Ricci flow does not necessarily preserve positive radial sectional curvature

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Thesis result

We will discuss the following result of my thesis:

Theorem (Main theorem)

There exists a Riemannian metric with positive radial sectional curvature on S^4 that attains a negative radial sectional curvature when evolved by the Ricci flow.

We remark that this theorem is an extension of a similar by Bettiol and Krishnan in 2019 using a metric with nonnegative sectional curvature on S^4. We also remark that, in December 2021, they have also recently extended their result to a metric with positive sectional curvature on S^4. This means that Theorem 1 is now a special case of their 2021 result. I will explain that my metric with positive radial sectional curvature contains properties that differ significantly from the metric of their 2021 result.

The group action on M

Let M be a compact Riemannian manifold equipped with a Riemannian metric g. Let G be a group that acts by isometries on M.

Definition

We say that the action by G on M is with ***cohomogeneity one*** *if the quotient space M/G is one-dimensional.*

Since we are assuming that M is compact, the quotient space M/G must be isometric to either a circle S^1 or a closed interval $[0, L]$ for some $L > 0$.

In this presentation, we will focus only on the closed interval $[0, L]$.

The SO(3) action on S^4

Define $V := \{A \in \mathbb{R}^{3\times 3} \mid A = A^T, \mathrm{tr}(A) = 0\}$. Then V is five-dimensional and its natural inner product is $g(A, B) := \mathrm{tr}(A^T B)$ for $A, B \in V$.

The $SO(3)$ action on S^4 is defined to be the restriction to the unit sphere in V of the group action of $SO(3)$ on V by conjugation: $SO(3) \times V \to V$ given by $h \cdot A = hAh^{-1}$ for all rotation matrices $h \in SO(3)$ and for all $A \in V$.

Geodesic through the orbits of S^4

The geodesic $\gamma : [0, \frac{\pi}{3}] \to V$ defined by

$$\gamma(s) := \begin{bmatrix} \frac{\cos(s)}{\sqrt{6}} + \frac{\sin(s)}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\cos(s)}{\sqrt{6}} - \frac{\sin(s)}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{2\cos(s)}{\sqrt{6}} \end{bmatrix} \tag{1}$$

runs orthogonally through all the principal orbits $SO(3)/SO(3)_{\gamma(s_{\mathrm{int}})}$ of S^4.

Basis of the Lie algebra of $SO(3)$

The Lie algebra of $SO(3)$ is

$$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} \mid A + A^T = 0\}. \tag{2}$$

The set $\{E_{23}, E_{31}, E_{12}\}$, where we define

$$E_{23} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \tag{3}$$

$$E_{31} := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{4}$$

$$E_{12} := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

is a basis of $\mathfrak{so}(3)$.

Killing fields

The corresponding Killing fields are X, Y, Z, defined by

$$X(p) := \frac{d}{ds} \exp(sE_{23}) \cdot p \Big|_{s=0}, \tag{6}$$

$$Y(p) := \frac{d}{ds} \exp(sE_{31}) \cdot p \Big|_{s=0}, \tag{7}$$

$$Z(p) := \frac{d}{ds} \exp(sE_{12}) \cdot p \Big|_{s=0}, \tag{8}$$

where

$$\begin{aligned} \exp : \ T_p(M \setminus (\pi^{-1}(\{0\}) \cup \pi^{-1}(\{L\}))) \\ \to M \setminus (\pi^{-1}(\{0\}) \cup \pi^{-1}(\{L\})) \end{aligned} \tag{9}$$

is the Lie group exponential map.

Killing fields

Since the $SO(3)$ action on S^4 is cohomogeneity one and by isometries, the metric g is determined completely by its restriction to the geodesic γ.

The lengths of the Killing fields are denoted

$$\varphi(s) := |X(\gamma(s))|, \tag{10}$$

$$\psi(s) := |Y(\gamma(s))|, \tag{11}$$

$$\xi(s) := |Z(\gamma(s))|. \tag{12}$$

The 2017 result by Renato Bettiol and Anusha Krishnan

Let $\frac{\partial}{\partial s}$ be the tangent vector of the geodesic γ at the point $\gamma(s)$.

Lemma (Bettiol and Krishnan, 2017)

Any $SO(3)$-invariant Riemannian metric g on S^4 takes the form

$$g = dr^2 + \varphi^2 dx^2 + \psi^2 dy^2 + \xi^2 dz^2, \tag{13}$$

where dr, dx, dy, dz are covectors corresponding respectively to the vector fields $\frac{\partial}{\partial s}, X, Y, Z$.

The entries off the diagonal of the 4×4 matrix associated with g are zero.

Applications of Ricci flow

Now we consider metrics $g(t)$ that evolve in time.

Definition

*A smooth time-dependent family of metrics $g(t)$ for all $t \geq 0$ is called the **Ricci flow** if it satisfies the partial differential equation*

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)}, \tag{14}$$

where $Ric_{g(t)}$ is the Ricci tensor.

Applications of Ricci flow

The Ricci flow equation (14), the diagonal form of g, the Ricci curvature expressions in terms φ, ψ, ξ, and some calculus all yield the system of partial differential equations associated with the Ricci flow $g(t)$:

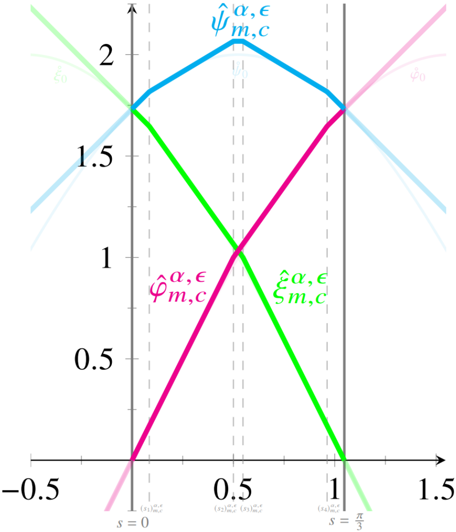
$$\zeta_t = -\left(\frac{\varphi_r}{\varphi} + \frac{\psi_r}{\psi} + \frac{\xi_r}{\xi}\right)\frac{\zeta_r}{\zeta^2} + \left(\frac{\varphi_{rr}}{\varphi} + \frac{\psi_{rr}}{\psi} + \frac{\xi_{rr}}{\xi}\right)\frac{1}{\zeta}, \tag{15}$$

$$\varphi_t = \frac{1}{\zeta^2}\varphi_{rr} + \frac{1}{\zeta\psi\xi}\left(\frac{\psi\xi}{\zeta}\right)_r \varphi_r - \frac{1}{2\psi^2\xi^2}\varphi^3 + \frac{(\psi^2 - \xi^2)^2}{2\psi^2\xi^2}\frac{1}{\varphi}, \tag{16}$$

$$\psi_t = \frac{1}{\zeta^2}\psi_{rr} + \frac{1}{\zeta\varphi\xi}\left(\frac{\varphi\xi}{\zeta}\right)_r \psi_r - \frac{1}{2\varphi^2\xi^2}\psi^3 + \frac{(\varphi^2 - \xi^2)^2}{2\varphi^2\xi^2}\frac{1}{\psi}, \tag{17}$$

$$\xi_t = \frac{1}{\zeta^2}\xi_{rr} + \frac{1}{\zeta\varphi\psi}\left(\frac{\varphi\psi}{\zeta}\right)_r \xi_r - \frac{1}{2\varphi^2\psi^2}\xi^3 + \frac{(\varphi^2 - \psi^2)^2}{2\varphi^2\psi^2}\frac{1}{\xi}. \tag{18}$$

Graphs of the functions for the metric



The middle metric

The *middle linearized metric* on **a neighborhood of** S^4 about $s = s_0$, for any fixed $0 < s_0 < \frac{\pi}{3}$ and for any $m, c \in \mathbb{R}$:

$$\bar{g}_{m,c} = ds^2 + \bar{\varphi}^2_{m,c}\, dx^2 + \bar{\psi}^2_{m,c}\, dy^2 + \bar{\xi}^2_{m,c}\, dz^2, \tag{19}$$

where we define $\bar{\varphi}_{m,c}, \bar{\psi}_{m,c}, \bar{\xi}_{m,c} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

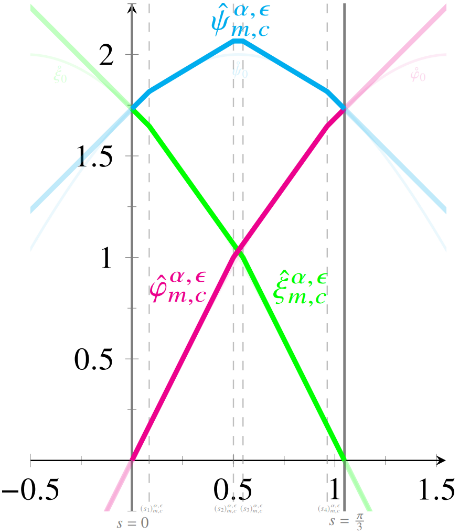
$$\bar{\varphi}_{m,c}(s) := c + ms \tag{20}$$

$$\bar{\psi}_{m,c}(s) := \frac{m\pi}{3} + 2c, \tag{21}$$

$$\bar{\xi}_{m,c}(s) := \frac{m\pi}{3} + c - ms. \tag{22}$$

A working example for our result is $\bar{g}_{m=\frac{7}{5}, c=\frac{3}{10}}$.

Graphs of the functions for the metric



First key lemma

We introduce a key lemma.

Lemma (First key lemma)

Let $\{(M, g_\tau)\}_{\tau \geq 0}$ *be a smooth family of Riemannian metrics. For simplicity of notation, we set* $g_0 = g$*. Suppose there exists a tangent plane* σ *that satisfies*

$$\sec_g(\sigma) = 0, \tag{23}$$

$$(\sec_{g(t)}(\sigma))_t|_{t=0} < 0. \tag{24}$$

If $\tau > 0$ *is sufficiently small, then* (M, g_τ) *evolves through Ricci flow to a metric with a negative sectional curvature; that is, there exists* $t_0 > 0$ *that satisfies*

$$\sec_{g_\tau(t_0)}(\sigma) < 0. \tag{25}$$

The middle metric

The middle metric $(S^4, \bar{g}_{m,c})$ has a negative temporal derivative of sectional curvature for suitable values of $m, c \in \mathbb{R}$.

Lemma

There exist $m \in (-\sqrt{2}, \sqrt{2})$, $c \in \mathbb{R}$, and a sufficiently small $\alpha > 0$ such that, for all $\frac{\pi}{6} - \alpha < s < \frac{\pi}{6} + \alpha$, the middle metric $\bar{g}_{m,c}$ has zero mixed curvatures and satisfies

$$\sec_{\bar{g}_{m,c}}\left(\frac{\partial}{\partial s}, X\right) = 0, \tag{26}$$

$$\frac{\partial}{\partial t}\left(\sec_{\bar{g}_{m,c}(t)}\left(\frac{\partial}{\partial s}, X\right)\right)\Bigg|_{t=0} < 0, \tag{27}$$

where $\bar{g}_{m,c}(t)$ is Ricci flow whose initial metric is $\bar{g}_{m,c}(0) = \bar{g}_{m,c}$.

A working example is $\bar{g}_{m=\frac{7}{5}, c=\frac{3}{10}}$.

The deformed middle metric

The corresponding family of *deformed linearized metrics* $\{\bar{g}^{\epsilon}_{m,c}\}_{\epsilon \geq 0}$ on **a neighborhood of** S^4 about $s = s_0$, for any fixed $0 < s_0 < \frac{\pi}{3}$ and for any $m, c \in \mathbb{R}$, defined by

$$\bar{g}^{\epsilon}_{m,c} = ds^2 + (\bar{\varphi}^{\epsilon}_{m,c})^2\, dx^2 + (\bar{\psi}^{\epsilon}_{m,c})^2\, dy^2 + (\bar{\xi}^{\epsilon}_{m,c})^2\, dz^2, \tag{28}$$

where we define $\bar{\varphi}^{\epsilon}_{m,c}, \bar{\psi}^{\epsilon}_{m,c}, \bar{\xi}^{\epsilon}_{m,c} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

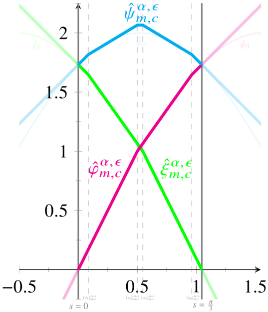
$$\bar{\varphi}^{\epsilon}_{m,c}(s) := c + ms - \epsilon s^3 \tag{29}$$

$$\bar{\psi}^{\epsilon}_{m,c}(s) := \frac{m\pi}{3} + 2c - \epsilon s^2 - \epsilon s^3, \tag{30}$$

$$\bar{\xi}^{\epsilon}_{m,c}(s) := \frac{m\pi}{3} + c - ms - \epsilon s^2. \tag{31}$$

Graphs of the functions for the metric

The figure below shows the graphs of $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\psi}^{\alpha,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\epsilon}_{m,c}$ when $\alpha > 0$ and $\epsilon > 0$ are very small.



As we will see, one of the tangent planes at $s = \frac{\pi}{6}$ has positive radial sectional curvature that will flow to a negative one through the Ricci flow.

The metrics we will use to construct a one-parameter family

The *modified round metric* on a neighborhood of the singular orbit of S^4 about $s = 0$, for any $\alpha > 0$:

$$\mathring{g}_0^\alpha = dr^2 + (\mathring{\varphi}_0^\alpha)^2\, d\mathring{x}^2 + (\mathring{\psi}_0^\alpha)^2\, d\mathring{y}^2 + (\mathring{\xi}_0^\alpha)^2\, d\mathring{z}^2, \tag{32}$$

where we define $\mathring{\varphi}_0^\alpha, \mathring{\psi}_0^\alpha, \mathring{\xi}_0^\alpha : [0, \frac{\pi}{3\alpha}] \to \mathbb{R}$ by

$$\mathring{\varphi}_0^\alpha(s) := \frac{2}{\alpha}\sin(\alpha s), \tag{33}$$

$$\mathring{\psi}_0^\alpha(s) := \sqrt{3}\cos(\alpha s) + \frac{1}{\alpha}\sin(\alpha s), \tag{34}$$

$$\mathring{\xi}_0^\alpha(s) := \sqrt{3}\cos(\alpha s) - \frac{1}{\alpha}\sin(\alpha s), \tag{35}$$

The metrics we will use to construct a one-parameter family

The *modified round metric* on a neighborhood of the singular orbit of S^4 about $s = \frac{\pi}{3}$, for any $\alpha > 0$:

$$\mathring{g}^{\alpha}_{\frac{\pi}{3}} = dr^2 + (\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}})^2 \, d\mathring{x}^2 + (\mathring{\psi}^{\alpha}_{\frac{\pi}{3}})^2 \, d\mathring{y}^2 + (\mathring{\xi}^{\alpha}_{\frac{\pi}{3}})^2 \, d\mathring{z}^2, \tag{36}$$

where we define $\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}, \mathring{\psi}^{\alpha}_{\frac{\pi}{3}}, \mathring{\xi}^{\alpha}_{\frac{\pi}{3}} : [0, \frac{\pi}{3\alpha}] \to \mathbb{R}$ by

$$\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}(s) := \mathring{\psi}^{\alpha}_{0}\left(s - \frac{\pi}{3}\right), \tag{37}$$

$$\mathring{\psi}^{\alpha}_{\frac{\pi}{3}}(s) := \mathring{\xi}^{\alpha}_{0}\left(s - \frac{\pi}{3}\right), \tag{38}$$

$$\mathring{\xi}^{\alpha}_{\frac{\pi}{3}}(s) := -\mathring{\varphi}^{\alpha}_{0}\left(s - \frac{\pi}{3}\right). \tag{39}$$

The metric is smooth at the singular orbits

Theorem

The SO(3)*-invariant metric* g *is smooth on the singular orbit* $(\mathrm{SO}(3) \times \mathbb{D}^2)/\,\mathrm{SO}(3)_{\gamma(0)}$ *if and only if the metric*

$$g|_{\mathbb{D}^2} := ds^2 + \varphi^2\, dx^2 \tag{40}$$

is smooth on $\mathbb{D}^2$ *and the extended functions* $\psi_{ext}, \xi_{ext} : [-L, L] \to \mathbb{R}$ *defined by*

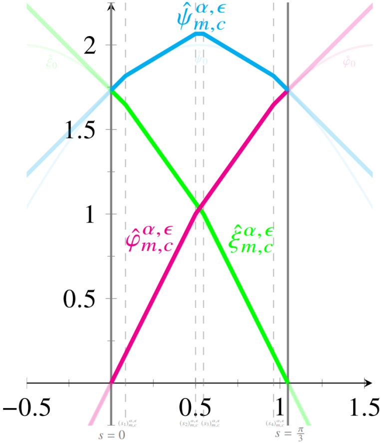
$$\psi_{ext}(s) := \begin{cases} \psi(s) & \text{for } s \geq 0, \\ \xi(-s) & \text{for } s < 0 \end{cases} \tag{41}$$

$$\xi_{ext}(s) := \begin{cases} \xi(s) & \text{for } s \geq 0, \\ \psi(-s) & \text{for } s < 0 \end{cases} \tag{42}$$

are also smooth on $[-L, L]$.

Graphs of the functions for the metric

We remind the audience of the graphs of $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\psi}^{\alpha,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\epsilon}_{m,c}$ when $\alpha > 0$ and $\epsilon > 0$ are very small.



The metrics we will use to construct a one-parameter family

The *deformed hat metric*, for any $\epsilon > 0$:

$$\hat{g}^{\alpha,\epsilon}_{m,c} = dr^2 + (\hat{\varphi}^{\alpha,\epsilon}_{m,c})^2\, dx^2 + (\hat{\psi}^{\alpha,\epsilon}_{m,c})^2\, dy^2 + (\hat{\xi}^{\alpha,\epsilon}_{m,c})^2\, dz^2, \tag{43}$$

where we define $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\psi}^{\alpha,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\epsilon}_{m,c} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\hat{\varphi}^{\alpha,\epsilon}_{m,c}(s) := \min\{\mathring{\varphi}^{\alpha}_{0}(s), \bar{\varphi}^{\epsilon}_{m,c}(s), \mathring{\varphi}^{\alpha}_{\frac{\pi}{3}}(s)\}, \tag{44}$$

$$\hat{\psi}^{\alpha,\epsilon}_{m,c}(s) := \min\{\mathring{\psi}^{\alpha}_{0}(s), \bar{\psi}^{\epsilon}_{m,c}(s), \mathring{\psi}^{\alpha}_{\frac{\pi}{3}}(s)\}, \tag{45}$$

$$\hat{\xi}^{\alpha,\epsilon}_{m,c}(s) := \min\{\mathring{\xi}^{\alpha}_{0}(s), \bar{\xi}^{\epsilon}_{m,c}(s), \mathring{\xi}^{\alpha}_{\frac{\pi}{3}}(s)\}. \tag{46}$$

Metric modification techniques

Consider a mollifier $\eta : \mathbb{R} \to \mathbb{R}$ that is positive and has compact support. Furthermore, consider for any $\delta > 0$ the function $\eta^\delta : \mathbb{R} \to \mathbb{R}$ given by

$$\eta^\delta(x) := \frac{1}{\delta}\eta\left(\frac{x}{\delta}\right) \tag{47}$$

that satisfies

$$\int_{-\infty}^{\infty} \eta^\delta(x)\, dx = 1. \tag{48}$$

Define for any $\delta > 0$ the *mollification* $f^\delta : \mathbb{R} \to \mathbb{R}$ by the convolution of η^δ and f; namely, we define

$$f^\delta(x) := (\eta^\delta * f)(x) = \int_{-\infty}^{\infty} \eta^\delta(x - z) f(z)\, dz. \tag{49}$$

Metric modification techniques

For any closed interval $[a, b] \subseteq \mathbb{R}$, let $H_{a,b} : \mathbb{R} \to \mathbb{R}$ be a smooth, nondecreasing function that is 0 on $(-\infty, a]$ and 1 on $[b, \infty)$. This allows us to glue the functions $f, h : \mathbb{R} \to \mathbb{R}$, which we write as

$$(f\,{}_a\!\diamond_b\, h)(x) := (1 - H_{a,b}(x))f(x) + H_{a,b}(x)h(x) \tag{50}$$

In particular, the glue function is smooth on $\mathbb{R}$ and satisfies

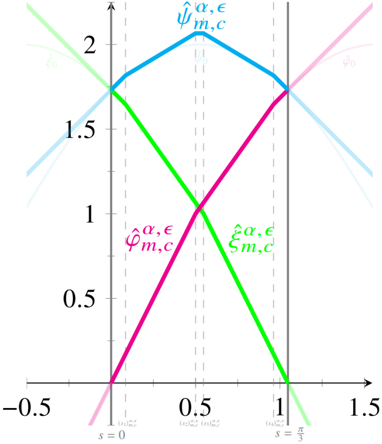
$$(f\,{}_a\!\diamond_b\, h)(x) = f(x) \text{ for all } x \leq a, \tag{51}$$

$$(f\,{}_a\!\diamond_b\, h)(x) = h(x) \text{ for all } x \geq b. \tag{52}$$

The glue operation allows us to glue the convolved functions with the modified round metrics using bump functions.

Graphs of the functions for the metric

Once again, we remind the audience of the graphs of $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\psi}^{\alpha,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\epsilon}_{m,c}$ when $\alpha > 0$ and $\epsilon > 0$ are very small.



Glued metric

In particular, define for any $\alpha > 0$, $\delta > 0, \epsilon > 0$ the glue functions $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{3,4,7,8}}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,5,6}} : [0, \frac{\pi}{3}] \to \mathbb{R}$ by

$$\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{3,4,7,8}}(s) := (\mathring{\varphi}^{\alpha}_{0}\,{}_{\rho_3}\diamond_{\rho_4}\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}\,{}_{\rho_7}\diamond_{\rho_8}\mathring{\varphi}^{\alpha}_{\frac{\pi}{3}})(s), \tag{53}$$

$$\hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,5,6}}(s) := (\mathring{\xi}^{\alpha}_{0}\,{}_{\rho_1}\diamond_{\rho_2}\hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}\,{}_{\rho_5}\diamond_{\rho_6}\mathring{\xi}^{\alpha}_{0})(s), \tag{54}$$

and $\hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,3,4,5,6,7,8}} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,3,4,5,6,7,8}}(s) := \hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{3,4,7,8}}(s) + \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,5,6}}(s). \tag{55}$$

Accordingly, we have the *glued metric* for any $m, c \in \mathbb{R}$ and for any $\delta > 0, \epsilon > 0$:

$$\begin{aligned}\hat{g}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,3,4,5,6,7,8}} = dr^2 &+ (\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{3,4,7,8}})^2\, dx^2 \\ &+ (\hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,3,4,5,6,7,8}})^2\, dy^2 \\ &+ (\hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,5,6}})^2\, dz^2.\end{aligned} \tag{56}$$

One-parameter family of metrics

Define the one-parameter family of metrics $\{g^{\tau}_{\mathsf{final}}\}_{\tau \geq 0}$ taking the form

$$g^{\tau}_{\mathsf{final}} = dr^2 + \varphi^{\tau}_{\mathsf{final}}\, dx^2 + \psi^{\tau}_{\mathsf{final}}\, dy^2 + \xi^{\tau}_{\mathsf{final}}\, dz^2, \tag{57}$$

where we define $\varphi^{\tau}_{\mathsf{final}}, \psi^{\tau}_{\mathsf{final}}, \xi^{\tau}_{\mathsf{final}} : \mathbb{R} \to \mathbb{R}$ by

$$\varphi^{\tau}_{\mathsf{final}} := \hat{\varphi}^{\alpha,\delta,\tau\epsilon}_{m=\frac{7}{5},c=\frac{3}{10},\rho_{3,4,7,8}}, \tag{58}$$

$$\psi^{\tau}_{\mathsf{final}} := \hat{\psi}^{\alpha,\delta,\tau\epsilon}_{m=\frac{7}{5},c=\frac{3}{10},\rho_{1,2,3,4,5,6,7,8}}, \tag{59}$$

$$\xi^{\tau}_{\mathsf{final}} := \hat{\xi}^{\alpha,\delta,\tau\epsilon}_{m=\frac{7}{5},c=\frac{3}{10},\rho_{1,2,5,6}}. \tag{60}$$

for all $0 < \delta < \delta_0$ and $0 < \epsilon < \epsilon_0$.

Outline of our proof of the main theorem

Now that we have introduced all the Riemannian metrics on S^4 that we need, we will now turn our attention to outlining our proof of the main theorem, which we will state again here:

Theorem (Main theorem)

There exists a Riemannian metric with positive radial sectional curvature on S^4 that attains a negative radial sectional curvature when evolved by the Ricci flow.

To prove this theorem, we will need two key lemmas.

Outline of our proof of the main theorem

Lemma (First key lemma)

Let $\{(M, g_\tau)\}_{\tau \geq 0}$ *be a smooth family of Riemannian metrics. For simplicity of notation, we set* $g_0 = g$*. Suppose there exists a tangent plane* σ *that satisfies*

$$\sec_g(\sigma) = 0, \tag{61}$$

$$(\sec_{g(t)}(\sigma))_t|_{t=0} < 0. \tag{62}$$

If $\tau > 0$ *is sufficiently small, then* (M, g_τ) *evolves through Ricci flow to a metric with a negative sectional curvature; that is, there exists* $t_0 > 0$ *that satisfies*

$$\sec_{g_\tau(t_0)}(\sigma) < 0. \tag{63}$$

Outline of our proof of the main theorem

Lemma (Second key lemma)

There exists g on S^4 of the form given by (13) such that:

1. *The functions φ, ψ, ξ are concave down.*
2. *For $k = 0, 1$ and for $s^* = 0, \frac{\pi}{3}$, the functions φ, ψ, ξ satisfy*

$$\varphi^{(k)}(s^*) = (\mathring{\varphi}_0^{\alpha})^{(k)}(s^*), \tag{64}$$

$$\psi^{(k)}(s^*) = (\mathring{\psi}_0^{\alpha})^{(k)}(s^*), \tag{65}$$

$$\xi^{(k)}(s^*) = (\mathring{\xi}_0^{\alpha})^{(k)}(s^*). \tag{66}$$

3. *There exists $t_0 \in (0, \frac{\pi}{3})$ such that, for any $p \in \mathrm{SO}(3)/\mathrm{SO}(3)_{\gamma(t_0)}$, there exists a radial plane $\sigma \in T_pS^4$ that satisfies $(\sec_{g(t)}(\sigma))_t|_{t=0} < 0$, where $g(t)$ solves Ricci flow near (t_0, p) whose initial metric is $g(0) = g$.*

Outline of our proof of the main theorem

Proposition

Let $f, h : \mathbb{R} \to \mathbb{R}$ be twice differentiable functions that are strictly concave on any closed interval $[a, b] \subseteq \mathbb{R}$. Then:

1. *The sum function $f + h$ is strictly concave on $\mathbb{R}$.*
2. *If $\delta > 0$ is sufficiently small, then the mollification $f^\delta := \eta^\delta * f$ is strictly concave on $[a, b]$.*
3. *For any $\beta > 0$, there exists $\epsilon > 0$ such that, if f, h satisfy*

$$\max(\max_{x\in[a,b]} f''(x), \max_{x\in[a,b]} h''(x)) \leq -\beta, \tag{67}$$

$$\|f - h\|_{C^1([a,b])} < \epsilon, \tag{68}$$

   then the glue function $f\,{}_a\!\diamond_b\, h$ is strictly concave on $[a, b]$.

In particular, $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{3,4,7,8}}, \hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,3,4,5,6,7,8}}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,5,6}} : [0, \frac{\pi}{3}] \to \mathbb{R}$ are all strictly concave on $[0, \frac{\pi}{3}]$, according to the Proposition.

Outline of our proof of the main theorem

Lemma

Let (S^4, g) be SO(3)*-invariant. Suppose g takes the form in* (13)*.*

1. *If φ, ψ, ξ are concave, then (S^4, g) has a nonnegative radial sectional curvature.*
2. *If φ, ψ, ξ are strictly concave, then (S^4, g) has positive radial sectional curvature.*

Lemma

Fix any number $\tau_0 > 0$. There exist $\delta > 0$ and $\epsilon > 0$ such that the following properties hold:

1. *The manifold (S^4, g^{τ}_{final}) has positive radial sectional curvature for all $0 < \tau < \tau_0$.*
2. *The family of manifolds $\{(S^4, g^{\tau}_{final})\}_{0 \leq \tau \leq \tau_0}$ satisfies the hypotheses of the first key lemma.*

Outline of our proof of the main theorem

Let us summarize what we have discussed before we prove the main theorem.

1. Our functions $\hat{\varphi}^{\alpha,\epsilon}_{m,c}, \hat{\psi}^{\alpha,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\epsilon}_{m,c}$, where $\alpha > 0$ and $\epsilon > 0$ are small, are continuous, piecewise smooth, and strictly concave.
2. Their corresponding mollifications $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}, \hat{\psi}^{\alpha,\delta,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}$ maintain the properties we mentioned in (1) and are also smooth.
3. The glue functions $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{3,4,7,8}}, \hat{\psi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,3,4,5,6,7,8}}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,5,6}}$ glue $\hat{\varphi}^{\alpha,\delta,\epsilon}_{m,c}, \hat{\psi}^{\alpha,\delta,\epsilon}_{m,c}, \hat{\xi}^{\alpha,\delta,\epsilon}_{m,c}$ with $\mathring{\varphi}^{\alpha}_{0}, \mathring{\psi}^{\alpha}_{0}, \mathring{\xi}^{\alpha}_{0}$, respectively. The glue functions maintain the properties we mentioned in (1) and (2), and they satisfy (64), (65), (66), as asserted by the second key lemma. We called this resulting metric $g^{\alpha,\delta,\epsilon}_{m,c,\rho_{1,2,3,4,5,6,7,8}}$.
4. Finally, we were able to construct our desired one-parameter family of metrics $\{g^{\tau}_{\text{final}}\}_{\tau \geq 0} := \{g^{\alpha,\delta,\tau\epsilon}_{m,c,\rho_{1,2,3,4,5,6,7,8}}\}_{\tau \geq 0}$.

Finally, we will write a short proof of the main theorem, citing our previous propositions and lemmas. We will also restate the main theorem one more time.

Theorem (Main theorem)

There exists a Riemannian metric with positive radial sectional curvature on S^4 that attains a negative radial sectional curvature when evolved by the Ricci flow.

Proof. Consider the one-parameter family of metrics $\{g_{\text{final}}^{\tau}\}_{\tau \geq 0}$. For all $0 < \tau < \tau_0$, the final key lemma asserts that the manifold $(S^4, g_{\text{final}}^{\tau})$ has positive radial sectional curvature and that the family $\{(S^4, g_{\text{final}}^{\tau})\}_{0 \leq \tau \leq \tau_0}$ satisfies the hypotheses of the first key lemma for the tangent plane $\text{span}(\frac{\partial}{\partial s}, X)$. Furthermore, the first key lemma asserts (25), which is precisely the assertion of our main theorem.

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