sum_{l=0}^{2} \binom{2}{l} H_{a,b}^{(l)} (h^{(k-l)} - f^{(2-l)}) + f'' \\
&= \binom{2}{0} H''_{a,b}(h - f) + \binom{2}{1} H'_{a,b}(h' - f') \\
&\qquad + \binom{2}{2} H_{a,b}(h'' - f'') + f'' \\
&= H''_{a,b}(h - f) + 2H'_{a,b}(h' - f') \\
&\qquad + H_{a,b}(h'' - f'') + f'' \\
&= H''_{a,b}(h - f) + 2H'_{a,b}(h' - f') \\
&\qquad + (1 - H_{a,b}) f'' + H_{a,b} h'' \\
&\leq H''_{a,b}(h - f) + 2H'_{a,b}(h' - f') \\
&\qquad + (1 - H_{a,b})(-\beta) + H_{a,b}(-\beta) \\
&= H''_{a,b}(h - f) + 2H'_{a,b}(h' - f') - \beta \\
&\leq |H''_{a,b}||h - f| + 2|H'_{a,b}||h' - f'| - \beta \\
&\leq |H''_{a,b}|\|h - f\|_{C^1([a,b])} + 2|H'_{a,b}|\|h - f\|_{C^1([a,b])} - \beta \\
&= (|H''_{a,b}| + 2|H'_{a,b}|)\|h - f\|_{C^1([a,b])} - \beta \\
&< (|H''_{a,b}| + 2|H'_{a,b}|)\epsilon - \beta \\
&\leq (C_2 + 2C_1)\epsilon - \beta.
\end{aligned} \tag{6.2.47}$$

Now, if we choose, for instance, $\epsilon := \frac{\beta}{2(C_2+2C_1)}$, then we would have

$$\begin{aligned}(f\,{}_a\!\diamond_b\, h)'' &\leq (C_2 + 2C_1)\epsilon - \beta \\ &= (C_2 + 2C_1)\frac{\beta}{2(C_2 + 2C_1)} - \beta \\ &= \frac{\beta}{2} - \beta \\ &= -\frac{\beta}{2} \\ &< 0.\end{aligned} \tag{6.2.48}$$

So we conclude that $f\,{}_a\!\diamond_b\, h$ is strictly concave on $[a, b]$. □

6.3 One-parameter family of smooth metrics

We are now ready to construct a smooth Riemannian metric for S^4. Fix $m, c \in \mathbb{R}$, and let $\alpha > 0$, $\delta > 0$, $\epsilon > 0$ all be sufficiently small. Following the definition of the mollification in (6.2.4), we introduce the mollifications $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}(s) := (\eta^{\delta} * \hat{\varphi}^{\alpha,\epsilon}_{m,c})(s), \tag{6.3.1}$$

$$\hat{\psi}^{\alpha,\delta,\epsilon}_{m,c}(s) := (\eta^{\delta} * \hat{\psi}^{\alpha,\epsilon}_{m,c})(s), \tag{6.3.2}$$

$$\hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}(s) := (\eta^{\delta} * \hat{\xi}^{\alpha,\epsilon}_{m,c})(s), \tag{6.3.3}$$

where we have defined $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\psi}^{\alpha,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\epsilon}_{m,c}$ by (6.1.2), (6.1.4), (6.1.3), respectively. According to Proposition 6.2.1, $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}$ are smooth functions. Now, we choose numbers $\rho_i \in (0, \frac{\pi}{3})$ for $i = 1, 2, 3, 4, 5, 6, 7, 8$ that satisfy the inequality relations

$$0 < \rho_1 < \rho_2 < (s_1)^{\alpha,\epsilon}_{m,c}, \tag{6.3.4}$$

$$(s_1)^{\alpha,\epsilon}_{m,c} < \rho_3 < \rho_4 < (s_2)^{\alpha,\epsilon}_{m,c} < \frac{\pi}{6}, \tag{6.3.5}$$

$$\frac{\pi}{6} < (s_3)^{\alpha,\epsilon}_{m,c} < \rho_5 < \rho_6 < (s_4)^{\alpha,\epsilon}_{m,c}, \tag{6.3.6}$$

$$(s_4)^{\alpha,\epsilon}_{m,c} < \rho_7 < \rho_8 < \frac{\pi}{3}, \tag{6.3.7}$$

where we have defined $(s_1)^{\alpha,\epsilon}_{m,c}, (s_2)^{\alpha,\epsilon}_{m,c}, (s_3)^{\alpha,\epsilon}_{m,c}, (s_4)^{\alpha,\epsilon}_{m,c}$ by (6.1.5), (6.1.6), (6.1.7), (6.1.8), respectively. Finally, we define $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_3,\rho_4,\rho_7,\rho_8}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_5,\rho_6} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_3,\rho_4,\rho_7,\rho_8}(s) := (\mathring{\varphi}^{\alpha}_{0}\,{}_{\rho_3}\!\diamond_{\rho_4} \hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}\,{}_{\rho_7}\!\diamond_{\rho_8} \mathring{\varphi}^{\alpha}_{\frac{\pi}{3}})(s), \tag{6.3.8}$$

$$\hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_5,\rho_6}(s) := (\mathring{\xi}^{\alpha}_{0}\,{}_{\rho_1}\!\diamond_{\rho_2} \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}\,{}_{\rho_5}\!\diamond_{\rho_6} \mathring{\xi}^{\alpha}_{0})(s), \tag{6.3.9}$$

where we have defined $\mathring{\varphi}^{\alpha}_{0}, \mathring{\xi}^{\alpha}_{0}, \mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}, \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}$ by (5.1.2), (5.1.4), (5.1.6), (5.1.8), respectively. Also, we define $\hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8}(s) := \hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_3,\rho_4,\rho_7,\rho_8}(s) + \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_5,\rho_6}(s), \tag{6.3.10}$$

which are all smooth, according to Proposition 6.2.1 and Lemma 6.2.3, and strictly concave, according to Proposition 6.2.5. Also, while this is not necessary for our dissertation, we can write $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_3,\rho_4,\rho_7,\rho_8}, \hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_5,\rho_6}$ more explicitly as

$$\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_3,\rho_4,\rho_7,\rho_8}(s) = \begin{cases} \mathring{\varphi}^{\alpha}_{0}(s) & \text{for } 0 \le s \le \rho_3, \\ (\mathring{\varphi}^{\alpha}_{0}{}_{\rho_3}\diamond_{\rho_4}\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c})(s) & \text{for } \rho_3 < s < \rho_4, \\ \hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}(s) & \text{for } \rho_4 \le s \le \rho_7, \\ (\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}{}_{\rho_7}\diamond_{\rho_8}\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}})(s) & \text{for } \rho_7 < s < \rho_8, \\ \mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}(s) & \text{for } \rho_8 \le s \le \frac{\pi}{3} \end{cases} \tag{6.3.11}$$

and

$$\hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8}(s) = \begin{cases} \mathring{\psi}^{\alpha}_0(s) & \text{for } 0 \le s \le \rho_1, \\ (\mathring{\psi}^{\alpha}_0{}_{\rho_1}\diamond_{\rho_2}(\mathring{\varphi}^{\alpha}_0 + \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}))(s) & \text{for } \rho_1 < s < \rho_2, \\ \mathring{\varphi}^{\alpha}_0(s) + \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}(s) & \text{for } \rho_2 \le s \le \rho_3, \\ ((\mathring{\varphi}^{\alpha}_0 + \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c})_{\rho_3}\diamond_{\rho_4}\hat{\psi}^{\alpha,\delta,\epsilon}_{m,c})(s) & \text{for } \rho_3 < s < \rho_4, \\ \hat{\psi}^{\alpha,\delta,\epsilon}_{m,c}(s) & \text{for } \rho_4 \le s \le \rho_5, \\ (\hat{\psi}^{\alpha,\delta,\epsilon}_{m,c}{}_{\rho_5}\diamond_{\rho_6}(\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c} + \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}))(s) & \text{for } \rho_5 < s < \rho_6, \\ \hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}(s) + \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}(s) & \text{for } \rho_6 \le s \le \rho_7, \\ ((\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c} + \mathring{\xi}^{\alpha}_{\frac{\pi}{3}})_{\rho_7}\diamond_{\rho_8}\mathring{\psi}^{\alpha}_{\frac{\pi}{3}})(s) & \text{for } \rho_7 < s < \rho_8, \\ \mathring{\psi}^{\alpha}_{\frac{\pi}{3}}(s) & \text{for } \rho_8 \le s \le \frac{\pi}{3} \end{cases} \tag{6.3.12}$$

and

$$\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_5,\rho_6}(s) = \begin{cases} \mathring{\xi}^{\alpha}_0(s) & \text{for } 0 \le s \le \rho_1, \\ (\mathring{\xi}^{\alpha}_0{}_{\rho_1}\diamond_{\rho_2}\hat{\xi}^{\alpha,\delta,\epsilon}_{m,c})(s) & \text{for } \rho_1 < s < \rho_2, \\ \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}(s) & \text{for } \rho_2 \le s \le \rho_5, \\ (\hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}{}_{\rho_5}\diamond_{\rho_6}\mathring{\xi}^{\alpha}_{\frac{\pi}{3}})(s) & \text{for } \rho_5 < s < \rho_6, \\ \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}(s) & \text{for } \rho_6 \le s \le \frac{\pi}{3}. \end{cases} \tag{6.3.13}$$

In particular, the glue functions $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_3,\rho_4,\rho_7,\rho_8}, \hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_5,\rho_6}$ are equal to the functions $\mathring{\varphi}^{\alpha}_0, \mathring{\psi}^{\alpha}_0, \mathring{\xi}^{\alpha}_0$, respectively, on $[0,\rho_1]$. Similarly, the glue functions $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_3,\rho_4,\rho_7,\rho_8}, \hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_5,\rho_6}$ are equal to $\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}, \mathring{\psi}^{\alpha}_{\frac{\pi}{3}}, \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}$, respectively, on $[\rho_6, \frac{\pi}{3}]$. Also, upon introducing a new parameter, which we will do after Lemma 6.3.1, these functions will allow us to discover a one-parameter family of smooth metrics that will facilitate the proof of Lemma 1.0.4 and, in turn, that of Theorem 1.0.3.

Now we will address the important consequences of the functions φ, ψ, ξ associated with the metric g being concave or strictly concave.

Lemma 6.3.1. *Let* (S^4, g) *be* SO(3)*-invariant. Suppose* g *takes the form in* (1.0.8).

(1) *If* φ, ψ, ξ *are concave, then* (S^4, g) *has a nonnegative radial sectional curvature.*

(2) *If* φ, ψ, ξ *are strictly concave, then* (S^4, g) *has positive radial sectional curvature.*

Proof. First, we will prove (1). We have assumed that φ, ψ, ξ are concave, their second derivatives are nonpositive; that is, we have

$$\varphi'' \leq 0, \tag{6.3.14}$$

$$\psi'' \leq 0, \tag{6.3.15}$$

$$\xi'' \leq 0, \tag{6.3.16}$$

which are equivalent to, respectively,

$$-\varphi'' \geq 0, \tag{6.3.17}$$

$$-\psi'' \geq 0, \tag{6.3.18}$$

$$-\xi'' \geq 0. \tag{6.3.19}$$

Also, by the fact that Riemannian metrics are nonnegative, it follows from (1.0.8) that

φ, ψ, ξ are nonnegative, which means that we have

$$\varphi \geq 0, \tag{6.3.20}$$

$$\psi \geq 0, \tag{6.3.21}$$

$$\xi \geq 0. \tag{6.3.22}$$

So we conclude that, invoking Proposition 3.1.1, we obtain

$$\begin{aligned} R\left(\frac{\partial}{\partial s}, X, X, \frac{\partial}{\partial s}\right) &= -\varphi\varphi'' \\ &\geq 0 \cdot 0 \\ &= 0 \end{aligned} \tag{6.3.23}$$

and

$$\begin{aligned} R\left(\frac{\partial}{\partial s}, Y, Y, \frac{\partial}{\partial s}\right) &= -\psi\psi'' \\ &\geq 0 \cdot 0 \\ &= 0 \end{aligned} \tag{6.3.24}$$

and

$$\begin{aligned} R\left(\frac{\partial}{\partial s}, Z, Z, \frac{\partial}{\partial s}\right) &= -\xi\xi'' \\ &\geq 0 \cdot 0 \\ &= 0. \end{aligned} \tag{6.3.25}$$

So, by also applying Proposition 3.1.2, we have, for any vector field V on S^4,

$$\begin{aligned}
R\left(\frac{\partial}{\partial s}, V, V, \frac{\partial}{\partial s}\right) &= R\left(\frac{\partial}{\partial s}, \sum_{i=1}^{3} V^i X_i, \sum_{j=1}^{3} V^j X_j, \frac{\partial}{\partial s}\right) \\
&= \sum_{i,j=1}^{3} V^i V^j R\left(\frac{\partial}{\partial s}, X_i, X_j, \frac{\partial}{\partial s}\right) \\
&= \sum_{i=1}^{3} (V^i)^2 R\left(\frac{\partial}{\partial s}, X_i, X_i, \frac{\partial}{\partial s}\right) \\
&\qquad + \sum_{\substack{i,j=1 \\ i\neq j}}^{3} V^i V^j R\left(\frac{\partial}{\partial s}, X_i, X_j, \frac{\partial}{\partial s}\right) \\
&= \sum_{i=1}^{3} (V^i)^2 R\left(\frac{\partial}{\partial s}, X_i, X_i, \frac{\partial}{\partial s}\right) + \sum_{\substack{i,j=1 \\ i\neq j}}^{3} V^i V^j \cdot 0 \\
&= \sum_{i=1}^{3} (V^i)^2 R\left(\frac{\partial}{\partial s}, X_i, X_i, \frac{\partial}{\partial s}\right) \\
&\geq \sum_{i=1}^{3} (V^i)^2 \cdot 0 \\
&= 0.
\end{aligned} \tag{6.3.26}$$

Finally, we obtain the sectional curvature

$$\begin{aligned}
\sec\left(\frac{\partial}{\partial s}, V\right) &= \frac{R(\frac{\partial}{\partial s}, V, V, \frac{\partial}{\partial s})}{|\frac{\partial}{\partial s}|^2 |V|^2 - g(\frac{\partial}{\partial s}, V)^2} \\
&= \frac{R(\frac{\partial}{\partial s}, V, V, \frac{\partial}{\partial s})}{1^2 |V|^2 - 0^2} \\
&= \frac{R(\frac{\partial}{\partial s}, V, V, \frac{\partial}{\partial s})}{|V|^2} \\
&\geq \frac{0}{|V|^2} \\
&= 0.
\end{aligned} \tag{6.3.27}$$

So we conclude that according to Definition 1.0.2, the radial sectional curvatures are nonnegative.

The argument for (2) is identical to that of (1), with "positive" and "negative" replacing all instances of "nonnegative" and "nonpositive", respectively, "strictly concave" replacing all instances of "concave", and the inequality signs $>$ and $<$ replacing all instances of $\geq$ and $\leq$, respectively. $\square$

For the reader's sake, we recapitulate our results of this chapter here. First, we recall the continuous, piecewise smooth, and strictly concave functions $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\epsilon}_{m,c}$ that we defined on the interval $[0, \frac{\pi}{3}]$ by (6.1.10) and (6.1.12), respectively, and $\hat{\psi}^{\alpha,\epsilon}_{m,c}$ that we defined on $[0, \frac{\pi}{3}]$ by (6.1.11) for all $\epsilon \in (0, \epsilon_0)$. We convolved them with the standard mollifier η^δ defined by (6.2.3) for all $\delta \in (0, \delta_0)$ in order to create their corresponding mollifications $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}, \hat{\psi}^{\alpha,\delta,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}$, which are smooth functions on $[0, \frac{\pi}{3}]$. Finally, by using the gluing binary operation that we introduced in (6.2.31), we were able to create the glue functions $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_3,\rho_4,\rho_7,\rho_8}, \hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_1,\rho_2,\rho_5,\rho_6}$ defined by (6.3.11), (6.3.12), (6.3.13), respectively, which are also smooth and strictly concave on $[0, \frac{\pi}{3}]$. Also, Lemma 5.1.1 asserts that $\hat{g}^{\alpha,\delta,\epsilon}_0$ is smooth on the singular orbits $(\mathrm{SO}(3) \times \mathbb{D}^2)/\mathrm{SO}(3)_{\gamma(0)}$ and $(\mathrm{SO}(3) \times \mathbb{D}^2)/\mathrm{SO}(3)_{\gamma(\frac{\pi}{3})}$, and so Lemma 6.3.1 asserts that the radial sectional curvature of S^4 with $\hat{g}^{\alpha,\delta,\epsilon}_0$ is strictly positive.

Finally, we now consider our desired one-parameter family of metrics $\{g^\tau_{\text{final}}\}_{\tau \geq 0}$ taking the form

$$g^\tau_{\text{final}} = ds^2 + \varphi^\tau_{\text{final}}\, dx^2 + \psi^\tau_{\text{final}}\, dy^2 + \xi^\tau_{\text{final}}\, dz^2, \tag{6.3.28}$$

where we define $\varphi^\tau_{\text{final}}, \psi^\tau_{\text{final}}, \xi^\tau_{\text{final}} : \mathbb{R} \to \mathbb{R}$ by

$$\varphi^\tau_{\text{final}} := \hat{\varphi}^{\alpha,\delta,\tau\epsilon}_{m=\frac{7}{5},c=\frac{3}{10},\rho_3,\rho_4,\rho_7,\rho_8}, \tag{6.3.29}$$

$$\psi^\tau_{\text{final}} := \hat{\psi}^{\alpha,\delta,\tau\epsilon}_{m=\frac{7}{5},c=\frac{3}{10},\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_7,\rho_8}, \tag{6.3.30}$$

$$\xi^\tau_{\text{final}} := \hat{\xi}^{\alpha,\delta,\tau\epsilon}_{m=\frac{7}{5},c=\frac{3}{10},\rho_1,\rho_2,\rho_5,\rho_6}. \tag{6.3.31}$$

for all $\delta \in (0, \delta_0)$ and $\epsilon \in (0, \epsilon_0)$. In order to keep our notation simple, we assume that the reader understands from this point on that g^τ_{final} depends on the parameters $\alpha, \delta, \epsilon, \tau, \rho_i$ for $i = 1, 2, 3, 4, 5, 6, 7, 8$, even if g^τ_{final} does not explicitly list any of these parameters apart from τ in its superscripts or subscripts. Also, we remark that our choice of $m = \frac{7}{5}$ and $c = \frac{3}{10}$ represents one example of a family of metrics out of infinitely many such examples. For instance, there exists a neighborhood of infinitely many points $(m, c) \in \mathbb{R}^2$ containing the point $(\frac{7}{5}, \frac{3}{10})$ on which their corresponding metrics equally facilitate the proof of Theorem 1.0.3.

We need just one more lemma here before we prove Theorem 1.0.3.

Lemma 6.3.2. *Fix any number $\tau_0 > 0$. There exist $\delta > 0$ and $\epsilon > 0$ such that the following properties hold:*

(1) *The manifold (S^4, g^τ_{final}) has positive radial sectional curvature for all $\tau \in (0, \tau_0)$.*

(2) *The family of manifolds $\{(S^4, g^\tau_{final})\}_{0\le\tau\le\tau_0}$ satisfies the hypotheses of Lemma 1.0.4.*

Proof. First, we will prove (1). As we mentioned in three paragraphs before the statement of this lemma, the functions $\varphi^\tau_{\text{final}}, \psi^\tau_{\text{final}}, \xi^\tau_{\text{final}}$ are smooth and strictly concave on $[0, \frac{\pi}{3}]$ for all $\tau \in (0, \tau_0)$. By Lemma 6.3.1, we conclude that $(S^4, g^\tau_{\text{final}})$ has positive radial sectional curvature for all $\tau \in (0, \tau_0)$.

Next, we will prove (2). Notice that, by construction (that is, according to (6.3.11), (6.3.12), (6.3.13) with $\tau\epsilon$ replacing all instances of ϵ), the metric g^0_{final} coincides with the metric $\bar{g}_{m,c}$ on the interval $[\rho_4, \rho_5]$. Furthermore, we claim $\frac{\pi}{6} \in [\rho_4, \rho_5]$. Indeed, the reader can numerically solve for

$$(s_2)^{\tau\epsilon}_{\text{final}} := (s_2)^{\alpha,\tau\epsilon}_{m=\frac{7}{5},c=\frac{3}{10}}, \tag{6.3.32}$$

$$(s_3)^{\tau\epsilon}_{\text{final}} := (s_3)^{\alpha,\tau\epsilon}_{m=\frac{7}{5},c=\frac{3}{10}} \tag{6.3.33}$$

from (6.1.6) and (6.1.7), respectively, as adapted to g^{τ}_{final}. Then one would observe the strict inequality relation

$$(s_2)^{\tau\epsilon}_{\text{final}} < \frac{\pi}{6} < (s_3)^{\tau\epsilon}_{\text{final}} \tag{6.3.34}$$

from the numerical values of $(s_2)^{\tau\epsilon}_{\text{final}}$ and $(s_3)^{\tau\epsilon}_{\text{final}}$. So we can choose ρ_4, ρ_5 such that they satisfy

$$(s_2)^{\tau\epsilon}_{\text{final}} < \rho_4 < \frac{\pi}{6} < \rho_5 < (s_3)^{\tau\epsilon}_{\text{final}}, \tag{6.3.35}$$

completing the proof our claim $\frac{\pi}{6} \in [\rho_4, \rho_5]$. So we conclude that g^0_{final} coincides with the metric $\bar{g}_{m,c}$ from Lemma 5.3.1 on a neighborhood of $s = \frac{\pi}{6}$, and so there exists a sufficiently small $\alpha > 0$ such that the sectional curvature of the tangent plane $\text{span}(\frac{\partial}{\partial s}, X)$ satisfies (1.0.5) on $(\frac{\pi}{6} - \alpha, \frac{\pi}{6} + \alpha) \subseteq [\rho_4, \rho_5]$, and so we have satisfied the hypotheses of Lemma 1.0.4. □

Finally, we will write a proof of Theorem 1.0.3.

Proof of Theorem 1.0.3. Consider the one-parameter family of metrics $\{g^{\tau}_{\text{final}}\}_{\tau \geq 0}$ with each metric g^{τ}_{final} defined by (6.3.28), which takes the form given by (1.0.8). For all $\tau \in (0, \tau_0)$, Lemma 6.3.2 asserts that the manifold $(S^4, g^{\tau}_{\text{final}})$ has positive radial sectional curvature and that the family $\{(S^4, g^{\tau}_{\text{final}})\}_{0 \leq \tau \leq \tau_0}$ satisfies the hypotheses of Lemma 1.0.4 for the tangent plane span$(\frac{\partial}{\partial s}, X)$. Furthermore, Lemma 1.0.4 asserts (1.0.6). Finally, since (1.0.6) is precisely the assertion of Theorem 1.0.3, our proof is complete. $\square$

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