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Some remarks on the cohomology of $SU(3)$ gauge orbit space

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Abstract: I compute the rational cohomology ring of the physical configuration space of gauge theories with structure group $SU(3)$ over a simply connected four-manifold. The consequences of this computation are analyzed, in relation with gauge anomalies of the Dirac operator coupled with gauge field and with possible definition of $SU(3)$-polynomial invariants for smooth manifolds.

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1. Introduction

In this letter I want to discuss the (rational) cohomological structure of the orbit space of Yang–Mills theory over a simply-connected four-manifold M with the structure group $SU(3)$. As is well known this orbit space $\mathcal{B}_M$ is mathematically defined as the quotient of the space of connections of a principal $SU(3)$-bundle over M modulo gauge transformations, i.e., vertical automorphisms of the bundle. The knowledge of the cohomology ring of the orbit space, and in particular of the second cohomology group, allows us to make some remarks in the study of two interesting problems, coming respectively from quantum field theory and from differential topology of four-manifolds.

First of all, the presence of the so called gauge anomalies in quantum field theories is directly related with the nonvanishing of the second cohomology group, which witnesses the presence of a nontrivial line bundle over $\mathcal{B}_M$ whose sections represent the regularized determinant of Dirac operator coupled with Yang–Mills field: when these sections have zeroes, the usual perturbative treatment of the quantum theory gives up, presenting "an anomaly". The general knowledge of the generators of the second cohomology group allows a complete treatment of the problem, at least at the topological level, for all simply connected four-manifolds. In particular I shall prove that, when we consider a general four-manifold, there are no significant differences, for what concerns anomalies of the Dirac operator, with the S^4 case.

The second problem is related with the definition of Donaldson polynomial invariants for smooth four-manifolds. These invariants, roughly speaking, are defined via a pairing of moduli spaces of $SU(2)$ anti-self-dual connections with polynomials in the second homology group of the four-manifold, suitably interpreted as polynomials in the second cohomology group of $SU(2)$ gauge orbit space. Donaldson asks whether the generalization of the structure group to $SU(n)$,

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$n > 2$, could give some significative upgrading of the results. In the last paragraph I will briefly comment on this.

2. Notation and preliminaries

In this paper I shall consider an $SU(3)$-principal bundle P over a simply connected four-manifold M. The space of connections $\mathcal{A}$ over this bundle is an affine space modeled over $\Omega^1(M, \text{ad } P)$. The physical configuration space appears as a quotient of $\mathcal{A}$ modulo the actions of vertical automorphisms group $\mathcal{G}$ of the bundle. To render the quotient a smooth Banach space the standard procedure is to complete the aforementioned spaces in a suitable Sobolev norm and to pick those vertical automorphisms which send a given point of M (the "point at infinity") to the identity of $SU(3)$. With this procedure the quotient space $\mathcal{B}_M$ is a smooth Banach manifold. This orbit space has weakly the homotopy type of $\text{Map}^0(M, BSU(3))$, base point preserving maps from M to the classifying space of $SU(3)$ (see ref. [1]). Now, for a general simply connected four-manifold whose intersection form has rank r, standard homotopical classification (see ref. [2]) implies that up to homotopy equivalence there's a cofibration

$$\bigvee_1^r S^2 \hookrightarrow M \longrightarrow S^4$$

which, under the application of the functor $\text{Map}^0(\,\cdot\,, BSU(3))$, induces the following fibration of orbit spaces:

$$\mathcal{B}_{S^4} \hookrightarrow \mathcal{B}_M \longrightarrow \prod_1^r \mathcal{B}_{S^2}.$$

With this fibration in mind it is possible to compute the cohomology ring of $\mathcal{B}_M$ once the cohomology ring of the base and the fibre is known. Our task is then to exhibit these two cohomology rings, which appear as the cohomology rings of $\prod_1^r \Omega SU(3)$ and of $\Omega^3 SU(3)$.

The Poincarè series of $\mathcal{B}_M$ has been computed, with different techniques and with structure group $U(n)$, in ref. [3].

3. Calculation

The technique involved in this calculation is basically that of Leray–Serre spectral sequence for fibrations, applied to the fibration

$$\Omega X \hookrightarrow PX \longrightarrow X$$

where X is a manifold and PX is the contractible space of loops over X starting from a fixed base point. This fibration, for simply connected X, gives rise to a spectral sequence with $E_2^{p,q} = H^p(X, \mathbb{Q}) \otimes H^q(\Omega X, \mathbb{Q})$ and $E_\infty^{p,q} = \mathbb{Q}$ for $p = q = 0$ and zero otherwise. We shall apply this to compute subsequently the cohomology of $\Omega SU(3)$, $\Omega^2 SU(3)$ and then, with a little trick, $\Omega^3 SU(3)$. We will prove the following

Lemma. *$H^*(\Omega SU(3))$ is a polynomial ring generated by two elements of degree 2 and degree 4; $H^*(\Omega^3 SU(3))$ is a polynomial ring generated by an element of degree 2.*

The input for the proof of this lemma resides in the structure of the cohomology ring for $SU(3)$; this group has the same cohomology ring of $S^3 \times S^5$, as could be easily proven via, e.g., the fibration $SU(2) \hookrightarrow SU(3) \to S^5$. Thus the 0th row of the E_2 term of the spectral sequence associated to the path fibration

$$\Omega SU(3) \hookrightarrow PSU(3) \longrightarrow SU(3)$$

appears as

$$H^p(SU(3)) = 1, 0, 0, x_3, 0, x_5, 0, 0, x_3 \cdot x_5, 0$$

where I have denoted by x_3 and x_5 the generators of the respective cohomology groups. (Note: I will frequently switch in the notation from cohomology groups to their generators).

As we require all elements of the spectral sequence to be killed (with the trivial exception of $E_2^{0,0}$) it is easy to verify that $H^*(\Omega SU(3))$ is a polynomial ring generated by two elements y_2 and y_4 such that $d_3(y_2) = x_3$ and $d_5(y_4) = x_5$, as the full E_2 term shows:

7	0								
6	y_2^3, y_4y_2			$y_2^3x_3, y_4y_2x_3$		.			.
5	0								
7	y_2^2, y_4			$y_2^2x_3, y_4x_3$		$y_2^2x_5, y_4x_5$			.
4	0								
3	y_2			y_2x_3		y_2x_5			$y_2x_3x_5$
2	0								
1	1	0	0	x_3	0	x_5	0	0	x_3x_5
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**

Iterating the previous procedure for the path fibration

$$\Omega^2 SU(3) \hookrightarrow P\Omega SU(3) \longrightarrow \Omega SU(3)$$

we obtain, from a careful study of the E_2 term of the spectral sequence, the result that the cohomology ring of $\Omega^2 SU(3)$ is an exterior algebra generated by two elements of degree 1 and 3, as the figure shows:

5	0							
4	z_1z_3		$z_1z_3y_2$		.		.	
3	z_3		z_3y_2		$z_3y_2^2, z_3y_4$		.	
2	0							
1	z_1		z_1y_2		$z_1y_2^2, z_1y_4$		$z_1y_2^3, z_1y_2y_4$	
0	1	0	y_2	0	y_2^2, y_4	0	y_2^3, y_2y_4	0
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

(as usual the 0th column represents the cohomology group of the fibre).

It is not possible now to apply directly the spectral sequence technique to the path fibration

$$\Omega^3 SU(3) \hookrightarrow P\Omega^2 SU(3) \longrightarrow \Omega^2 SU(3)$$

as the base space is not simply connected: but the difficulty is mild and can be solved, following the trick of ref. [1], passing to the universal covering of the base space, whose cohomology ring is an exterior algebra generated by an element of degree 3 (the element of degree 1, cause of the non simply connectedness, is killed by the covering procedure—the case is perfectly analogous to the covering $S^3 \times \mathbb{R} \to S^3 \times S^1$) and interpreting the kth component of $\Omega^3 SU(3)$ as the paths starting from the base point in $\Omega^2 SU(3)$ and ending at the kth translate. With this trick the cohomology ring of $(\Omega^3 SU(3))_k$ appears immediately as a polynomial ring in one generator of degree 2.

The consequence of the lemma is that we can compute the cohomology ring of the orbit space for the simply connected M in the light of the fibration of orbit spaces which, up to homotopy equivalence, appears as

$$\Omega^3 SU(3) \hookrightarrow \mathcal{B}_M \longrightarrow \prod_1^r \Omega SU(3).$$

From the form of the E_2 term, which contains nonzero elements only in all entries of type $(2n, 2m)$, is immediate to verify that the spectral sequence associated to this fibration degenerates at $E_2 = E_\infty$, as all derivations must be zero.

This proves the main aim:

Proposition. *The cohomology ring of the orbit space is a polynomial ring generated by* $r+1$ *forms of degree* 2 *and* r *forms of degree* 4.

4. Anomalies

My aim is now to discuss the result trying to identify the geometrical and physical meaning of the generators of the cohomology ring: first of all I will relate the result with the a priori study of gauge anomalies. Basically these anomalies appear when we consider a spin four-manifold and try to define a regularized determinant of the Dirac operator $\partial\!\!\!/_A$ which maps spinors (i.e., sections of a spin bundle over M) of opposite chiralities. The natural treatment of this problems consists in considering the determinant as a section of the determinant bundle of the index bundle of the Dirac operator over the gauge orbit space. When this determinant bundle is not trivial (i.e., the first Chern character of the index bundle, computed via Atiyah-Singer theorem, does not vanish), every section has zeroes and the quantization procedure, which requires the definition of the logarithm of $\det \partial\!\!\!/_A$, fails. To obtain a similar case it is thus necessary, a priori, that $H^2(\mathcal{B}_M, \mathbb{Q})$ does not vanish, otherwise all line bundles would be trivial. It is apparent from my construction how $H^2(\mathcal{B}_M, \mathbb{Q})$ is related to $H^2(\mathcal{B}_{S^4}, \mathbb{Q})$: this witnesses the fact that the determinant index bundle over $\mathcal{B}_{S^4}$—which generates, as proven in ref. [4], $H^2(\mathcal{B}_{S^4}, \mathbb{Q})$—enters in play for a general M giving a generator of degree 2 (which is not present in the case of $SU(2)$-principal bundles; this is related to the fact that for $SU(2)$ there are not standard anomalies but only global ones, which being torsion terms are invisible for a rational cohomology ring). Moreover it is apparent from the construction that the topological nature of the gauge anomaly is related with the 4-cell composing the four-manifold. Note that our procedure can be easily extended to a general structure group of the kind $SU(m)$, which cohomologically appears as $S^3 \times S^5 \times \ldots \times S^{2m-1}$: it is an easy (but lengthy) exercise to generalize the previous proposition at least for $SU(m)$ gauge theories over $(n-1)$-connected $2n$-manifolds. On the same vein, the results can be obtained by computing the rational homotopy type of the orbit space, but at the price of losing somehow the clearness of the relation with the S^4 case.

5. Conclusions

In this paper I have shown how the gauge anomaly, which is usually studied, in physical literature, for gauge theories over S^4, appears for general simply connected four-manifold. The presence of other nontrivial line bundles over the orbit space suggests the presence of other observables, whose definition is affected of anomaly when we consider a four-manifold different from S^4: it might be tempting to understand if these new anomalies could be of physical relevance. The result I have obtained might be of some interest also in the construction of smooth invariants of the type defined by Donaldson. It is easily recognizable that the r "horizontal" generators of degree 2 are of the same origin of those of the $SU(2)$ case, i.e., images under slant product, evaluated on the homology generators of M, of the second chern class of the universal bundle over $\mathcal{B}_M \times M$; similarly the generators of degree 4 appear as images under slant product of the third chern class of the same bundle. Instead the anomaly generator has not such an origin, as its presence is explained in terms of the four dimensional cell. The possibility of defining new polynomial invariants, which is suggested by these results, faces up immediately with the analytical difficulties involved in the study of $SU(3)$ moduli spaces.

References

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