Days 3-4: Metric spaces

Why study metric spaces? A metric space is an abstract generalization of the Euclidean space, such as $\mathbb{R}^3$. Sometimes, mathematicians would like to work in abstract terms. A new mathematical result is more powerful whenever it can be presented in an abstract sense. For instance, one can construct a sequence of points that converges to a limit point in not only Euclidean spaces, but also metric spaces in general.

Definitions. *Let X be set of points. Then X is called a* metric space *if, for any two points $x, y \in X$, there exists a positive real number $d(x, y)$—the distance from x to y—that satisfies*

(1) *positivity: $d(x, y) > 0$ if $x \neq y$, $d(x, x) = 0$,*

(2) *symmetry: $d(x, y) = d(y, x)$,*

(3) *triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$ for all $z \in X$.*

Finally, any function $d(x, y)$ that satisfies the above three properties is called a metric.

Definition. *The absolute value of $x \in \mathbb{R}$, denoted $|x|$, is defined by*

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Theorem (Triangle Inequality). *For all $x, y \in \mathbb{R}$, we have*

$$|x + y| \leq |x| + |y|.$$

Proof. If we assume $x \geq 0$ and $y \geq 0$, then we have $|x| = x$ and $|y| = y$, and so we have

$$\begin{aligned} |x + y| &= \begin{cases} x + y & \text{if } x + y \geq 0, \\ -(x + y) & \text{if } x + y \leq 0, \end{cases} \\ &= \begin{cases} x + y & \text{if } x + y \geq 0, \\ -x - y & \text{if } x + y \leq 0, \end{cases} \\ &\leq \begin{cases} |x| + |y| & \text{if } x + y \geq 0, \\ -|x| - |y| & \text{if } x + y \leq 0, \end{cases} \\ &\leq \begin{cases} |x| + |y| & \text{if } x + y \geq 0, \\ |x| + |y| & \text{if } x + y \leq 0, \end{cases} \\ &= |x| + |y|. \end{aligned}$$

The case $x < 0$ and $y < 0$ requires a similar argument. Now, if we assume $x \geq 0$ and $y < 0$, then we have $|x| = x$ and $|y| = -y$, and so we have

$$\begin{aligned}
|x+y| &= \begin{cases} x+y & \text{if } x+y \geq 0, \\ -(x+y) & \text{if } x+y \leq 0, \end{cases} \\
&= \begin{cases} x+y & \text{if } x+y \geq 0, \\ -x-y & \text{if } x+y \leq 0, \end{cases} \\
&\leq \begin{cases} |x|-|y| & \text{if } x+y \geq 0, \\ -|x|+|y| & \text{if } x+y \leq 0, \end{cases} \\
&\leq \begin{cases} |x|+|y| & \text{if } x+y \geq 0, \\ |x|+|y| & \text{if } x+y \leq 0, \end{cases} \\
&= |x|+|y|.
\end{aligned}$$

The case $x < 0$ and $y \geq 0$ requires a similar argument. □

Example. *Let $x, y \in \mathbb{R}$ be points. Show that $d(x, y) = |x - y|$ is a metric in $\mathbb{R}$. This would make $\mathbb{R}$ a metric space.*

Proof. We will satisfy the three properties of the definition of a metric.

(1) First, we will show that $d(x, y)$ satisfies positivity. We have

$$\begin{aligned}
d(x,x) &= |x-x| \\
&= |0| \\
&= 0.
\end{aligned}$$

Also, if $x \neq y$, or equivalently $x - y \neq 0$, then the law of trichotomy—every number is positive, negative, or zero—implies either $x - y > 0$ or $x - y < 0$. In either case, we have $|x - y| > 0$. In other words, if $x \neq y$, then we have

$$\begin{aligned}
d(x,y) &= |x-y| \\
&> 0,
\end{aligned}$$

as desired.

(2) Next, we will show that $d(x, y)$ satisfies symmetry. If $x - y > 0$, then we have $|x - y| = x - y$ and $|y - x| = -(y - x)$, and so we have

$$\begin{aligned}
d(x,y) &= |x-y| \\
&= x-y \\
&= -(y-x) \\
&= |y-x| \\
&= d(y,x),
\end{aligned}$$

as desired.

(3) Finally, we will show that $d(x, y)$ satisfies the triangle inequality. For all $z \in X$, we have

$$\begin{aligned} d(x, y) &= |x - y| \\ &= |(x - z) + (z - y)| \\ &\leq |x - z| + |z - y| \\ &= d(x, z) + d(z, y), \end{aligned}$$

as desired.

Since the three properties of a metric are satisfied, we conclude that $d(x, y) = |x - y|$ is a metric in $\mathbb{R}^2$. □

Example. *Let $x, y \in \mathbb{R}$ be points. Show that $d(x, y) = x - y$ is NOT a metric in $\mathbb{R}$.*

Proof. To show that $d(x, y) = x - y$ is NOT a metric in $\mathbb{R}$, we must show by an explicit example that one of the properties of the metric is not satisfied. Let $x = 1 \in \mathbb{R}$ and $y = -1 \in \mathbb{R}$. Then we have $1 \neq -1$ and

$$\begin{aligned} d(x, y) &= d(1, -1) \\ &= 1 - (-1) \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} d(y, x) &= d(-1, 1) \\ &= (-1) - 1 \\ &= -2. \end{aligned}$$

In other words, we have shown $d(x, y) \neq d(y, x)$, meaning that $d(x, y) = x - y$ does NOT satisfy the property of symmetry in the definition of a metric. □

Exercise. *Let $x, y \in \mathbb{R}$ be points. Some of these functions $d(x, y)$ are metrics in $\mathbb{R}$, but others are not. Determine which of the following are metrics. For the ones that are not, show an explicit example that violates one of the properties in the definition of a metric.*

(1) $d(x, y) = (x - y)^2$,

(2) $d(x, y) = \sqrt{|x - y|}$,

(3) $d(x, y) = |x^2 - y^2|$,

(4) $d(x, y) = |x - 2y|$,

(5) $d(x, y) = \dfrac{|x - y|}{1 + |x - y|}$.