Days 1-2: Sequences and subsequences

Definition. *A* sequence $\{a_n\}_{n=1}^{\infty}$ *is a function defined on the set of all positive integers* $n = 1, 2, 3, \ldots$ *to real numbers.*

Examples (Example 1)**.** *The sets*

$$\{n\}_{n=1}^{\infty}, \{n^2\}_{n=1}^{\infty}, \left\{\frac{1}{n}\right\}_{n=1}^{\infty}, \left\{\frac{n^2+n+2}{2n^2-3}\right\}_{n=1}^{\infty}, \{\sin(2\pi n)\}_{n=1}^{\infty}, \left\{\frac{n}{(n!)^{\frac{1}{n}}}\right\}_{n=1}^{\infty}$$

are all examples of sequences.

Definition. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence. Consider a sequence* $\{n_k\}_{k=1}^{\infty}$ *of positive integers that satisfy the infinite chain of inequalities* $n_1 < n_2 < n_3 < \cdots$*. Then* $\{a_{n_k}\}_{k=1}^{\infty}$ *is called a* subsequence *of* $\{a_n\}_{n=1}^{\infty}$.

Examples. *The sets*

$$\{3n-1\}_{n=1}^{\infty}, \{4n^2\}_{n=1}^{\infty}, \left\{\frac{1}{2n+6}\right\}_{n=1}^{\infty}, \left\{\frac{4n^2+2n+2}{8n^2-3}\right\}_{n=1}^{\infty}, \{\sin(8\pi n)\}_{n=1}^{\infty}, \left\{\frac{2n}{((2n)!)^{\frac{1}{2n}}}\right\}_{n=1}^{\infty}$$

are all respective examples of subsequences of the sequences in Example 1.

Exercise. *Find the limits of each of the sequences in Example 1. If a limit of a certain sequence does not exist, state "DNE".*

Definition. *A sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers is said to* converge *if there exists a real number* a *with the following property: For all* $\epsilon > 0$*, there exists an integer* N *such that* $n \geq N$ *implies* $|a_n - a| < \epsilon$*. In this case, we say that* a *is the limit of* $\{a_n\}_{n=1}^{\infty}$*, writing* $\lim\limits_{n\to\infty} a_n = a$*.*

Example. *Find the limit of the sequence* $\left\{\dfrac{n^2+n}{n^2-1}\right\}_{n=1}^{\infty}$*. Then prove it using the definition of the limit of a sequence.*

Proof. The limit of the sequence is

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2+n}{n^2-1} &= \lim_{n\to\infty} \frac{n^2+n}{n^2-1}\,\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1-\frac{1}{n^2}} \\
&= \frac{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n}}{\lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty} \frac{1}{n^2}} \\
&= \frac{1+0}{1-0} \\
&= 1.
\end{aligned}$$

Now we will prove $\lim_{n\to\infty} \frac{n^2+n}{n^2-1} = 1$ using the definition of the limit of a sequence. Let $\epsilon > 0$ be given. Choose $N = 1 + \frac{1}{\epsilon}$. If $n \geq N$, then we have

$$\begin{aligned}
\left|\frac{n^2+n}{n^2-1} - 1\right| &= \left|\frac{n^2+n}{n^2-1} - \frac{n^2-1}{n^2-1}\right| \\
&= \left|\frac{(n^2+n)-(n^2-1)}{n^2-1}\right| \\
&= \left|\frac{n+1}{(n+1)(n-1)}\right| \\
&= \left|\frac{1}{n-1}\right| \\
&= \frac{1}{n-1} \\
&\leq \frac{1}{N-1} \\
&= \epsilon,
\end{aligned}$$

as desired. □

Exercise. *For the first three sequences of Example 1, use the definition of sequence convergence to show that the sequences indeed converge to their respective limits.*

Now, consider the Euler constant

$$e \approx 2.71828182845904523536028747135266249775724709369995...$$

Example. *Compute the limit of the sequence* $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$. *The answer is*

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

You may use all the tools you have learned in first-year calculus. Note, however, that we are NOT asking you here to prove this limit.

Solution. Let

$$y = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Then we can take the natural log of both sides to obtain

$$\begin{aligned}
\ln(y) &= \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right) \\
&= \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n \\
&= \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right).
\end{aligned}$$

Recall for all $x \in \mathbb{R}$ the Taylor series

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots.$$

Then we have

$$\begin{aligned} n\ln\left(1+\frac{1}{n}\right) &= n\left(\frac{1}{n}-\frac{1}{2}\left(\frac{1}{n}\right)^2+\frac{1}{3}\left(\frac{1}{n}\right)^3-\frac{1}{4}\left(\frac{1}{n}\right)^4+\cdots\right)\\ &= n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\cdots\right)\\ &= 1-\frac{1}{2n}+\frac{1}{3n^2}-\frac{1}{4n^3}+\cdots . \end{aligned}$$

So we have

$$\begin{aligned} \ln(y) &= \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)\\ &= \lim_{n\to\infty}\left(1-\frac{1}{2n}+\frac{1}{3n^2}-\frac{1}{4n^3}+\cdots\right)\\ &= \lim_{n\to\infty} 1-\lim_{n\to\infty}\frac{1}{2n}+\lim_{n\to\infty}\frac{1}{3n^2}-\lim_{n\to\infty}\frac{1}{4n^3}+\cdots\\ &= 1-0+0-0+\cdots\\ &= 1. \end{aligned}$$

Finally, we can exponentiate both sides to obtain $y = e$, or equivalently

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,$$

as desired. □

Now also recall that, if n is a positive integer, then its factorial is defined

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1.$$

Exercise (very hard)**.** *Compute the limit of the sequence* $\left\{\dfrac{n}{(n!)^{\frac{1}{n}}}\right\}_{n=1}^{\infty}$. *The answer is*

$$\lim_{n\to\infty}\frac{n}{(n!)^{\frac{1}{n}}} = e.$$

You may use all the tools you have learned in first-year calculus. Note, however, that we are NOT asking you here to prove this limit.