Proceedings of the Royal Society of Edinburgh, **142A**, 359–370, *2012*

Construction of symplectic structures on 4-manifolds with a free circle action

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(MS received 24 April 2010; accepted 6 April 2011)

Let M be a closed 4-manifold with a free circle action. If the orbit manifold N^3 satisfies an appropriate fibering condition, then we show how to represent a cone in $H^2(M;\mathbb{R})$ by symplectic forms. This generalizes earlier constructions by Thurston, Bouyakoub and Fernández *et al*. In the case that M is the product 4-manifold $S^1 \times N$, our construction complements our previous results and allows us to determine completely the symplectic cone of such 4-manifolds.

1. Introduction and main results

Let M be a closed 4-manifold with a free circle action. We denote the orbit space by N and we denote by $p\colon M \to N$ the quotient map that defines a principal S^1-bundle over N. We denote by $p_*\colon H^2(M;\mathbb{R}) \to H^1(N;\mathbb{R})$ the map given by integration along the fibre. Our main result (which will be proved in § 2) is the following existence theorem.

Theorem 1.1. *Let M be a closed, oriented 4-manifold admitting a free circle action. Let $\psi \in H^2(M;\mathbb{R})$ such that $\psi^2 > 0 \in H^4(M;\mathbb{R})$ and such that $p_*(\psi) \in H^1(N;\mathbb{R})$ can be represented by a non-degenerate closed 1-form. Then there exists an S^1-invariant symplectic form ω on M with $[\omega] = \psi \in H^2(M;\mathbb{R})$.*

Remark 1.2.

(i) Note that, given $\phi \in H^1(N;\mathbb{R})$, we can represent ϕ by a non-degenerate (i.e. nowhere zero) closed 1-form if and only if ϕ lies in the cone on a fibred face of the Thurston norm ball (see [13] for details). Therefore, the theorem assumes implicitly that N admits a fibration over S^1.

(ii) This theorem generalizes work by Thurston [12], Bouyakoub [2] and Fernández *et al*. [4]. More precisely, Thurston first constructed symplectic forms on product manifolds $S^1 \times N$ for fibred 3-manifolds N. Bouyakoub generalized Thurston's results and showed that, given ψ as in the theorem, there exists an S^1-invariant symplectic form ω with $p_*([\omega]) = p_*(\psi)$. Finally, Fernández *et al*. proved the theorem in the case where $p_*(\psi)$ is rational.

Let W be a 4-manifold. The set of all elements of $H^2(W;\mathbb{R})$ which can be represented by a symplectic form is called the *symplectic cone* of W. Note that this is indeed a cone, i.e. if ψ can be represented by a symplectic form, then any non-zero scalar multiple can also be represented by a symplectic form. Determining the symplectic cone of 4-manifolds is a fundamental problem, but little seems to be known in general. We refer the reader to [10, § 3] for more information.

In [6] we showed that if N is a closed 3-manifold, then $S^1 \times N$ is symplectic if and only if N fibres over S^1. (In the case that $b_1(N) = 1$ this also follows from combining the work of Kutluhan and Taubes [8] with the work of Kronheimer and Mrowka [7] and Ni [11].) In fact, a slightly more precise version of this result [6, theorems 1.2 and 1.4] will allow us to determine, in § 3, the symplectic cone of closed 4-manifolds of the form $S^1 \times N$.

THEOREM 1.3. *Let N be a closed, oriented* 3*-manifold. Then, given $\psi \in H^2(S^1 \times N;\mathbb{R})$, the following are equivalent:*

(i) *ψ can be represented by a symplectic structure;*

(ii) *ψ can be represented by a symplectic structure which is S^1-invariant;*

(iii) *$\psi^2 > 0$ and the Künneth component $\phi = p_*(\psi) \in H^1(N;\mathbb{R})$ of ψ lies in the open cone on a fibred face of the Thurston norm ball of N.*

REMARK 1.4.

(i) Note that we are not claiming that *any* symplectic form is isotopic, or even homotopic to an S^1-invariant form, although this might be the case.

(ii) We expect a very similar theorem to hold for closed 4-manifolds with a free circle action. In fact, the proof of theorem 1.3 together with work of Bowden [3] and the authors [5] shows that an analogous statement holds for circle bundles $M \to N$ whenever N has vanishing Thurston norm or N is a graph manifold.

Convention. All maps are assumed to be C^∞ unless stated otherwise. All manifolds are assumed to be connected, compact, closed and orientable. All homology and cohomology groups are with integral coefficients, unless it says specifically otherwise.

2. Construction of symplectic forms

2.1. Outline of the proof of theorem 1.1

In this section we shall give a proof of theorem 1.1 modulo some technical lemmas which will be proved in §§ 2.2–2.4.

For the remainder of this section let M be an oriented 4-manifold admitting a free S^1-action. We denote the orbit space by N and we denote by $p\colon M \to N$ the quotient map that defines a principal S^1-bundle over N.

In the following we denote by $e \in H^2(N)$ the Euler class of the S^1-bundle $M \to N$. (Note that M decomposes as a product $M = S^1 \times N$ if and only if $e = 0$.) Recall the Gysin sequence

$$\mathbb{Z} = H^0(N;\mathbb{R}) \xrightarrow{\cdot e} H^2(N;\mathbb{R}) \xrightarrow{p^*} H^2(M;\mathbb{R}) \xrightarrow{p_*} H^1(N;\mathbb{R}) \xrightarrow{\cup e} H^3(N;\mathbb{R}) = \mathbb{R}. \tag{2.1}$$

Here $p_*\colon H^2(M;\mathbb{R}) \to H^1(N;\mathbb{R})$ is the map given by integration along the fibre. The same sequence can be considered for cohomology with integral coefficients. Note that the map $p_*\colon H^4(M) \to H^3(N)$ is an isomorphism, and we endow N with the orientation given by the image of the orientation of M under p_*.

Throughout this section we assume that $\psi \in H^2(M;\mathbb{R})$ is such that $\psi^2 > 0 \in H^4(M;\mathbb{R})$ and such that $p_*(\psi) \in H^1(N;\mathbb{R})$ can be represented by a non-degenerate closed 1-form α.

LEMMA 2.1. *There exists a* 1*-form* β *on* N *such that* $\alpha \wedge \beta$ *is closed and* $[\beta \wedge \alpha] = e \in H^2(N;\mathbb{R})$*.*

In the case that $p_*(\psi)$ is integral, this lemma is stated in [4, lemma 15]. We give the proof of lemma 2.1 in § 2.3.

Now let $\gamma = \beta \wedge \alpha$. Since $[\gamma] = e \in H^2(N;\mathbb{R})$, we can find a 1-form η (namely a connection 1-form for $M \to N$) on M with the following properties:

(i) η is invariant under the S^1-action;

(ii) the integral of η over a fibre (which inherits an orientation from S^1) equals 1;

(iii) $\mathrm{d}\eta = p^*(\gamma)$.

This form is often referred to as the *global angular form*. We refer the reader to [1] for more details. Note that (i) and (ii) imply that η is non-trivial on any non-trivial vector tangent to a fibre.

Note that $\mathrm{d}(p^*(\alpha) \wedge \eta) = p^*(\alpha \wedge \gamma) = p^*(\alpha \wedge \alpha \wedge \beta) = 0$. We can therefore consider $\psi - [p^*(\alpha) \wedge \eta] \in H^2(M;\mathbb{R})$. It follows easily from $p_*(\psi) = [\alpha]$ and the second property of η that $p_*(\psi - [p^*(\alpha) \wedge \eta]) = 0 \in H^1(N;\mathbb{R})$. By the exact sequence (2.1) we can therefore find $h \in H^2(N;\mathbb{R})$ with $p^*(h) = \psi - [p^*(\alpha) \wedge \eta]$. By assumption we have $\psi^2 > 0$. Note that

$$\begin{aligned}\psi^2 &= (p^*(h) + [p^*(\alpha) \wedge \eta]) \cup (p^*(h) + [p^*(\alpha) \wedge \eta]) \\ &= p^*(h^2) + [p^*(\alpha) \wedge \eta] \cup [p^*(\alpha) \wedge \eta] + 2p^*(h) \cup [p^*(\alpha) \wedge \eta] \\ &= p^*(h^2) + [p^*(\alpha) \wedge \eta \wedge p^*(\alpha) \wedge \eta] + 2p^*(h) \cup [p^*(\alpha) \wedge \eta].\end{aligned}$$

The first term is zero since N supports no 4-forms, and the second term is zero since η and $p^*(\alpha)$ are 1-forms. It follows that

$$p^*(h) \cup [p^*(\alpha) \wedge \eta] = \tfrac{1}{2}\psi^2 > 0 \in H^4(M;\mathbb{R}).$$

Recall that the map $p_*\colon H^4(M;\mathbb{R}) \to H^3(N;\mathbb{R})$ is an orientation-preserving isomorphism. In particular, we therefore get that

$$h \cup [\alpha] = p_*(p^*(h) \cup [p^*(\alpha) \wedge \eta]) > 0 \in H^3(N;\mathbb{R}).$$

We shall prove the following lemma in § 2.4.

Lemma 2.2. *Given $h \in H^2(N;\mathbb{R})$ with $h \cup [\alpha] > 0$, we can find a representative Ω of h such that $\Omega \wedge \alpha > 0$ everywhere.*

It is now clear that the following claim concludes the proof of theorem 1.1.

Claim 2.3.

$$\omega = p^*(\Omega) + p^*(\alpha) \wedge \eta$$

is an S^1-invariant symplectic form on M which represents ψ.

It is clear that ω is S^1-invariant. We compute

$$\mathrm{d}\omega = \mathrm{d}(p^*(\Omega) + p^*(\alpha) \wedge \eta) = p^*(\alpha) \wedge \mathrm{d}\eta = p^*(\alpha \wedge \gamma) = p^*(\alpha \wedge \alpha \wedge \beta) = 0,$$

i.e. ω is closed. Also note that

$$\psi = p^*(h) + [p^*(\alpha) \wedge \eta] = [p^*(\Omega) + p^*(\alpha) \wedge \eta],$$

i.e. ω represents ψ. It remains to show that $\omega \wedge \omega$ is positive everywhere. For any point $q \in M$, pick a basis a, b, c, d for the tangent space T_qM such that

(a) $p_*(a)$, $p_*(b)$ are a basis for the tangent space $\ker \alpha|_{p(q)}$ of a leaf of the foliation on N determined by α (in other words, $\alpha(p_*(a)) = \alpha(p_*(b)) = 0$ and $p_*(a)$, $p_*(b)$ are linearly independent),

(b) $\alpha(p_*(c)) > 0$,

(c) d is tangent to the fibres of the S^1-fibration $M \to N$ and $\eta(d) > 0$.

Note that $p_*(d) = 0$ and $p^*(\alpha)$ vanishes on a, b, d. It is now easy to see that

$$\begin{aligned}(\omega \wedge \omega)(a,b,c,d) &= 2(p^*(\Omega) \wedge p^*(\alpha) \wedge \eta)(a,b,c,d) \\ &= 2p^*(\Omega)(a,b) \cdot p^*(\alpha)(c) \cdot \eta(d) \\ &= 2\Omega(p_*(a), p_*(b)) \cdot \alpha(p_*(c)) \cdot \eta(d) \\ &= 2(\Omega \wedge \alpha)(p_*(a), p_*(b), p_*(c)) \cdot \eta(d).\end{aligned}$$

Since $\Omega \wedge \alpha$ is a non-zero 3-form and since $p_*(a)$, $p_*(b)$, $p_*(c)$ form a basis for the tangent space of N we see that the last expression is in fact non-zero. This shows that $\omega \wedge \omega$ is non-zero everywhere, but since $\omega \wedge \omega$ represents the positive class ψ^2 we see that $\omega \wedge \omega$ is in fact positive throughout. This concludes the proof of the claim and hence the proof of theorem 1.1.

2.2. Non-degenerate closed 1-forms and dual curves

Throughout this section α will be a non-degenerate closed 1-form on N. Note that α (or strictly speaking $\mathrm{Ker}(\alpha)$) defines a foliation which we denote by $\mathcal{F}$. Before we can prove lemmas 2.1 and 2.2 we need a preliminary result regarding representability of homology classes in N by smooth embedded curves transverse to, or contained in a leaf of, the foliation $\mathcal{F}$. The following lemma is presumably known (its existence is discussed in, for example, [9]) but we include a proof for completeness.

LEMMA 2.4. *Let α be a non-degenerate closed 1-form on N with corresponding foliation $\mathcal{F}$ and let $p \in N$. For every $h \in H^2(N;\mathbb{Z})$ with $h \cup [\alpha] \neq 0$ (respectively, $h \cup [\alpha] = 0$) there exists a smoothly embedded closed (possibly disconnected) curve c with $PD([c]) = h$ transverse to (respectively, contained in a leaf of) the foliation $\mathcal{F}$ and that goes through p.*

Proof. Let α be a non-degenerate closed 1-form on N with corresponding foliation $\mathcal{F}$. We first pick a metric g on N. We let v' be the unique vector field on N with the property that for any $p \in N$ and any $w \in T_pN$ we have $g(v'(p), w) = \alpha(w)$. Note that this implies that $\alpha(v'(p)) \neq 0$ for all p. We then define a new vector field v by

$$v(p) = \frac{v'(p)}{\alpha(v'(p))}.$$

Note that $\alpha(v(p)) = 1$ for all $p \in N$. We denote by $F\colon N \times \mathbb{R} \to N$ the flow corresponding to $-v$, i.e. for any $p \in N, s \in \mathbb{R}$ we have

$$\frac{\partial}{\partial t} F(p,t)\Big|_{t=s} = -v(F(p,s)) \tag{2.2}$$

with initial condition $F(p,0) = p$ (as N is compact, the flow is defined for all $s \in \mathbb{R}$). Observe that Cartan's formula implies that $L_v\alpha = \mathrm{d}(i_v\alpha) + i_v(\mathrm{d}\alpha) = \mathrm{d}(1) = 0$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}(F_s^*\alpha) = 0,$$

where $F_s\colon N \to N$ is the map defined by $F_s(q) = F(q,s)$. Hence,

$$F_s^*\alpha = F_0^*\alpha = \alpha.$$

We shall repeatedly make use of the following formula: given a path

$$(\gamma,\rho)\colon [0,1] \to N \times \mathbb{R},$$

by the chain rule the induced path $\eta := F(\gamma,\rho)\colon \mathbb{R} \to N$ has tangent vector

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t),\rho(t)) = (F_{\rho(t)})_*\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) + \frac{\partial}{\partial s} F(\gamma(t),s)\Big|_{s=\rho(t)} \frac{\mathrm{d}\rho}{\mathrm{d}t}$$

and as usual the derivatives at the endpoints are interpreted as being one-sided. Using (2.2) we can rewrite this vector as

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = (F_{\rho(t)})_* \frac{\mathrm{d}\gamma}{\mathrm{d}t} - v(\eta(t)) \frac{\mathrm{d}\rho}{\mathrm{d}t} \in T_{\eta(t)}N. \tag{2.3}$$

Let now $\gamma\colon [0,1] \to N$ be any smoothly embedded loop with $\gamma(0) = \gamma(1) = p$ whose image (which by abuse of notation we shall also denote by γ), is dual to a class $h \in H^2(N;\mathbb{Z})$ such that

$$h \cup [\alpha] = \int_\gamma \alpha = m \in \mathbb{R}.$$

Let $\rho(t) = mt$ and denote, as above, $\eta(t) = F(\gamma(t), mt)$. Define a map $\Phi\colon [0,1] \to \mathbb{R}$ as

$$\Phi(t) = \int_{\eta|_{[0,t]}} \alpha,$$

where $\eta|_{[0,t]}$ denotes the restriction of the map $\eta\colon [0,1] \to N$ to the interval $[0,t]$. Note that, by (2.3),

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \alpha\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right) = \alpha\left((F_{mt})_* \frac{\mathrm{d}\gamma}{\mathrm{d}t} - mv\right).$$

Using the identities $\alpha((F_s)_*) = F_s^* \alpha = \alpha$ and $\alpha(v) = 1$, we therefore obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \alpha\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) - m.$$

In particular, it follows that

$$\begin{aligned} \Phi(1) = \int_\eta \alpha &= \int_0^1 \alpha\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right) \mathrm{d}t \\ &= \int_0^1 \alpha\left((F_{mt})_* \frac{\mathrm{d}\gamma}{\mathrm{d}t} - mv\right) \mathrm{d}t \\ &= \int_0^1 \alpha\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) \mathrm{d}t - m \\ &= \int_\gamma \alpha - m = 0. \end{aligned}$$

We consider now the following homotopy

$$\begin{aligned} H\colon [0,1] \times [0,1] &\to N, \\ (t,s) &\mapsto F(\gamma(t), s\Phi(t)). \end{aligned}$$

This is clearly a smooth map. Since $\Phi(1) = 0$, this descends in fact to a homotopy $H\colon S^1 \times [0,1] \to N$. Note that $H(t,0) = \gamma(t)$ for all t. We now consider the path $\tilde\gamma(t)$ defined by $\tilde\gamma(t) = H(t,1)$. Note that $\tilde\gamma(0) = \tilde\gamma(1) = p$. The map $\tilde\gamma(t)$ is smooth, and we claim that the image $\tilde\gamma$ of $\tilde\gamma(t)$ is transverse to the foliation $\mathcal{F}$ if $m \neq 0$, and is contained in the leaf through p if $m = 0$.

In fact, as $\tilde\gamma(t) = F(\gamma(t), \Phi(t))$, we have by (2.3),

$$\frac{\mathrm{d}\tilde\gamma}{\mathrm{d}t} = (F_{\Phi(t)})_* \left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) - v(\tilde\gamma(t)) \frac{\mathrm{d}\Phi}{\mathrm{d}t} \in T_{\tilde\gamma(t)} N.$$

Applying α pointwise, we get

$$\begin{aligned} \alpha\left(\frac{\mathrm{d}\tilde\gamma}{\mathrm{d}t}\right) &= (F_{\Phi(t)}^* \alpha)\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) - \alpha(v)\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \alpha\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) - \frac{\mathrm{d}\Phi}{\mathrm{d}t} \\ &= \alpha\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) - \alpha\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) + m = m, \end{aligned}$$

so $\mathrm{d}\tilde{\gamma}/\mathrm{d}t$ is pointwise transverse to or contained in $\ker\alpha$, depending on the value of m.

Note that $\tilde{\gamma}$ may have self-intersection and (when $m = 0$) may fail to be an immersion. However, using a local model, we can use a general position argument to further homotope $\tilde{\gamma}$ (at the price perhaps of increasing the number of components, when $\tilde{\gamma}$ sits on a leaf) to get the curve c that satisfies the conclusions of the lemma. □

2.3. Proof of lemma 2.1

We are now ready to prove the first of the two auxiliary lemmas, i.e. we shall prove the following claim.

Claim 2.5. *Let α be a non-degenerate closed* 1*-form on N and $e \in H^2(N;\mathbb{Z})$ such that $e \cup [\alpha] = 0$. There exists a* 1*-form β on N such that $\alpha \wedge \beta$ is closed and $[\beta \wedge \alpha] = e \in H^2(N;\mathbb{R})$.*

By lemma 2.4 we can find an oriented smoothly embedded curve c dual to $e \in H^2(N;\mathbb{Z})$ such that $\alpha|_c \equiv 0$. We denote the components of c by $c_1, \dots, c_m$. We now consider $S^1 \times D^2$ with the coordinates $(\mathrm{e}^{2\pi \mathrm{i}t}, x, y)$ and we orient $S^1 \times D^2$ by picking the equivalence class of the basis $\{\partial_x, \partial_y, \partial_t\}$.

Using the orientability of the N and of the leaves of the foliation we use a standard argument to show that for $i = 1, \dots, m$ we can pick a map

$$f_i \colon S^1 \times D^2 \to N$$

with the following properties:

(i) f_i is an orientation-preserving diffeomorphism onto its image;

(ii) f_i restricted to $S^1 \times 0$ is an orientation-preserving diffeomorphism onto c_i;

(iii) $\alpha((f_i)_*(\partial_t)) = 0$;

(iv) $\alpha((f_i)_*(\partial_x)) = 0$;

(v) there exists an $r_i \in (0,\infty)$ such that $\alpha((f_i)_*(\partial_y)) = r_i$ everywhere.

Note that (iii), (iv) and (v) are equivalent to $f_i^*(\alpha) = r_i \cdot \mathrm{d}y$.

For $i = 1, \dots, m$ we now pick a function $\rho_i \colon D^2 \to \mathbb{R}_{\geqslant 0}$ such that the closure of the support of ρ_i lies in the interior of D^2 and such that

$$\int_{D^2} \rho_i(x,y)\,\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{r_i}.$$

We define the following 1-form on $S^1 \times D^2$:

$$\beta_i'(t,x,y) = \rho_i(x,y) \cdot \mathrm{d}x.$$

Note that

$$\mathrm{d}(\beta_i' \wedge f_i^*(\alpha)) = \mathrm{d}(\beta_i' \wedge r_i \cdot \mathrm{d}y) = \mathrm{d}(r_i \rho_i(x,y) \cdot \mathrm{d}x \wedge \mathrm{d}y) = 0. \tag{2.4}$$

Furthermore, for any $z \in S^1$ we have

$$\int_{z\times D^2} \beta_i' \wedge f_i^*(\alpha) = \int_{z\times D^2} r_i\rho_i(x,y)\cdot \mathrm{d}x\wedge \mathrm{d}y = 1. \tag{2.5}$$

For $i = 1,\ldots,m$ we now define the following 1-form on N:

$$\beta_i(p) = \begin{cases} 0 & \text{if } p \in N \setminus f_i(S^1\times D^2), \\ (f_i^{-1})^*(\beta_i'(q)) & \text{if } p = f_i(q) \quad \text{for some } q \in S^1 \times D^2. \end{cases}$$

Furthermore, we let $\beta = \sum_{i=1}^m \beta_i$. We claim that β has all the required properties.

First note that β is C^∞ by our condition on the support of ρ_i. Furthermore, it follows immediately from (2.4) that $\beta\wedge\alpha$ is closed. Finally, we have to show that $\beta\wedge\alpha$ represents e.

In order to show that $\beta\wedge\alpha$ represents e in $H^2(N;\mathbb{R}) = \hom(H_2(N;\mathbb{Z}),\mathbb{R})$ it is enough to show that, for any embedded oriented surface $S \subset N$, we have

$$\int_S \beta\wedge\alpha = e([S]).$$

We first note that $e([S]) = c\cdot s$. It is therefore enough to show that for any embedded oriented surface $S \subset N$, we have

$$\int_S \beta_i\wedge\alpha = c_i\cdot S.$$

In fact, given such a surface we can isotope S in such a way that S intersects the curve c 'vertically', i.e. we can assume that

$$f_i(S^1\times D^2)\cap S = \coprod_{j=1}^k \epsilon_j\cdot f_i(z_1\times D^2)$$

for disjoint z_i and $\epsilon_i \in \{-1,1\}$. We view this equality as an equality of oriented manifolds, where we give $z_i\times D^2$ the orientation given by the basis $\{\partial_x,\partial_y\}$. In particular, S is transverse to c_i. In this case we have

$$c_i\cdot S = \sum_{j=1}^k \epsilon_j.$$

On the other hand, it follows from (2.5) that

$$\int_S \beta_i\wedge\alpha = \sum_{j=1}^k \int_{\epsilon_j\cdot(z_j\times D^2)} f_i^*(\beta_i)\wedge f_i^*(\alpha) = \sum_{j=1}^k \int_{\epsilon_j\cdot(z_j\times D^2)} \beta_i'\wedge f_i^*(\alpha) = \sum_{j=1}^k \epsilon_j.$$

This concludes the proof that β has all the required properties.

2.4. Proof of lemma 2.2

The following claim is the last missing piece in the proof of theorem 1.1.

CLAIM 2.6. *Let α be a non-degenerate closed 1-form on N. Given $h \in H^2(N;\mathbb{R})$ with $h\cup[\alpha] > 0$, we can find a representative Ω of h such that $\Omega\wedge\alpha > 0$ everywhere.*

We first consider the case that h is represented by an integral class, i.e. by an element in the image of the map $H^2(N;\mathbb{Z}) \to H^2(N;\mathbb{R})$. Let $\mathcal{F}$ be the foliation corresponding to α.

Using lemma 2.4 we can pick for each $p \in N$ a curve c_p transverse to $\mathcal{F}$ which goes through p and which represents h. Since N is orientable we can pick maps

$$f_p\colon S^1 \times D^2 \to N$$

such that

(i) f_p is an orientation-preserving diffeomorphism onto its image (where we again view $S^1 \times D^2$ with the orientation given by $\{\partial_x, \partial_y, \partial_t\}$),

(ii) f_p restricted to $S^1 \times 0$ is an orientation-preserving diffeomorphism onto c_p,

(iii) $\alpha((f_p)_*(\partial_x)) = 0$,

(iv) $\alpha((f_p)_*(\partial_y)) = 0$,

(v) $\alpha((f_p)_*(\partial_t)) > 0$.

Note that (iii) and (iv) are equivalent to saying that $(f_p)_*(\partial_x)$ and $(f_p)_*(\partial_y)$ are tangent to the leaves of the foliation $\mathcal{F}$. Also note that on $S^1 \times D^2$ we have $\mathrm{d}x \wedge \mathrm{d}y \wedge (f_p)^*(\alpha) \neq 0$.

By compactness we can find $p_1, \ldots, p_k$ such that

$$\bigcup_{j=1}^{k} f_{p_i}(S^1 \times \tfrac{1}{2}D^2) = N. \tag{2.6}$$

We write $f_i = f_{p_i}, i = 1, \ldots, k$. Now we pick a function $\rho\colon D^2 \to \mathbb{R}_{\geqslant 0}$ such that the following conditions hold:

(a) $\displaystyle\int_{D^2} \rho = \frac{1}{k}$;

(b) ρ is strictly positive on $\frac{1}{2}D^2$;

(c) the closure of the support of ρ lies in the interior of D^2.

Let Ω' be the 2-form on $S^1 \times D^2$ given by

$$\Omega'(z, x, y) = \rho(x, y)\mathrm{d}x \wedge \mathrm{d}y.$$

Clearly, Ω' is closed and for any $z \in S^1$ we have

$$\int_{z \times D^2} \Omega' = \frac{1}{k}.$$

For $i = 1, \ldots, k$ we now define the following 2-form on N:

$$\Omega_i(p) = \begin{cases} 0 & \text{if } p \in N \setminus f_i(S^1 \times D^2), \\ (f_i^{-1})^*(\Omega'(q)) & \text{if } p = f_i(q) \quad \text{for some } q \in S^1 \times D^2. \end{cases}$$

As in the proof of lemma 2.1 we see that Ω_i is smooth, Ω_i is closed and

$$[\Omega_i] = \frac{1}{k} h \in H^2(N;\mathbb{R}).$$

Now let $\Omega(h) = \sum_{i=1}^{k} \Omega_i$. Clearly, $[\Omega(h)] = h \in H^2(N;\mathbb{R})$, and it easily follows from (2.6) and all the other conditions that $\Omega(h) \wedge \alpha > 0$ everywhere.

We now turn to the general case, i.e. to the case that $h \in H^2(N;\mathbb{R})$ is not necessarily integral.

LEMMA 2.7. *Let $h \in H^2(N;\mathbb{R})$ with $h \cup [\alpha] > 0$. Then we can find $m \in \mathbb{N}$, integral $h_1, \dots, h_m$ and $a_1, \dots, a_m \in \mathbb{R}_{\geqslant 0}$ such that $h_i \cup [\alpha] > 0$ for all i and such that $h = \sum_{i=1}^{m} a_i h_i$.*

We first show that lemma 2.7 implies lemma 2.2. Indeed, given $h \in H^2(N;\mathbb{R})$ with $h \cup [\alpha] > 0$, we pick integral $h_1, \dots, h_m$ and $a_1, \dots, a_m \in \mathbb{R}_{\geqslant 0}$ as above. Then we define $\Omega(h_1), \dots, \Omega(h_m)$ as above. We let

$$\Omega = \sum_{i=1}^{m} a_i \Omega(h_i).$$

We see that

$$\Omega(h) \wedge \alpha = \sum_{i=1}^{m} a_i \Omega(h_i) \wedge \alpha > 0$$

everywhere. This concludes the proof of lemma 2.2, assuming lemma 2.7 holds.

We now turn to the proof of lemma 2.7. It is easy to see that we can pick a basis $e_1, \dots, e_n$ for $H^1(N;\mathbb{Q})$ such that $e_i \cup [\alpha] > 0$ for all $i = 1, \dots, m$. We use this basis to identify $H^2(N;\mathbb{R})$ with $\mathbb{R}^n$. We say that $h \in H^2(N;\mathbb{R})$ with $h \cup [\alpha] > 0$ has property $(*)$ if there exist $m \in \mathbb{N}$, integral $h_1, \dots, h_m$ and $a_1, \dots, a_m \in \mathbb{R}_{\geqslant 0}$ such that $h_i \cup [\alpha] > 0$ for all i and such that $h = \sum_{i=1}^{m} a_i h_i$. Note that if h_1, h_2 have property $(*)$, then $h_1 + h_2$ also has property $(*)$.

Given $m \in \{0, \dots, n\}$ we now say $P(m)$ *holds if $(*)$ holds for all $g = (g_1, \dots, g_n) \in H^2(N;\mathbb{R}) = \mathbb{R}^n$ with $g_1, \dots, g_m \in \mathbb{Q}$*. Clearly, we have to show that $P(0)$ holds. Note that $P(n)$ holds since any rational element of $H^2(N;\mathbb{R})$ is a non-negative multiple of an integral element.

We now show that $P(m+1)$ implies that $P(m)$ holds as well. So assume $P(m+1)$ holds and that we have

$$g = (g_1, \dots, g_m, g_{m+1}, \dots, g_n)$$

with $g_1, \dots, g_m \in \mathbb{Q}$ and $h \cdot [\alpha] > 0$. By continuity we can find $r > 0$ such that $g_{m+1} - r \in \mathbb{Q}$ and with the property that

$$(g_1, \dots, g_m, g_{m+1} - r, \dots, g_n) \cdot [\alpha] > 0.$$

We write

$$(g_1, \dots, g_m, g_{m+1}, \dots, g_n) = (g_1, \dots, g_m, g_{m+1} - r, \dots, g_n) + r e_{m+1}.$$

The claim now follows from $P(m+1)$ and $e_{m+1} \cup [\alpha] > 0$.

3. Proof of theorem 1.3

We first prove the following proposition.

PROPOSITION 3.1. *Let M be a 4-manifold with a free circle action. Denote by $p\colon M \to N$ the projection map to the orbit space. Assume that $(N, p_*([\omega]))$ fibres over S^1 for any symplectic form ω such that $p_*([\omega])$ is an integral class which is primitive in $H^1(N;\mathbb{Z})$. Then for any symplectic form ω the class $p_*([\omega]) \in H^1(N;\mathbb{R})$ can be represented by a non-degenerate closed 1-form.*

Proof. First let ω be a symplectic form such that $p_*([\omega]) \in H^1(N;\mathbb{Q})$. We can find $s \in \mathbb{Q}$ such that $sp_*([\omega]) = p_*([s\omega])$ is a primitive element in $H^1(N)$. By assumption $(N, sp_*([\omega]))$ fibres over S^1, in particular $sp_*([\omega])$ (and hence $p_*([\omega])$) can be represented by a non-degenerate closed 1-form.

Now let ω be a symplectic form such that $p_*([\omega]) \in H^1(N;\mathbb{R}) \setminus H^1(N;\mathbb{Q})$, and let C be the open cone over the face of the unit ball of the Thurston norm in which C lies. (Note that C is *a priori* not necessarily top dimensional.) Since the vertices of the Thurston norm ball are rational [13, § 2], and by the openness of the symplectic condition, we can find a symplectic form ω' on M such that $p_*([\omega'])$ is in $H^1(N;\mathbb{Q})$ and is contained in the cone C as well. By the previous observation it follows that there exists at least one element of C (namely $p_*([\omega'])$ itself) that can be represented by a non-degenerate closed 1-form. But then by [13, theorem 5] all elements in C, in particular $p_*([\omega])$, can be represented by non-degenerate closed 1-forms. □

We can now prove theorem 1.3.

Proof of theorem 1.3. Let N be a closed oriented 3-manifold and let $\psi \in H^2(S^1 \times N;\mathbb{R})$. We have to show that the following are equivalent:

(i) ψ can be represented by a symplectic structure;

(ii) ψ can be represented by a symplectic structure which is S^1-invariant;

(iii) $\psi^2 > 0$ and the Künneth component $\phi \in H^1(N;\mathbb{R})$ of ψ lies in the open cone on a fibred face of the Thurston norm ball of N.

Clearly, (ii) implies (i). Theorem 1.1 shows that (iii) implies (ii). By the results of [6, theorems 1.2 and 1.4] we know that, for any symplectic form ω with $p_*([\omega]) \in H^1(N)$ primitive, the pair $(N, p_*([\omega]))$ fibres over S^1. (Note that this is stated only for integral forms $[\omega]$, but the argument in [6] carries through for any $[\omega]$ such that $p_*([\omega])$ is primitive.) Proposition 3.1 then asserts that for any symplectic form ω the class $p_*([\omega]) \in H^1(N;\mathbb{R})$ can be represented by a non-degenerate closed 1-form. □

Theorem 1.3 lets us determine the symplectic cone for a significant class of 4-manifolds. Our result suggests that the symplectic cone shares the properties of the fibred cone of a 3-manifold. We propose the following conjecture.

Conjecture 3.2. *Let W be a symplectic 4-manifold. Then there exists a (possibly non-compact) polytope $C \subset H^2(W;\mathbb{R})$ with the following properties:*

(i) *the dual polytope in $H_2(W;\mathbb{R})$ is compact, symmetric, convex and integral;*

(ii) *there exist open top-dimensional faces $F_1, \ldots, F_s$ of C such that the symplectic cone coincides with all non-degenerate elements in the cone on $F_1, \ldots, F_s$.*

Acknowledgements

This paper is one section of our unpublished 2007 preprint [5]. S.V. was partly supported by NSF Grant no. 0906281. We thank Paolo Ghiggini and Ko Honda for helpful conversations.

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(*Issued 6 April 2012*)