SYMPLECTIC 4-MANIFOLDS AND 3-DIMENSIONAL TOPOLOGY

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Abstract. This is the text of a talk given in November 1999 at École Polytechnique for the *Groupe de travail en Topologie Symplectique* on some results, principally due to C.McMullen and C.Taubes, about some classes of symplectic 4-manifolds and their relation with three dimensional topology, through the eyes of Seiberg-Witten theory.

1. Introduction

The last five years have brought to an impressive improving of our knowledge and comprehension of symplectic 4-manifolds. The work of Donaldson, Auroux and others has given us, on one side, a certain feeling of how a symplectic 4-manifold is structured, providing a qualitative description which recalls the one appearing in the case of algebraic surfaces. On the other side, the symplectic surgery theory developed by Gompf, and fruitfully applied by Fintushel and Stern, has provided us of several examples of symplectic manifolds which go far beyond the usual ones inherited from Kähler geometry. Seiberg-Witten theory, moreover, mainly thanks to the work of Taubes, has provided a set of smooth invariants which fits particularly well in the world of symplectic 4-manifolds.

In spite of this almost idyllic scheme, we have to admit that, when we are faced with very simple questions like "does the manifold M admit a symplectic structure ω", "can ω be chosen in a given cohomology class $h \in H^2(M, \mathbb{R})$" and "is such an ω unique (in some suitable sense)", the criteria in our posses are in general insufficient to answer the question, and we are far from having a realistic conjectural picture of symplectic 4-manifolds.

There are some cases, anyhow, where, by a process of "dimensional reduction", we can rewrite some questions of four dimensional symplectic topology in terms of three dimensional topology, and eventually obtain some results.

Two cases appear particularly important; an obvious one, when the 4-manifold has the form $S^1 \times N$, and a more involved one, given by Fintushel-Stern manifolds (or variations on the theme, that we will introduce), where a four manifold is constructed by surgery from an algebraic surface and a manifold of the type $S^1 \times N$.

The interest in the study of manifolds of the classes above is due to the fact that they allow to give examples of symplectic manifolds which escape from the usual class (namely Kähler surfaces) and because of this are a bit more "generic" as symplectic manifolds.
In this paper, following C.McMullen and C.Taubes, we will address to the problem whether a 4-manifold can admit more, inequivalent, symplectic structure (we will give later the precise definition of equivalence). There has been a kind of common belief of the uniqueness of these structures: the main result we will discuss concerns the answer, in the negative, of this conjecture, quite unjustified as we will see:

Proposition 1.1. *(McMullen-Taubes): there exist symplectic 4-manifolds, even simply-connected, which admit inequivalent symplectic structures.*

In fact, it is quite likely that this result (at least for non simply-connected examples) was already known to someone, but the inexplicable laws of mathematical oblivion made conjecture its converse. This result seems to support the following general

Metaconjecture 1.2. *All conjectures in 4-dimensional symplectic topology are false.*

I am indebted to Paul Seidel for this illuminating observation.
The ingredients of the proof of Proposition 1.1 will consist in a specific construction of examples, and then the use of Seiberg-Witten invariant to distinguish symplectic structures.
The paper is divided as follows: in Sect. 2 we will review the definition of Alexander and Thurston norms; in Sect. 3 we will analyse some properties of three manifolds which fiber over the circle, bulk of McMullen-Taubes example, and discuss an explicit case. Sect. 4 and 5 contain a description of Fintushel-Stern manifolds, and some generalisations which include McMullen-Taubes manifold. In Sect. 6 we will describe some symplectic structures on McMullen-Taubes manifold and prove then Proposition 1.1
The sources of this paper are, principally, the works of McMullen-Taubes ([McMT]) and Fintushel-Stern ([FS]). Here and there some remark has been added (in particular Lemma 5.2) and some proof modified. As the purpose of this paper is purely expository, the reader and the authors will indulge us if we will be a bit careless in referring punctually to the literature.

2. Alexander and Thurston norms

We start with a brief recall of the definition of two (semi)norms on the first cohomology group of a three manifold N, with $b_1(N) > 1$, with boundary an (eventually empty) disjoint union of tori. The first one is the Thurston norm ([Th]), defined in terms of the complexity of the Poincaré dual homology class: if σ is a class in $H_2(N, \partial N; \mathbb{Z})$ we consider properly embedded Riemann surfaces Σ that represent σ and we define their Euler characteristic as the sum of the Euler characteristic over all components of positive genus. Denoting $\chi_-(\Sigma) = -\chi(\Sigma)$, minus the Euler characteristic of Σ, we define

$$\|\sigma\|_T = min\{\chi_-(\Sigma) | (\Sigma, \partial\Sigma) \hookrightarrow (N, \partial N), [\Sigma] = \sigma\}, \tag{1}$$

and we use Poincaré duality to induce a norm on $H^1(N; \mathbb{Z})$. As happens with minimal genus problems in 4-dimensional topology, there is a relation between this norm and SW theory; if we consider the set of monopole classes $L_i \in H^2(N, \partial N, \mathbb{Z})$ of N, we have

$$\|\phi\|_T \geq max L_i \cup \phi \tag{2}$$

and equality holds for irreducible closed manifolds ([KM]) (we presume, in fact, it holds even in the nonclosed case, but a complete proof does not exists, and it is probably not straightforward). The second norm is the Alexander norm, defined in [McM] from the Alexander polynomial of the manifold; denote by G the free abelian group $G := H_1(N; \mathbb{Z})/Tor$; its rank coincides, by definition, with the first Betti number of N. The Alexander polynomial of N appears as an element of the group ring $\mathbb{Z}[G]$, i.e. a finite sum $\Delta_N = \sum a_t t$ where the t are elements of G and the a_t are integer coefficients. The Alexander polynomial is well defined up to the multiplication by the units of $\mathbb{Z}[G]$; if we allow multiplication by formal square roots of element of G, we can in fact choose a representative which is symmetric with respect to the involution of G (up to a sign which, in the case of our interest, is $+$ in the case of closed manifolds and knots and $(-)^n$ for n-components links, $n > 1$). The Alexander polynomial coincides, up to sign, with a suitably defined polynomial related to Seiberg-Witten invariants ([MT]). As in the cases we will consider all three manifolds will have no torsion in homology, this polynomial coincides with the usual SW invariant, seen as a function over the set of spin^c structures trivial on the boundary, an affine $H^2(N, \partial N, \mathbb{Z})$; for the general case, see [MT].
For any element $\phi \in H^1(N; \mathbb{Z})$ we define the norm

$$\|\phi\|_A := max_{t,t'} \phi(t - t'). \tag{3}$$

where the indexes run over the support of (some representative of) Δ_N. The aforementioned relation with SW theory allows in fact to rewrite this norm in terms of (some) SW basic classes K_i; in particular, as we assume that N has no torsion in homology, we have

$$\|\phi\|_A = max\{K_i \cup \phi, SW_N(K_i) \neq 0\} \tag{4}$$

where the spin^c structure is bijectively defined by its determinant bundle.
These norms are related by McMullen's inequality

$$\|\phi\|_A \leq \|\phi\|_T, \tag{5}$$

which becomes an equality when ϕ represents a fibration of M over S^1. This inequality can be proved using the very definition of the norms ([McM]), or using their relation with Seiberg-Witten theory ([V]).
Both norms can be continuously extended to the cohomology with real coefficients and the unit balls of these norms are finite, convex, possibly non compact polyhedra.

3. 3-manifolds which fiber over S^1

A class of three manifolds which presents, for several reasons, particular interest, is that given by the manifolds N which admit a fibration over the circle with map $\pi : N \to S^1$. If we consider the cohomology class $\pi^*[dt] \in H^1(N, \mathbb{Z})$, that we can identify with the datum of the fibration, this is contained in the cone over an open face of the Thurston unit sphere which has the property that all integral points in it represent fibrations. Passing to DeRahm representatives, this fibered cone is characterized by the fact that all its point admit a representative 1-form which does never vanish on the three manifold.
In the case of a closed N, this fact gives an explicit way to construct a symplectic four manifold of the form $S^1 \times N$; take a point on a fibered face, and choose a nonvanishing representative $\rho \in \Omega^1(N, \mathbb{R})$; it is possible to endow N with a metric which makes ρ harmonic; with this choice, the form

$$dt \wedge \rho + *\rho \in \Omega^2(S^1 \times N, \mathbb{R}) \tag{6}$$

is a symplectic form. The canonical class $K_\rho \in H^2(N, \mathbb{Z}) \subset H^2(S^1 \times N, \mathbb{Z})$ of this symplectic structure restricts to the canonical bundle of any fiber $\Sigma_\phi = \phi^{-1}(1)$, for $\phi \in H^1(N, \mathbb{Z}) = [N, S^1]$ contained in the same fibered cone of ρ, in such a way that

$$\|\phi^*(dt)\|_A = \|\phi^*(dt)\|_T = \chi_-(\Sigma_\phi) = K_\rho \cup \phi^*(dt). \tag{7}$$

Note that the class K_ρ depends in fact only on the fibered face containing ρ. Taubes' results on Seiberg-Witten theory for symplectic manifolds insure that $SW_N(K_\rho) = \pm SW_{S^1 \times N}(K_\rho) = \pm 1$ (there can be differences in sign due to orientation of moduli spaces). These results correspond to the fact that the canonical class K_ρ is the vertex of the Newton polyhedron for the SW (or Alexander) polynomial, dual to the fibered face containing ρ.

Assume that $H_1(N,\mathbb{Z})$ is a free group of rank $n>1$: the SW invariants of $S^1\times N$, seen as a polynomial on determinants of spin^c structures (trivial on the boundary if $\partial N\neq\emptyset$) have the form

$$SW_{S^1\times N}=\pm\Delta_N(s^2) \tag{8}$$

where $s=(s_1,...,s_n)$ is the image of the generators of $H_1(N,\mathbb{Z})$ under the map

$$H_1(N,\mathbb{Z})\to H_2(S^1\times N,\mathbb{Z})\overset{PD}{\to}H^2(S^1\times N,\partial(S^1\times N),\mathbb{Z}). \tag{9}$$

(If $\partial N\neq\emptyset$ the invariants are defined using finite energy moduli spaces.) The presence of the "power 2" is due to the fact that we are considering determinants of spin^c structures.

The group of diffeomorphisms of N acts, on $H^1(N,\mathbb{Z})$, by isometries w.r.t. the Thurston norm, and the Alexander (and SW) polynomial is invariant by diffeomorphisms; this implies that it acts by automorphisms on the unit balls of the Thurston and Alexander norms, which are finite, possibly non compact, polyhedra.

Thurston has shown that it is possible to construct manifolds which have several (pairs of) fibered faces, and an example of this type is the bulk of McMullen-Taubes construction. We will discuss now this example.

Consider, in S^3, the 4-components oriented link K given by the union of the Borromean link $K_1\cup K_2\cup K_3$ and its axis of $\mathbb{Z}_3$-symmetry K_4; this last component is an unknot which has linking number 1 with the other three components. Denote $N=S^3\setminus\nu K$. There is another possible description for N; perform 0-surgery on S^3 along the components of the Borromean link; it is well known that this surgery yields T^3 that we assume endowed of flat euclidean metric. We can write thus

$$N=S^3\setminus\nu K=T^3\setminus\nu L \tag{10}$$

where L is the union of four oriented disjoint closed framed geodesics L_i in T^3, whose first three give a basis of $H_1(T^3,\mathbb{Z})$. In fact, when we perform the 0-surgery on the Borreomean link, the three meridians $m_i=\mu(K_i)$, $i=1,2,3$ to the components of the Borromean link go over longitudes of L_i, while the preferred longitudes $\lambda(K_i)$ are sent to meridians of L_i; concerning the axis K_4 of the Borromean link, this goes over the isotopy class of $L_4\subset T^3$, which must satisfy the relation

$$L_4=L_1+L_2+L_3\in H_1(T^3,\mathbb{Z}) \tag{11}$$

(this relation follows from the fact that K_4 is homologous, in $S^3\setminus\nu(K_1\cup K_2\cup K_3)$, to the sum of the meridians of the Borromean link); the meridian m_4 of K_4, instead, goes quietly to a meridian of L_4 and is nullhomologous in T^3, having 0 linking number with the components of the Borromean link (in fact, it is nullhomotopic). It is instead non trivial in $H_1(N, \mathbb{Z})$, where the four generators are given by the m_i. As a consequence of this we have $H^1(N, \mathbb{Z}) \supset H^1(T^3, \mathbb{Z}) = (\mathbb{Z} < m_4 >)^{\perp}$.

It is possible to compute explicitly the Alexander polynomial of N; denoting $(m_1, m_2, m_3, m_4) = (x, y, z, t)$, we have

$$\begin{aligned} \Delta_N(x, y, z, t) &= -4 + (t + t^{-1}) + (x + x^{-1} + y + y^{-1} + z + z^{-1}) + \\ &-(xy + (xy)^{-1} + yz + (yz)^{-1} + xz + (xz)^{-1}) + (xyz + (xyz)^{-1}). \end{aligned} \tag{12}$$

We will use the features of the Newton polyhedron of Δ_N and of its dual polyhedron (the unit ball of the Alexander norm) to analyse the action of the group of diffeomorphism of N on the homology and cohomology of N. We have the following

Proposition 3.1. *The action of the group of diffeomorphisms of N on $H_1(N, \mathbb{Z})$ has disjoint orbits; in particular $\{t^{\pm 1}\}$, $\{(xyz)^{\pm 1}\}$ and $\{(xy)^{\pm 1}\}$ must lie in different orbits.*

Proof: we can observe that the Newton polyhedron $N(\Delta_N)$ has vertices which are combinatorially distinct: first, the simple dependence on t, t^{-1} makes clear that $N(\Delta_N)$ is the suspension of the Newton polyhedron $N(\Delta_N(t = 1))$; this is built by juxtaposing two cubes (inherited from the Newton polyhedron of the Borromean link), on the common vertex which corresponds to the origin (the zero class) and taking the smallest convex set containing them. This shows quite clearly that there are vertices, for $N(\Delta_N(t = 1))$, which are combinatorially different: although the value of the Alexander invariant coincides for all vertices, there are some vertices which have, with respect to this polyhedron, valence 3 and others which have valence 4. The valence of the vertices t and t^{-1} of $N(\Delta_N)$ is 14 (equal to the number of vertices of $N(\Delta_N(t = 1))$); all the vertices of $N(\Delta_N(t = 1))$ have valence 5 or 6 instead. As a consequence of this, any diffeomorphism of N will preserve $\{t, t^{-1}\}$, and will thus preserve $N(\Delta_N(t = 1))$. In this polyhedron the vertices combinatorially distinct (e.g. those indicated in the statement) lie in different orbits. $\square$

As $H^1(T^3, \mathbb{Z}) = (\mathbb{Z} < t >)^{\perp}$, the polyhedron $N(\Delta_N(t = 1))$ appears as the dual polyhedron to the unit ball of the Alexander norm restricted to pull black classes under $H^1(T^3, \mathbb{Z}) \xrightarrow{i^*} H^1(N, \mathbb{Z})$. We will concentrate on these classes.

Consider, for every link $L \in T^3$ composed of closed oriented geodesics, the norm on $H^1(T^3, \mathbb{Z})$

defined as

$$\|\phi\|_L = \sum |\phi(L_i)|. \tag{13}$$

We have the following

Lemma 3.2. *(*[McMT]*, Theorem 2.3; see also* [Th]*, Sect. 4) Let L be a link in T^3 composed of disjoint, oriented geodesics and consider the inclusion $i : N = T^3 \setminus \nu L \to T^3$; for any class $\phi \in H^1(T^3, \mathbb{Z})$ the following three statements are equivalent:*
a) $i^*\phi$ is represented by a fibration; b) $\forall i$, $\phi(L_i) \neq 0$; c) ϕ is in the open cone over a top dimensional face of the unit sphere of the norm $\|\cdot\|_L$.
We have moreover

$$\|i^*\phi\|_A = \|i^*\phi\|_T = \|\phi\|_L. \tag{14}$$

We omit the proof of this Lemma, which can be found in [McMT] or adapted from [Th]. Therefore the unit ball of the Thurston norm $\|\cdot\|_T$, restricted to $(\mathbb{Z} < t >)^{\perp}$, coincide with the Alexander unit ball; we point out that the knowledge of the Alexander norm, together with the previous Lemma, is an essential ingredient to compute explicitely the Thurston norm on $i^*H^1(T^3, \mathbb{Z})$ and analyse the fibered faces.

4. Fintushel-Stern manifolds

In the beginning of '97 Fintushel and Stern showed, in [FS], how to construct some new smooth four manifolds, starting from some particular algebraic surfaces (usually, the simply connected elliptic surfaces without multiple fibers $E(n)$) and a knot or link in S^3. We will briefly recall here their construction (we omit full generality, referring to the original paper). First, consider an elliptic surface X of type $E(n)$: this can be constructed by fiber summing n times the rational elliptic surface $\mathbf{P^2}\#_9\bar{\mathbf{P}}^2$. $E(n)$ admits a singular fibration over $\mathbf{P}^1$ with elliptic fiber F and $6n$ cusp fibers. It has moreover a section σ of self intersection $-n$. The cusp fiber is the central fiber of the family of elliptic curves with monodromy

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \tag{15}$$

F has self-intersection 0, is homologically nontrivial (it is a divisor of $E(n)$) and its essential cycles are homotoped to zero in the cusp neighborhood. Moreover, the fiber structure endows F of a canonical framing. F is what is called a *c-embedded* torus. The previous properties imply that $E(n)$ is simply connected; because of the existence of a section, even $E(n) \setminus \nu F$ is simply connected (the normal circle to F bounds a disk, restriction of the section).

Next take a knot K in S^3 and denote by N_K its 0-surgery, a three manifold having the homology of $S^1 \times S^2$. The meridian $m = \mu(K)$ represents a generator of $H_1(N_K, \mathbb{Z})$ and inherits a canonical framing coming from the surgery operation. In $S^1 \times N_K$ there is therefore a preferred torus $T_m = S^1 \times m$, non-nullhomologous, having self-intersection 0 and canonical framing. We can define the normal connected sum of the couples (X, F) and $(S^1 \times N, T_m)$, that we denote by

$$X_K = X \#_{F=T_m} S^1 \times N_K, \tag{16}$$

where the pieces are glued together in such a way to preserve the homology class $[pt \times \partial D^2]$. In general this prescription is insufficient to define in an unique way the manifold. Before discussing this point, let's analyse another description of these (families of) manifolds. First, notice that the zero surgery operation induces a diffeomorphism $S^3 \setminus \nu K = N_K \setminus \nu\{m\}$. Using this diffeomorphism we can write

$$X_K = (X \setminus \nu F) \cup (S^1 \times (S^3 \setminus \nu K)); \tag{17}$$

the gluing prescription corresponds in this notation to the request that the homology class of the longitude $\lambda(K)$ goes to the homology class of the normal circle to F.

Let's come now to the issue of the indeterminacy in the definition. The isotopy class of the gluing map $\phi : S^1 \times \partial\nu K \to \partial\nu F$ between the boundary 3-tori is defined up to the action of the matrices of $SL(3, \mathbb{Z})$ of the form

$$\begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 1 \end{pmatrix}; \tag{18}$$

in reality, whenever F is in the neighborhood of a cusp fiber in X (something which follows from the very definition of c-embedded torus) we can dispose of the indeterminacy corresponding to the upper left $SL(2, \mathbb{Z})$ factor, originated from the absence of a canonical choice for the basis of $H_1(F, \mathbb{Z})$, as any fiber and orientation preserving diffeomorphism of $\partial(X \setminus \nu F)$ extends to a (fiber preserving) diffeomorphism of $(X \setminus \nu F)$, see Chapter 8 of [GS]. This follows from the fact that the monodromy around a cusp fiber generates the whole mapping class group of a regular fiber, namely $SL(2, \mathbb{Z})$. The remaining indeterminacy, instead, cannot be disposed of in general, but in the case where $X = E(1)$ we can do it, as *any* orientation preserving diffeomorphism of $\partial(E(1) \setminus \nu F)$ extends to an orientation preserving diffeomorphisms of $(E(1) \setminus \nu F)$ (see [GS]): no matter how we glue the manifold $S^1 \times (S^3 \setminus \nu K)$, the resulting four manifolds are smoothly equivalent.

In any case, the statements that follow will hold for *any* manifold constructed this way. Application of Van Kampen theorem shows X_K is simply connected, as the the meridian, which normally generates $\pi_1(S^3 \setminus K)$, is glued to a nullhomotopic loop; moreover, as we remove and glue an homology $T^2 \times D^2$, the homology (and the intersection form) of X and X_K coincide. By Freedman's classification theorem of topological four manifolds, X and X_K are homeomorphic, careless of the knot used to build them.
It is natural to ask whether the smooth structure of X_K can in some way reflect the properties of the knot. We have a first, strong result that tells that this question is in the right direction:

Theorem 4.1. *(Fintushel-Stern) Assume $b_2^+(X) > 1$ and denote by $\Delta_K(s)$ the symmetrized Alexander polynomial of the knot K (defined up to sign); then the Seiberg-Witten invariant of X_K, written as a (symmetric) polynomial in the basic classes, has the form*

$$SW_{X_K} = \pm SW_X \cdot \Delta_K(s^2), \tag{19}$$

where we identify s with the cohomology class of the Poincarè dual of the fiber F.

This result shows that knots having different Alexander polynomial give inequivalent smooth structure to the same topological manifold. We could be lead to state a quite optimistic conjecture, namely that inequivalent knots (even with the same Alexander polynomial) induce inequivalent structures, but we have to recall that the fundamental group of the knot is somehow "dissolved" in this construction, so it is possible that some information is lost; probably, something weaker than this conjecture holds true, but this is a very difficult point, by lack of other effective smooth invariants.
The surfaces $E(n)$, being algebraic, are symplectic; we want to analyze now whether X_K admits a symplectic structure (the reader familiar with Seiberg-Witten theory for symplectic manifolds has certainly recognized that a necessary condition for X_K to be symplectic is that $\Delta_K(s)$ has to be monic). This question is, by large, unsolved, but we can explicitly exhibit examples of symplectic X_K using Gompf's results on symplectic surgery. First, we fall back to three dimensional topology and we consider fibered knots; a non obvious result of Gabai states that a knot is fibered if and only if the 0-surgery N_K is so. Now, as discussed above, if N_K is fibered, $S^1 \times N_K$ is symplectic, and admits as symplectic section the torus $T_m = S^1 \times m$. Gompf's theory assures then that X_K, constructed as normal connected sum along the symplectic torus T_m, admits a symplectic structure with canonical bundle $K_{X_K} = K_X + K_{S^1 \times N_K} + 2T$ (by T_m we mean the (Poincaré dual of) homology class identified by $F = T_m$ in X_K). Elementary

calculations show that $K_{S^1\times N_K} = (2g-2)S^1\times m$, where g is the genus of the knot (equal to the minimal genus in N_K), so that $K_{X_K} = K_X + 2gF$ (in X_K we have $F = S^1\times m$). In particular $K_{E(n)_K} = (n-2+2g)F$.

Concerning the converse question, i.e. whether it is possible to construct a symplectic X_K only starting from a fibered K, this question is likely related to the question whether the only symplectic $S^1\times N$ arise from fibered N; we have to admit that, although in the latter case it is possible to prove some result ([K], [V]) on the structure of the Alexander and Thurston unit spheres which go in the direction of the stated conjecture (and which is, in the case of 0-surgery of a knot, monic Alexander polynomial and the condition $deg\Delta_K = 2g$), in the case of X_K the answer seems more problematic: the only known constraint concerns, as mentioned, the leading coefficient of Δ_K.

We can extend the above construction to the case of an n-component oriented link $L\subset S^3$. Denote $p_i = -\sum_j lk(L_i,L_j)$; the manifold M_L obtained performing p_i-surgery on each component has the property that the image of each meridian $\mu(L_i)$ has infinite order in $H_1(M_L,\mathbb{Z})$ and is canonically framed; in $S^1\times M_L$ the tori $S^1\times\mu(L_i)$ have self-intersection zero, are framed and non trivial in homology. Next take n copies of the surface X, containing each one an elliptic fiber F_i, and construct the manifold

$$X_L = \coprod X_i \#_{F_i=S^1\times\mu(L_i)} S^1\times M_L. \tag{20}$$

Similarly to the case of a sigle component (where $p_1 = 0$) the gluing is made in such a way to send the homology class of the normal circle to the i-th torus $S^1\times\mu(L_i)$ to the class of a normal circle to the i-th elliptic fiber. As in the case of X_K the manifold is not uniquely defined. This manifold is simply connected, and the image of the tori $S^1\times\mu(L_i)$ define n homology classes in X_L, without relations. If the link has more that one component then the manifold X_L has $b_2^+(X_L)$ strictly greater than 1, as the images of the $S^1\times\mu(L_i)$ in X_L constitute an independent set of generators of a subspace of $H_2(X_L,\mathbb{Z})$ for which the intersection form is trivial: as the intersection form is non-degenerate, it must have at least n positive and negative eigenvalues. We have an alternative description of these manifold, analogous to the one of eq. 17; we can represent X_L as

$$X_L = (\coprod X_i\setminus\nu F_i)\cup(S^1\times(S^3\setminus\nu L)) \tag{21}$$

and the gluing is made lifting a diffeomorphism between $S^1\times\mu(L_i)$ and F_i to an orientation reversing diffeomorphism of the boundary tori so that the homology class of $p_i\mu(L_i)+\lambda(L_i)$ is sent to the class of the normal circle to F_i. Concerning these manifolds, the following statement holds true:

Theorem 4.2. *(Fintushel-Stern) The Seiberg-Witten invariants of X_L, written as a (symmetric) polynomial in the basic classes, have the form*

$$SW_{X_L} = \pm \prod_i SW_{X\#_T E(1)} \cdot \Delta_L(s^2), \tag{22}$$

where $\Delta_L(s)$ is the symmetric multivariable Alexander polynomial of the link L, with $s = (s_1, ..., s_n)$ and we identify s_i with the cohomology class of the Poincarè dual of the fiber F_i.

Note that the symmetry up to sign of the multivariable Alexander polynomial follows the same rules of the symmetry up to sign of the SW polynomial (the charge conjugation invariance depends on a sign).

5. McMullen-Taubes examples

It is possible to generalise the construction of Fintushel-Stern in such a way to be able to deal with manifolds other than complements of links in S^3 or to change the gluing rules. This generalisation will include also the example built by McMullen and Taubes in their proof of Proposition 1.1.

In order to describe this generalisation, start with a three manifold N with boundary the disjoint union of n tori. Assume that the image of $\pi_1(\partial N) \to \pi_1(N)$ generates normally $\pi_1(N)$ and that the boundary tori have a well defined ordered oriented homology basis denoted by (α_i, β_i).

Next, after the choice of an elliptic surface $X = E(n)$, define the manifold

$$X_N(\alpha_i, \beta_i) = (\coprod X_i \setminus \nu F_i) \cup (S^1 \times N) \tag{23}$$

where the gluing is made in such a way that the homology class of β_i is sent over the homology class of the normal circle to the i-th elliptic fiber. The hypothesis on the fundamental group of N guarantees that the resulting manifold is simply connected.

The gluing condition is not enough to define completely the manifold; as in the case of Fintushel-Stern manifolds, the ambiguity related to the absence of a chosen basis in $H_1(F_i, \mathbb{Z})$ is only apparent, while the remaining ambiguity is effective. Moreover the smooth manifold, as the notation suggests, can depend on the choice of the (α_i, β_i), with the noteworthy exception considered in this

Lemma 5.1. *Let $X_N(\alpha_i, \beta_i)$ be defined as above with $X = E(1)$; then the manifold is well defined and moreover its definition depends uniquely on N, i.e. is unaffected by the choice of the basis on ∂N.*

Proof: the proof should be clear from the discussion of Section 4: on each boundary component we can reabsorb any orientation preserving self-diffeomorphism of $S^1 \times \alpha_i \times \beta_i$ by an orientation preserving self-diffeomorphism of $\partial\nu F_i$, which extends to $E(1)_i \setminus \nu F_i$. □

Fintushel-Stern manifolds enter in this definition, as they correspond to the choice $N = S^3 \setminus \nu L$ and $(\alpha_i, \beta_i) = (\mu(L_i), p_i\mu(L_i) + \lambda(L_i))$ as eq. 21 and the following comments show.

The example of McMullen and Taubes is constructed as follows: let $N = S^3 \setminus \nu K = T^3 \setminus \nu L$ be the three manifold defined in Section 3. We define the normal connected sum

$$\coprod X_i \#_{F_i = S^1 \times L_i} S^1 \times T^3. \tag{24}$$

Usual remarks on the ambiguity of the definition apply (in fact, the specific example of [McMT] is defined as fiber sum, i.e. so that the identification of the boundary 3-tori $F_i \times \partial D^2 \to S^1 \times L_i \times \partial D^2$ has the form $(x, t) \mapsto (x', -t)$ so that no ambiguity arises).

We can show that this manifold is of the type we have introduced: first, the condition on the fundamental group is satisfied, as N is complement of a link. We have still to specify the choice of a basis on the homology of ∂N and we do it as follows. We recall that the image of the meridians $\mu(K_i)$ in S^3, that we have denoted by m_i, goes to longitudes of L_i for $i = 1, 2, 3$ in T^3 and to a meridian of L_4 for $i = 4$; the images l_i of the preferred longitudes $\lambda(K_i)$ go to meridians of L_i for $i = 1, 2, 3$ and to a longitude of L_4 on the fourth component. An analysis of the definitions of normal connected sum (keeping track of the framing of L_i) and of the $X_N(\alpha_i, \beta_i)$ shows that McMullen-Taubes' manifold corresponds to the choice of the basis is given by $(\alpha_i, \beta_i) = (m_i, l_i)$ for $i = 1, 2, 3$ and $(\alpha_4, \beta_4) = (l_4, -m_4)$.

Note that the definition of X_N differs from the Fintushel-Stern one, applied to the link $K \subset S^3$, by the fact that we glue differently the manifold N along the boundary components. Anyhow, in a particular case, we have:

Lemma 5.2. *Let $X = E(1)$; then the McMullen-Taubes manifold $E(1)_N$ is diffeomorphic to the Fintushel-Stern manifold $E(1)_L$.*

Proof: the proof is just application of Lemma 5.1. □

For what concerns the Seiberg-Witten invariants, the proof of the aforementioned theorem of Fintushel-Stern passes word by word to prove that the SW invariant of X_N has the form expressed in

Theorem 5.3. *Let N be the three manifold discussed in Section 3; then we have*

$$SW_{X_N} = \pm\Delta_N(x^2, y^2, z^2, t^2) \tag{25}$$

where (x, y, z, t) are linearly independent cohomology classes Poincarè dual to the image of $S^1 \times m_i$.

We define now a norm, on the second homology group of a 4-manifold, which reflects (in the case of manifolds which have no torsion in homology, as in our case) the definition of Alexander norm in dimension three through three dimensional SW basic classes (the general case requires a slight modification). Namely, we define

$$\|\Sigma\|_{SW} = max\{K_i \cup \Sigma, SW_M(K_i) \neq 0\}. \tag{26}$$

Theorem 5.3 translates to the fact that the injective map

$$H_2(N, \mathbb{Z}) \to H_2(S^1 \times N, \mathbb{Z}) \to H_2(X_N, \mathbb{Z}) \tag{27}$$

is an isometry (up to a factor 2) between the norms $\|\cdot\|_T$ and $\|\cdot\|_{SW}$. We have moreover

$$N(SW_{X_N}) = 2N(\Delta_N). \tag{28}$$

6. Inequivalent symplectic structures

Let M a smooth closed manifold, which admits two symplectic structures ω_0 and ω_1. We introduce the following

Definition 6.1. *We say that ω_0 and ω_1 are homotopic (or deformation equivalent) if there exist a path ω_t of symplectic forms connecting them.*

For example, for sufficiently small values of ϵ, any form of the type $\omega_0 + \epsilon\eta$ is homotopic to ω_0, for any closed form η. It is not difficult to prove that homotopic symplectic forms have the same canonical bundle. We have the following, simple

Lemma 6.2. *Let N be a closed 3-manifold which fibers over S^1 and let be ρ_0, ρ_1 be never vanishing closed 1-forms whose cohomology classes belong to the cone over the same fibered face of the unit sphere of Thurston norm; then, on $S^1 \times N$, the symplectic forms $\omega_i = dt \wedge \rho_i + *\rho_i$ are homotopic.*

In fact, as we already noticed, the canonical class K associated to these symplectic structures depends only on the face we are considering. We need to introduce another equivalence relation.

Definition 6.3. *We say that ω_0 is pull back of ω_1 if there exist a diffeomorphism $f : M \to M$ such that $\omega_0 = f^*\omega_1$.*

We combine the equivalence relations introduced in Definitions 6.1 and 6.3 in the following

Definition 6.4. *Two symplectic forms ω and $\hat{\omega}$ on a manifold M are said to be equivalent if there exist a path of symplectic forms ω_t and a diffeomorphism $f : M \to M$ such that $\omega = \omega_0$ and $\hat{\omega} = f^*\omega_1$.*

There is a first question, that arises quite naturally, for which I don't know the answer.

Problem 6.5. *Do there exist a symplectic four manifold (M, ω) with $b_2^+(M) > 1$ and $K_\omega \neq 0$ such that ω is not equivalent to $-\omega$?*

The condition on the canonical bundle, by applying Taubes' constraints, implies ω and $-\omega$ are not homotopic (maybe easier proofs of this fact exist). Therefore, for an M satisfying the condition required, we require that there exist no diffeomorphism sending ω to a symplectic form which can be deformed to ω. A positive answer to Problem 6.5 would require, in my opinion, the reformulation of Definition 6.3, adding the sentence "up to sign". (Note that ω can be deformation equivalent to $-\omega$, for example on the surface $K3 = E(2)$, but then we have $K = 0$.) McMullen and Taubes have been able to show that the manifold denoted by X_N has two inequivalent symplectic structures, inherited by inequivalent fibrations on N. We have the following

Theorem 6.6. *Let X_N be the four manifold defined above; there exist two non equivalent symplectic structures ω_0, ω_1.*

Proof: the proof is in two steps: first we construct several symplectic structures on X_N, non homotopic as distinguished by the different canonical class, and then we show that there can not exist a diffeomorphism of X_N which connects them.

Step 1: we start by recalling that for every never vanishing 1-form ρ on T^3, representing a fibration transverse to the link $L \subset T^3$ we can construct a symplectic two form ω_ρ on $S^1 \times T^3$, never vanishing on L. We can assume that the link $L \subset T^3$ is framed and oriented in such a way that the tori $S^1 \times L_i$ have an orientation and a well defined normal bundle in $S^1 \times T^3$. Lemma 3.2 tells that the cone in which lie the fibrations of N, induced from fibrations of T^3, is identified from the sign of the value of ρ on the different components of L_i. Put another way, the symplectic forms ω_ρ induce by definition an orientation on symplectic submanifolds, and according whether this orientation is consistent with the orientation of $S^1 \times L_i$ we can characterize the cone in which ρ lies. By scaling by a real number the Kähler forms Ω_i on the $E(1)$'s we can obtain that, for every component of the link, we have $\omega_i(F_i) = \omega_\rho(S^1 \times L_i)$. Introduce the $\mathbb{Z}_2$-valued function $\epsilon_{\rho,i} = sign(\omega_\rho(S^1 \times L_i))$; according to the value of this expression, we can make a symplectic normal connected sum, following Gompf's theory, between $(S^1 \times T^3, \omega_\rho)$ and four copies of $E(1)$ (whose symplectic form will be, up to scale, the ordinary one or the the reversed one) along the symplectic tori $\epsilon_{\rho,i} S^1 \times L_i$.

The cohomology class of the canonical bundle of X_N will depend on the previous choice, in particular on ρ. The general formula for the canonical class of a symplectic connected sum along a torus of zero self-intersection ($K = K_a + K_b + 2T$) gives, in this case,

$$K = \sum_{i=1}^{4} (K_{E(1)_i} + \epsilon_{\rho,i} F_i) + K_{S^1 \times T^3} + \sum_{i=1}^{4} \epsilon_{\rho,i} S^1 \times L_i = \sum_{i=1}^{4} \epsilon_{\rho,i} S^1 \times L_i. \tag{29}$$

Some explanation on this cryptic expression; the surfaces $E(1)$ do not contribute because, careless of the sign of the symplectic form, its canonical and its symplectic fiber enter with opposite sign ($K_{E(1)_i} = -\epsilon_{\rho,i} F_i$); the canonical of the four torus is of course trivial, and the only remaining term is the cohomology class Poincaré dual to the symplectic torus $\epsilon_{\rho,i} S^1 \times L_i$. This is the image, under the injective map

$$H_1(N, \mathbb{Z}) \to H_2(S^1 \times N, \mathbb{Z}) \to H_2(X_N, \mathbb{Z}) \overset{PD}{\to} H^2(X_N, \mathbb{Z}), \tag{30}$$

of the the homology class $\pm_i L_i \in H_1(N, \mathbb{Z})$ (by this we mean the homology classes of a cycle parallel to L_i). As previously noted, the classes L_i are related by the formula $L_4 = L_1 + L_2 + L_3$, both seen as cycles in T^3 and in N; this fact says, in particular, that the signs of $\rho(L_i)$ can be related. An easy check shows that exactly 14 combinations can arise, which correspond (of course!) to the vertices of $N(\Delta_N(t = 1))$. These 14 combinations describe the 14 canonical bundles arising, on the manifold X_N, according to the choice of ω_ρ from a ρ contained in one of the 14 fibered faces of the Thurston norm on the pull-back classes of $H^1(N, \mathbb{R})$. We write them, together with the corresponding class of the Alexander polynomial (written in the "algebraic" notation of eq. 12):

$$\begin{array}{cc} \pm S^1 \times (2L_1) = x^{\pm 2} & \pm S^1 \times (2L_2) = y^{\pm 2} \\ \pm S^1 \times (2L_3) = z^{\pm 2} & \pm S^1 \times (2L_1 + 2L_2) = (xy)^{\pm 2} \\ \pm S^1 \times (2L_2 + 2L_3) = (yz)^{\pm 2} & \pm S^1 \times (2L_1 + 2L_3) = (xz)^{\pm 2} \\ \pm S^1 \times (2L_1 + 2L_2 + 2L_3) = (xyz)^{\pm 2} & \end{array} \tag{31}$$

As all these 14 canonical classes are non homologous, there are 14 deformation classes of symplectic structures, distinguished by their canonical class.

Step 2: let's consider the action of the group of diffeomorphisms of X_N on these classes. The analysis is the same we did in Proposition 3.1: any diffeomorphism of X_N must preserve $N(SW_{X_N})$ and preserve the couple of extremal vertices $t^{\pm 2}$; moreover, as it must preserve all other valences of vertices, it must keep vertices of valence 5 and 6 in different orbits. For example, the vertex $(xyz)^2$ and $(xz)^2$ lie in different orbits. Considering them as canonical classes of the respective (homotopy class of) symplectic forms, this implies that there does not exist any diffeomorphism sending one homotopy class in the other. This completes the proof of the theorem. □

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