THE ALEXANDER NORM IS SMALLER THAN THE THURSTON NORM: A SEIBERG-WITTEN PROOF

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Abstract. In this paper we give a proof, within the framework of SW theory, of a theorem of C.McMullen on the relation of Alexander and Thurston norm on the first cohomology group of a three manifold. We then study these norms in the particular case of three manifolds whose product with the circle is symplectic: we show that, under some standard conjectures, these manifolds are prime. Assuming primeness and $b_1 > 1$, we show moreover that their Thurston and Alexander norm coincide on a cone over a closed face of the Thurston unit sphere; an analogous statement holds when $b_1 = 1$. These results provide some evidence to the conjecture that these manifolds fiber over the circle.

1. Introduction

It has recently been proven, in [11], that two natural seminorms defined on the first cohomology group of a three dimensional manifold (eventually with boundary), namely the Alexander norm $\|\cdot\|_A$, defined from the Alexander polynomial of the manifold, and the Thurston norm $\|\cdot\|_T$, defined in terms of minimal genus of the representatives of the Poincaré dual two dimensional homology class, satisfy a relation expressed in the following

Theorem 1.1. *(McMullen) Let N be a compact, connected, oriented three manifold (eventually with boundary a union of tori); then the Alexander and Thurston norm satisfy*

$$\|\cdot\|_A \leq \|\cdot\|_T + \begin{cases} (1+b_3(N))\,div(\cdot) & \text{if } b_1(N)=1 \\ 0 & \text{if } b_1(N)>1, \end{cases} \tag{1}$$

where $div(\cdot)$ denotes the divisibility of an element in $H^1(N,\mathbb{Z})$.

This inequality, applied to the particular case where the three manifold is the complement of a knot K, reduces to the well known fact that the degree of the Alexander polynomial of the knot (i.e. the difference between highest and lowest power) is bounded from above by twice the genus of the knot, i.e. the lowest value of the genus of a Seifert surface of the knot.
The proof of this inequality given in [11], is purely geometrical, but it is suggested that at least

in the case of closed manifolds there should exist a proof based on three dimensional Seiberg-Witten theory, using the results of [12] on the relation between three dimensional SW theory and Alexander polynomial of a manifold and of [6] on the relations between Thurston norm and the monopole classes of SW theory. In [5] it is discussed the fact that, when a three manifold N is the result of 0-surgery on a knot K, the above relation can be proved within SW theory. Our aim here is to prove this relation for any closed three manifold; the case with boundary, that we omit by sake of brevity, can be proven along the same lines.

We will analyse then the case of three manifolds whose product with the circle admits a symplectic structure. It has been conjectured that these manifolds must fiber over the circle. We will show that, assuming Thurston geometrisation conjecture, these manifolds are prime. Whenever these manifolds are irreducible, moreover, we show that on a cone over a closed face of the Thurston unit sphere in $H^1(N,\mathbb{R})$, the relation in Theorem 1.1 is in fact an equality, providing some evidence to the conjecture.

Some of our results appear as a consequence of the ideas contained in the aforementioned [6],[12], our contribution consisting in making explicit this consequence.

2. Alexander and Thurston norms

We start by briefly recalling the definition of Alexander and Thurston (semi) norms on the first cohomology group of a three manifold N. Denote by F the free abelian group $F := H_1(N;\mathbb{Z})/Tor$; its rank coincides, by definition, with the first Betti number of N. The Alexander polynomial of N appears as an element of the group ring $\mathbb{Z}[F]$, i.e. a finite sum $\Delta_N = \sum_i a_i f_i$ where the f_i are elements of F and the a_i are integer coefficients. The Alexander polynomial is well defined up to the multiplication by the units of $\mathbb{Z}[F]$. Now for any element $\varphi \in H^1(N;\mathbb{Z})$ we can define the norm

$$\|\varphi\|_A := max_{ij}\varphi(f_i - f_j). \tag{2}$$

where the indexes run over all i, j such that a_i, a_j are non zero. It is clear that this definition is unaffected by the indeterminacy in the Alexander polynomial and moreover does not depend on the (nonzero) values of the coefficients. Note that this norm can be degenerate.

When N is the complement of a knot K, the Alexander polynomial of N is the Alexander polynomial of K; the first homology group of N can be identified with $\mathbb{Z}$ and if we denote by t a generator of $\mathbb{Z}$ all the f_i are powers (using the multiplicative notation for $\mathbb{Z}$) of t, so that Δ_N takes the familiar form $\Delta_N = \sum_k a_k t^k$ and is well defined up to the multiplication by $\pm t^m$.

If we compute the Alexander norm of ϕ, the generator of $H^1(N;\mathbb{Z})$ such that $\phi(t) = 1$, we have $\|\phi\|_A = 2r(N)$ where $2r(N)$ is the difference between the highest and lowest power of t appearing in a given presentation of Δ_N. This is, as the notation suggests, an even number by well known symmetry properties of the Alexander polynomial.

For what concerns the Thurston norm, introduced in [16], it is usually defined on the second (relative) homology group of N as follows: if σ is a class in $H_2(N, \partial N; \mathbb{Z})$ we consider properly embedded Riemann surfaces Σ that represent σ and we define their Euler characteristic as the sum of the Euler characteristic over all components of positive genus. Denoting $\chi_-(\Sigma) = -\chi(\Sigma)$, minus the Euler characteristic of Σ, we define

$$\|\sigma\|_T = min\{\chi_-(\Sigma) | (\Sigma, \partial\Sigma) \hookrightarrow (N, \partial N), [\Sigma] = \sigma\}, \tag{3}$$

and we use Poincaré duality to induce a norm on $H^1(N;\mathbb{Z})$. Also this norm can be degenerate. It is not difficult to verify that both norms are linear on rays and satisfy the triangle inequality. It is possible to extend continuously these norms to the cohomology with real coefficient; in the sequel we will make use of this.

3. Averaged basic classes and monopole classes

In this section we will discuss the way the Alexander and Thurston norms are related to Seiberg-Witten theory, in the form of some particular basic classes, respectively monopole classes, of a three manifold. Essentially the relation between Alexander norm and SW theory will be deduced from Meng-Taubes proof of the equivalence of a SW invariant of a three manifold and refined Reidemeister-Franz torsion (see [12]) which is in turn related, by a result of Turaev ([17]), to the Alexander polynomial of the manifold. The relation of Thurston norm and monopole classes, instead, is the content of [6].

We start with a brief review of SW theory in dimension three, in order to have the formulation which is the more suitable for our purposes. Let (N, g) be a smooth, closed, oriented, riemannian three dimensional manifold. We will assume that $b_1(N) > 0$, as this is the only case of interest for us. We equip N with the canonical homology orientation induced by a basis $(t_1, ..., t_{b_1(N)})$ of F. Once N is endowed with a spin^c structure P_N, i.e. a $U(1)$ lifting of the $SO(3)$ frame bundle, we can consider the three dimensional SW equations

$$F_A = q(\psi) - i\eta, \quad \not{\partial}_A\psi = 0, \tag{4}$$

where A is a connection on the determinant bundle of the spin^c structure, $q(\cdot)$ is an $\Omega^2(N; i\mathbb{R})$-valued bilinear on the sections of the spinor bundle associated to P_N, η is a perturbation term that lives in $\Omega^2(N;\mathbb{R}) \cap ker\, d$, $\partial\!\!\!/_A$ is the Dirac operator that acts on the the spinors. These equations are invariants under the gauge group of those automorphisms of P_N that act trivially on the frame bundle. This group acts freely away from reducible couples. The latter are solutions of the equations whenever the real first Chern class of the spin^c structure is cohomologous to $\frac{1}{2\pi}\eta$. We will call a couple (P_N, η) *bad* whenever this happens. It is clear from the definition that good couples have codimension $b_1(N)$ in the space of all couples. We denote by $\mathcal{M}(P_N, g, \eta)$ the moduli space of solutions of eq. 4, omitting the arguments whenever unnecessary.

We want use SW equations to extract information on the set $\mathcal{S}$ of spin^c structures. In order to do so we choose, in correspondence of a fixed spin^c structure, a good perturbation class. Under this hypothesis, the usual techniques of gauge theory allow to prove that the gauge equivalence classes of irreducible solutions of equations 4 define a compact, oriented, generically (in η) smooth, 0-dimensional moduli space. The equations, and consequently the solution spaces, depend on the choice of the metric and the perturbation. We want to keep track of this dependence. Two cases can arise, depending whether the spin^c structure is torsion or not. In the case where the spin^c structure is not torsion it is natural to use an exact perturbation to define the moduli space: no reducible solutions can arise for any metric and, as usual, moduli spaces corresponding to different metric are compactly cobordant. This allows an unambiguous definition of invariants very much as in the four dimensional case, counting the solutions with a sign. However it turns out to be interesting also to consider the use of a non exact perturbation, which allows to deal also with the case of torsion structures. For a spin^c structure, taking a fixed generic good perturbation, the moduli space of solutions corresponding to different metric are smoothly cobordant, while by a change of the perturbation class the spaces remain cobordant as long as there exist a path entirely composed of good couples; this is always possible if and only if $b_1(N) > 1$.

Summing up, it is possible to define the following invariants:

- if $b_1(N) > 1$ we define $SW(P_N)$ = algebraic sum of points of $\mathcal{M}(P_N, g, \eta)$ where (P_N, η) is any good couple;

- if $b_1(N) = 1$ let ϕ be the generator of $H^1(N;\mathbb{Z}) = \mathbb{Z}$ which satisfies $\phi(t) = 1$: we define $SW^{\pm}(P_N, \phi)$ = algebraic sum of points of $\mathcal{M}(P_N, g, \eta)$ where (P_N, η) is a good couple and $\pm(\eta - 2\pi c_1(P_N)) \cdot \phi > 0$. The values of SW^+ and SW^- are in fact related by a wall crossing formula, that can be extracted from the four dimensional one by considering the four manifold (with $b_+ = 1$) $S^1 \times N$ and the correspondence of SW theories by dimensional reduction (see

[13]). This formula reads

$$SW^+(P_N, \phi) - SW^-(P_N, \phi) = \frac{1}{2} c_1(P_N) \cdot \phi. \tag{5}$$

- if $b_1(N) = 1$, $c_1(P_N)$ non torsion, $SW(P_N) =$ algebraic sum of points of $\mathcal{M}(P_N, g, \eta)$ where η is an exact form; if $c_1(P_N)$ is torsion, $SW(P_N) =$ algebraic sum of points of $\mathcal{M}(P_N, g, \eta)$ where η is any nonexact form: the formula of eq. 5 tells in fact that being P_N torsion this value does not depend on the sign of $\eta \cdot \phi$ for any (non exact) perturbation η, so that the value of SW does not change crossing the wall. Note that this definition of SW is equivalent to the one we would get considering (even non exact) suitably small perturbations.

We are interested, when $b_1(N) = 1$, in the relation between SW and $SW^\pm$. From the argument above, if $c_1(P_N)$ is torsion, $SW(P_N) = SW^\pm(P_N, \phi)$; in the other cases it follows from the definition that if $\pm c_1(P_N) \cdot \phi < 0$ then $SW(P_N) = SW^\pm(P_N)$; in fact, according to the sign of $c_1(P_N) \cdot \phi$, an exact form is an allowed perturbation for defining $SW^\pm(P_N, \phi)$. We introduce now the following definition:

Definition 3.1. *Let $c \in H^2(N; \mathbb{Z})$ be an integral cohomology class that arises as first Chern class of a spin^c structure P_N such that the invariant $SW(P_N)$ is nonzero. Then c is called a basic class of N.*

It is quite clear from this definition that the SW equations for a basic class admit a solution for any metric and a generic perturbation. Whenever the small forms are a good perturbation, moreover, as the compactness of the equations implies that non emptiness is an open condition, also the unperturbed equations have a solution for any metric, i.e. $\mathcal{M}(P_N, g, 0) \neq \emptyset$ (note that this space can be nonsmooth). This makes natural to introduce the

Definition 3.2. *Let $c \in H^2(N; \mathbb{Z})$ an integral cohomology class that arises as Chern class of a spin^c structure P_N such that $\mathcal{M}(P_N, g, 0) \neq \emptyset$ for any metric g. Then c is called a monopole class.*

From the previous observation, the set of monopole classes, that we denote by $\mathcal{C}(N)$, contains all the basic classes; the torsion classes, moreover, are always monopole classes (whether or not basic), because of the presence of reducible solutions.

We now introduce, following ref. [12], an element in $\mathbb{Z}[[F]]$, defined from the family of SW invariants of the spin^c structures.

The set $\mathcal{S}$ of spin^c-structures on N is an affine $H^2(N; \mathbb{Z})$. There is a natural way to define a map which goes from $\mathcal{S}$ to F which is constructed as follows. Fix a reference spin^c-structure Q_N (for example the product structure). Any other structure P_N differs from it by the action of an element of $H^2(N,\mathbb{Z})$. Consider now the composed map

$$H^2(N,\mathbb{Z}) \xrightarrow{PD} H_1(N,\mathbb{Z}) \xrightarrow{\pi} F. \tag{6}$$

Using this map we can construct a map s_Q (which depends from Q_N) which goes from $\mathcal{S}$ to F. The fiber of this map is given by the order of the torsion of $H_1(N;\mathbb{Z})$, that we will denote now on by $ord(N)$. Note that twice this map gives, up to torsion, the Poincaré dual of the difference between the Chern classes of P_N and Q_N. Consider for any $f \in F$ the set $s_Q^{-1}(f) \in \mathcal{S}$. These are the spin^c structures that have the same real Chern class. Define now

$$\begin{aligned} SW(f,Q_N) &:= \textstyle\sum_{s_Q^{-1}(f)} SW(P_N), \quad \text{if } b_1(N) > 0, \\ SW^+(f,Q_N,\phi)) &:= \textstyle\sum_{s_Q^{-1}(f)} SW^+(P_N,\phi), \quad \text{if } b_1(N) = 1 \end{aligned} \tag{7}$$

(we will omit ϕ from the notation, when unnecessary).
We can now define from this the functions

$$\begin{aligned} SW(N,Q_N) &= \textstyle\sum_F SW(f,Q_N)f \in \mathbb{Z}[[F]], \quad \text{if } b_1(N) > 0, \\ SW^+(N,Q_N) &= \textstyle\sum_F SW^+(f,Q_N)f \in \mathbb{Z}[[F]], \quad \text{if } b_1(N) = 1. \end{aligned} \tag{8}$$

It is clear from the definition that changing the reference structure Q_N changes the function by the action of F. In this sense there are well defined elements $SW(N), SW^+(N) \in \mathbb{Z}[[F]]/F$. When $b_1(N) = 1$ there is a relation between $SW(N)$ and $SW^+(N)$, which is induced by the relation between $SW(P_N)$ and $SW^{\pm}(P_N)$ and by the wall crossing formula. An explicit analisys shows that, if we denote by $F' = \{f \in F | \phi(f) > -\frac{1}{2}c_1(Q_N)\cdot\phi\}$ (this is the image, under s_Q, of the set of spin^c structures which have Chern class that satisfies $(c_1(P_N) - c_1(Q_N))\cdot\phi > 0$), then we have

$$SW^+(N,Q_N) = SW(N,Q_N) + ord(N)\sum_{f\in F'}(\phi(f) + \frac{1}{2}c_1(Q_N)\cdot\phi)f. \tag{9}$$

In this formula $ord(N)$ appears because each structure in $s_Q^{-1}(f)$ gives a contribution depending on the formula in eq. 5. It might be useful to write the formula above for a specific choice of Q_N, that we fix to be the product structure Q_N^o: if we denote, as above, by t the generator of F (so that $\phi(t) = 1$), then $F' = \{t^k, |k > 0\}$ and we have the much clearer expression

$$SW^+(N,Q_N^o) = SW(N,Q_N^o) + ord(N)\sum_{k>0} kt^k. \tag{10}$$

A well known fact of SW theory is that the number of spin^c structures for which unperturbed SW equations admit solutions is bounded. This observation, together with the definition of the function SW, yields the fact that $SW(N) \in \mathbb{Z}[F]/F$. Concerning $SW^+(N)$ we have a weaker result: in this case the relation between SW and SW^+ of eq. 9 shows that $SW^+(N) \in Nov(F)/F$ where $Nov(F)$ is the Novikov ring of formal series in F s.t. the coefficients vanish for all but a finite number of negative powers of t.

The previous definitions, in the case where $ord(N) = 1$, are a simple reformulation of SW theory. In the other cases, instead, they define a kind of "average" over all structures which have the same real Chern class. The function SW introduced above, in fact, is not (at least in general) sufficient to reconstruct the complete theory. In particular it does not contain enough information to know which are the basic classes of N. We can introduce a new definition that is quite practical for treating the information on spin^c structures contained in the SW functions of eq. 8. For any element $\gamma \in H^2(N, \mathbb{Z})$, we denote by γ^F its projection to $H^2(N, \mathbb{Z})/Tor(= F^{PD})$.

Definition 3.3. *Let $c \in H^2(N, \mathbb{Z})/Tor$ be a cohomology class such that*

$$\sum_{c_1^F(P_N)=c} SW(P_N) \neq 0. \tag{11}$$

Then c is called a-basic class (where the "a" stands for averaged).

Let us see what does this definition mean: if c is an a-basic class, then there exist a basic class $\gamma \in H^2(N, \mathbb{Z})$ such that $\gamma^F = c$, as at least one of the spin^c structures appearing under summation in equation 11 must have non zero SW invariant. What is not guaranteed, instead, is that any basic class gives rise to an a-basic class. In sum, we have the following inclusions (with self-explaining notation)

$$\mathcal{A}(N) = (a - basic\ classes) \subset (basic\ classes)^F \subset (monopole\ classes)^F = \mathcal{C}(N)^F \tag{12}$$

(this relation could eventually exclude the zero class, as noted earlier). We want to relate now a-basic classes with the SW function $SW(N)$: let c be an a-basic class; then the sum appearing in equation 11 coincides with $SW(f, Q_N)$ where f is defined by the relation $2f = PD(c - c_1^F(Q_N))$. Therefore in correspondence of an a-basic class c the SW function $SW(N)$ has a non zero coefficient.

Note that, according to our definitions, the information on a-basic classes is not contained directly in the coefficients $SW^+(N)$ (if we don't use the relation of eq. 9, of course).

4. Relation between the norms

Our aim now is to relate a-basic classes of N with its Alexander polynomial, and then to the Alexander norm. In this section we will give a proof of the following

Proposition 4.1. *Let N be a closed three manifold with $b_1(N) > 0$; then the Alexander norm of an element $\varphi \in H^1(N;\mathbb{Z})$ is given by*

$$\|\varphi\|_A = \begin{cases} max_{\mathcal{A}(N)}(c\cdot\varphi) & \text{if } b_1(N) > 1 \text{ or } b_1(N) = 1 \text{ and } r(N) = 0 \\ max_{\mathcal{A}(N)}(c\cdot\varphi) + 2div\varphi & \text{if } b_1(N) = 1 \text{ and } r(N) > 0 \end{cases} \tag{13}$$

where the maximum is taken over all a-basic classes of N and $2r(N)$ denotes the degree of the one variable Alexander polynomial.

Proof: the basic ingredient for the proof is provided by the theorem of Meng and Taubes which identifies the SW function with the (sign-refined) Reidemeister-Franz torsion introduced by Milnor. This is related, on its own, to the (sign-refined) Alexander polynomial, denoted by Δ^s_N. More precisely, we have the

Theorem 4.2. *(Meng-Taubes, Turaev) Let N be a closed three manifold equipped of the canonical homology orientation, with $b_1(N) > 0$; then we have, in $\mathbb{Z}[[F]]/F$,*

$$\begin{aligned} SW(N) &= \Delta^s_N \quad if \;\; b_1(N) > 1, \\ (1-t)^2 SW^+(N) &= \Delta^s_N \quad if \;\; b_1(N) = 1. \end{aligned} \tag{14}$$

In the first case the a-basic classes can be immediately identified: we take the symmetric presentation of Δ^s_N, whose existence is guaranteed by Franz-Milnor duality (see e.g. [17]). This is a fixed element of $\mathbb{Z}[F]$ that we can present in the form of multivariable symmetric Alexander polynomial

$$\Delta^s_N = \sum_I a_I t^I \tag{15}$$

where $I = (i_1, ..., i_{b_1(N)})$ is a multi-index of cardinality $b_1(N)$ and $t = (t_1, ..., t_{b_1(N)})$ with $\{t_i\}$ a basis of F. In the same way $SW(N)$ has a symmetric representative in $\mathbb{Z}[F]$ which corresponds to choosing Q_N to be the product structure. Then, the a-basic classes $c_I \in \mathcal{A}(N)$ are exactly twice the Poincaré duals of the elements of F with nonvanishing coefficients in eq. 15, i.e.

$$c_I \in H^2(N;\mathbb{Z})/Tor \text{ is a-basic} \Longleftrightarrow a_I \neq 0 \quad \text{where} \quad t^{2I} = PD(c_I). \tag{16}$$

The use of the first of the relations of equation 14 and the symmetry properties of the Alexander polynomial allow to write the Alexander norm in terms of a-basic classes: we can write

$$\|\varphi\|_A = max_{ij}\varphi(g_i - g_j) = max_{\mathcal{A}(N)}(c_I \cdot \varphi) \quad \text{if } b_1(N) > 1 \tag{17}$$

which constitutes part of the identities we have to prove.

In the case where $b_1(N) = 1$ the shape of equation 14 is a bit more complicated, but the use of equation 10, working again with the symmetric $\Delta^s_N = a_0 + \sum_{i>0} a_i(t^i + t^{-i})$, still allows to identify basic classes; we have, in $Nov(\mathbb{Z})$,

$$(1-t)^2(SW(N, Q^o_N) + ord(N)\sum_{k>0} kt^k) = t(a_0 + \sum_{i>0} a_i(t^i + t^{-i})) \tag{18}$$

(the extra t factor is readily explained in terms of the symmetry properties). Now, using the property that $a_0 + 2\sum_i a_i = ord(N)$ and applying the fact that the inverse, in $Nov(\mathbb{Z})$, of $(1-t)^2$ is given by $\sum_{k>0} kt^{k-1}$, we obtain after explicit computation that

$$SW(N, Q^o_N) = \sum_k (\sum_{j>0} ja_{j+|k|})t^k \tag{19}$$

(this formula appears in [5] for the case of 0-surgery of a knot, but as we have seen it is in fact general; note moreover that although a_0 does not appear explicitly in equation 19, it can be reconstructed from the relation with the other coefficients). The set of a-basic classes $\mathcal{A}(N)$ is composed by terms which are therefore twice the Poincaré duals of those t^k for which $\sum_{j>0} ja_{j+|k|}$ is non zero. In particular, if $2r(N) > 0$ is the degree of the Alexander polynomial, $PD(t^{2r-2})$ is an a-basic class, as the corresponding coefficient is $a_r \neq 0$.

This discussion boils down to the fact that

$$\|\varphi\|_A = max_{ij}\varphi(g_i - g_j) = \begin{cases} max_{\mathcal{A}(N)}(c \cdot \varphi) & \text{if } b_1(N) = 1 \text{ and } r(N) = 0 \\ max_{\mathcal{A}(N)}(c \cdot \varphi) + 2div\varphi & \text{if } b_1(N) = 1 \text{ and } r(N) > 0 \end{cases} \tag{20}$$

The presence of the "$+ 2div\varphi$" factor in the $r(N) > 0$ case corresponds, roughly speaking, to the fact that, because of the relation of eq. 18, we "lose", from the set of a-basic classes, the class corresponding to $t^{r(N)}$. This does not happen, of course, when $r(N) = 0$. This completes the proof of Proposition 4.1. □

The content of Proposition 4.1 is the good one for our purpose, at the light of the results of [6] on the relations between monopole classes and Thurston norm: we have now all we need to prove Theorem 1.1.

We start with the observation that is possible to restrict the class of three manifolds for which is necessary to prove the inequality. Write in fact N as connected sum of prime factors N_i.

It is a well known fact that if there are two or more factors having $b_1(N_i) > 0$ then the SW invariants are trivial. This implies that the Alexander norm is trivial too, and the inequality 1 is trivially true. Therefore the only interesting decomposition is when N admits a form $N = N_1 \# \Sigma$ where Σ is a rational homology sphere (eventually S^3). Under this hypothesis the connected sum theorems applied to this context guarantee that $SW(N) = ord(\Sigma) SW(N_1)$; the homology groups $H^1(N;\mathbb{Z})$ and $H^1(N_1;\mathbb{Z})$ are identified and the Alexander norms coincide. As the Thurston norms defined for N and N_1 coincide too, we deduce that is enough to prove the inequality 1 for a prime manifold that we continue denoting by N. A bit more is true: we can also assume that the Thurston norm is not entirely degenerated: otherwise there is a basis of $H_2(N;\mathbb{Z})$ which is composed by tori and spheres. But in this case (see [6]) there are no non torsion monopole classes, and the Alexander norm is degenerate too. We are reduced therefore to the prove of the inequality for irreducible three manifolds which have a basis for $H_2(N;\mathbb{Z})$ which is not entirely composed of tori. Under this hypothesis, we have the following deep result, that we rewrite in the form which is suitable for our purposes:

Theorem 4.3. *(Kronheimer-Mrowka) Let N be a manifold as above: then the Thurston norm of a class $\varphi \in H^1(N,\mathbb{Z})$ is given by*

$$\|\varphi\|_T = max_{\mathcal{C}(N)}(c \cdot \varphi). \tag{21}$$

In fact this result is even stronger than what we need, being enough an inequality, easier to obtain also in the case of a general three manifold.

Putting together the inclusion $\mathcal{A}(N) \subset \mathcal{C}(N)^F$, Theorem 4.3 and Proposition 4.1, we deduce the inequality 1 for any closed three manifold.

Remark: It comes natural, by looking at the inclusion of equation 12, to define an intermediate norm on cohomology, considering the maximum over *all* basic classes, instead than using only averaged classes. In light of the equivalence of the full SW theory (i.e. without averaging) with the Turaev torsion (see [18]) this corresponds to defining a norm as in equation 2, with the suitable modifications, using the Turaev torsion instead than the Milnor torsion. Applications of the previous results shows that this norm is naturally greater or equal than the Alexander norm and satisfies inequalities, analogous to those of Theorem 1.1, with respect to the Thurston norm.

5. A particular case: symplectic $S^1 \times N$

In this section we will consider a particular class of three manifolds, the ones whose product with S^1 admit a symplectic structure. The case when N is the 0-surgery of a knot has been considered in [5]. In the rest of the section all the three manifolds N we consider are such that $S^1 \times N$ admits a symplectic structure ω_0, so we will omit to mention explicitly this when unnecessary. We remark that the only known examples of manifolds of this type are the three manifolds which fiber over a circle, and it is possible that these are the only ones.
Manifolds which fiber over the circle are prime w.r.t. connected sum, and their Thurston and Alexander norm satisfy a quite particular relation. It is interesting therefore to prove that the manifolds N such that $S^1 \times N$ is symplectic satisfy these properties.
The first result we will obtain, and which restricts the class of possible N, is that N must be connected sum of a prime manifold (therefore $S^1 \times S^2$ or a $K(\pi, 1)$) with an homology sphere which has not finite covers; in particular, if we assume Thurston geometrisation conjecture, this homology sphere is S^3 and N is prime. We will then prove that for a cone $\mathcal{C}$ in $H^1(N, \mathbb{Z})$ it is possible to write the Alexander norm in terms of a preferred a-basic class, induced from the canonical class of $S^1 \times N$. This allows to deduce, under the hypothesis of irreducibility, that in a cone over a closed face of the unit sphere of the Thurston norm the Alexander norm coincides with the Thurston norm (plus the correction term in the case where $b_1(N) = 1$ and $r(N) > 0$). This condition, as we mentioned, can be proved by direct topological arguments for manifolds which fiber over the circle, and provides some further evidence to the conjecture that the two classes coincide. But let's proceed in order. As claimed, we have the following

Theorem 5.1. *Let N be a three dimensional manifold such that $S^1 \times N$ admits a symplectic structure. Then N is equal to the connected sum of a prime manifold with an homology sphere not admitting nontrivial finite covers.*

Proof: the proof of this theorem will be obtained using quite standard tools like vanishing theorem for SW invariants of manifolds decomposed along three manifold of positive scalar curvature and an adaptation of a result of [7] on the decomposition of fundamental group of a symplectic four manifold (note: a different proof of this statement appears also in [10]).
Let's start by recalling the Milnor decomposition theorem, which states that any (compact, smooth, closed, oriented) three manifold N admits an unique (up to the order) factorization in

prime factors that we write in the form

$$N = (\#_{i=1}^{p} Y_i)\#(\#_{j=1}^{q} S^1 \times S^2)\#(\#_{k=1}^{r} K(\pi_k, 1)) \tag{22}$$

where the Y_i are rational homology spheres and the three dimensional Eilenberg-Maclane spaces $K(\pi_k, 1)$ on the r.h.s. have $b_1 > 0$. The theorem is equivalent to proving that $(q, r) = (1, 0)$ or $(q, r) = (0, 1)$, and that the Y_i are of the form discussed.

Step 1: we start by proving the first point. By contradiction assume that N contains more than one factor with positive b_1: we can factorise it in two factors (generally non prime) $N = N_1 \# N_2$ which have both positive first Betti number. We deduce that we can write $M = S^1 \times N = M_1 \cup_{S^1 \times S^2} M_2$ where the decomposition of the four manifold is determined by the connected sum decomposition of N and the two factors M_i are the manifolds with boundary $M_i = S^1 \times (N_i \setminus intD^3)$. It is clear from the fact that N_i have positive first Betti number that the intersection forms of M_i, although in general degenerate, have positive b_2^+. The standard proof of the Donaldson vanishing theorem (for S^3) applies *mutatis mutandis* for decomposition along any three manifold of positive scalar curvature (for a proof see e.g. [19]) and so applying it to $S^1 \times S^2$ we deduce that M has vanishing Seiberg-Witten invariants. The theorem of Taubes in [14] rules out therefore that such a M could admit a symplectic structure. We are lead to assume therefore that N has the form $Y \# Z$ where Y is a rational homology sphere and Z is one of $S^1 \times S^2$ or a $K(\pi, 1)$.

Step 2: we want now show that Y cannot admit finite covers: we proceed like above, decomposing M in the form $M = S^1 \times N = M_1 \cup_{S^1 \times S^2} M_2$ (we use same notation as there no risk of confusion) where now $M_1 = S^1 \times (Y \setminus intD^3)$ and $M_2 = S^1 \times (Z \setminus intD^3)$; now, as Y is a rational homology sphere M_1 has $b_2^+ = 0$ and there is no room to apply the previous vanishing theorem (and in fact, the invariants will not vanish in general). But the decomposition retains enough information to rule out, when $\pi_1(Y) \neq 0$, that M is symplectic. In fact, if it were symplectic, so would be any finite cover of it. But we have the following

Claim 5.2. *Let Y be a three manifold with nontrivial finite covers: then M defined as above has a finite cover with vanishing SW invariants.*

Proof: we will modify the idea of [7], which deals with connected sum, to deal with the case of sum over $S^1 \times S^2$. Let's denote $\pi = \pi_1(Y)$. It is immediate verifying that the fundamental group of M is given by $\mathbb{Z} \oplus (\pi \star \pi_1(Z))$ and that the π quotient of $\pi_1(M)$ corresponds to nontrivial finite covers of order $d > 1$ of M which have the form

$$\tilde{M} = d[S^1 \times (Z \setminus intD^3)] \cup_{dS^1 \times S^2} S^1 \times (\tilde{Y} \setminus d\, intD^3) \tag{23}$$

where $\tilde{Y}$ is a degree d cover of Y. The formula above means that, after removing to $\tilde{Y}$ d 3-disks, we multiply it with S^1 obtaining a manifold with d boundary components isomorphic to $S^1 \times S^2$, that we glue to the d copies of $S^1 \times (Z \setminus D^3)$ (for sake of comprehension it might be useful to forget the S^1 factor and look at the connected sum in dimension three where all becomes quite transparent). The manifold $\tilde{M}$ is a manifold which decomposes along a three manifold of positive curvature in two factors with $b_2^+ > 0$, as $d > 1$, and so by the vanishing theorem quoted before its Seiberg-Witten invariant vanish. This proves the Claim, and completes the proof of Theorem 5.1 □

Three manifolds with nontrivial fundamental group and without finite covers are excluded by Thurston geometrisation conjecture; the same assumption excludes homotopy spheres different from S^3. We can further remark that, according to Kneser conjecture in dimension 3, Theorem 5.1 would imply that $\pi_1(N)$ is prime w.r.t. free product.

We recall now some general results we will apply to our case. The first is the Donaldson theorem on the existence of symplectic submanifolds ([1]). We start from a symplectic structure ω_0 on $S^1 \times N$. We can identify an open neighborhood $\Omega \subset H^2(S^1 \times N, \mathbb{R})$ of $[\omega_0]$ which still admits symplectic representative, as nondegeneracy is an open condition. Now take any rational point in Ω, with symplectic representative ω'; Donaldson theorem assures that, eventually passing to an integral class $[\omega]$, multiple of $[\omega']$, there exist a connected symplectic submanifold $H_\omega \subset S^1 \times N$ such that

$$[H_\omega] = [\omega]^{PD} = [S^1] \times \gamma_\omega + \tau_\omega \in H_2(S^1 \times N, \mathbb{Z}) \tag{24}$$

where $\gamma_\omega \in H_1(N, \mathbb{Z})$, $\tau_\omega \in H_2(N, \mathbb{Z})$ and $\gamma_\omega \cdot \tau_\omega > 0$ (for sake of notation we will denote all products, both on N and on M, with a dot, the distinction being clear from the context). The neighborhood Ω defines two open sets $\Omega_1 \subset H_1(N, \mathbb{R})$, $\Omega_2 \subset H_2(N, \mathbb{R})$; by changing the rational point $[\omega'] \in \Omega$ we can construct different integral classes γ_ω, τ_ω in the respective homology groups which lie in a cone over Ω_1, respectively Ω_2, and which exhaust all the rays with rational slope passing through the Ω_i. Of course, when $b_1(N) = 1$, this construction becomes trivial and we can assume a symplectic form ω representing an integral class to be fixed.

The second result is that the spin^c structures on $S^1 \times N$ for which the unperturbed four dimensional SW equations have solutions (the four dimensional monopole classes) must be pull back of spin^c structures on N (a proof of this can be found in [4]); moreover there is an identification between the moduli spaces for a spin^c structure P_N on N and the moduli space for the pull-back structure on $S^1 \times N$ (that we will usually denote with the same symbol), once a suitable correspondence of the perturbation terms is set (see [13]). This allows to identify, up to a sign determined by the choice of the homology orientations, the SW invariants associated to these moduli spaces.

The third point concerns spin^c structures on a symplectic four manifold (M, ω) with canonical bundle K_M. There exist, in that case, a canonical spin^c structure that decomposes as $\mathbb{C} \oplus K_M^{-1}$ (and has first Chern class equal to $-K_M$). Any other spin^c structure can be written as $E \oplus (K_M^{-1} \otimes E)$ for an $E \in H^2(M, \mathbb{Z})$. There are some constraints on spin^c structures with nonvanishing invariants that arises from Taubes' work (see [14],[15]). In the case of $b_+(M) > 1$ the canonical spin^c structure has SW invariant ± 1 and for any other structure $E \oplus (K_M^{-1} \otimes E)$ with nonzero invariants we have $0 \leq E \cdot \omega \leq K_M \cdot \omega$. Equality implies, respectively, $E = \mathbb{C}$ or $E = K_M$. The case of $b_+(M) = 1$ is more subtle, because of the dependence on the chambers for the SW invariant. An analysis of this case appears in [2]. Assume there exist an embedded torus $T \subset M$ of zero self intersection s.t. $\omega \cdot [T] > 0$; the sign of $(\eta_4 - 2\pi c_1(P_M)^+) \cdot [T]$, where η_4 is a self dual perturbation, allows to distinguish the chambers. Endow M of a metric such that ω is self dual and denote by $SW_{s\omega}$ the invariants corresponding to a perturbation of the four dimensional equations given by $\eta_4 = s\omega$, $s \gg 0$. Taubes' theorems imply that the invariants $SW_{s\omega}$ satisfy a property similar to the $b_+ > 1$ case, namely we have $SW_{s\omega}(\mathbb{C} \oplus K_M^{-1}) = \pm 1$ and $E \cdot \omega \geq 0$ for any structure $E \oplus (K_M^{-1} \otimes E)$ with non zero $SW_{s\omega}$ invariants, the equality implying $E = \mathbb{C}$.

We want apply these results to $S^1 \times N$. We proceed first with the easier case, namely $b_1(N) > 1$; in this case if, for some spin^c structure $P_{S^1 \times N}$ on $S^1 \times N$, we have $SW(P_{S^1 \times N}) \neq 0$ then also the unperturbed four dimensional equations must have a solution. This implies that $P_{S^1 \times N}$ is a pull back structure. It follows that the canonical class and all the other basic classes are pull back: there exists a preferred line bundle $K_N \in H^2(N, \mathbb{Z})$ and a preferred spin^c structure on N of the form $\mathbb{C} \oplus K_N^{-1}$ with SW invariant ± 1 such that any other spin^c structure on N appears as $E \oplus (K_N^{-1} \otimes E)$ for $E \in H^2(N, \mathbb{Z})$ and if it has nonzero invariant must satisfy $0 \leq E \cdot \tau_\omega^{PD} \leq K_N \cdot \tau_\omega^{PD}$, the equalities implying respectively $E = \mathbb{C}$ or $E = K_N$. We will call the line $\mathcal{L}_\omega = \{\lambda \tau_\omega^{PD} \cap H^1(N, \mathbb{Z}), \lambda \in \mathbb{Q}\}$ a *symplectic line.* Changing $[\omega']$ in Ω we identify a *symplectic cone* $\mathcal{C}$ in $H^1(N, Z)$.

The case of $b_1(N) = 1$ is not immediate, as we have information, in this case, only on the invariants $SW_{s\omega}$ and these will not in general provide information on solutions for the unperturbed equation. In particular there is no *a priori* evidence that the canonical spin^c structure on $S^1 \times N$ arises as pull back of a structure on N. We need a slightly more refined analysis to prove this result. We start by leaving aside from our discussion the case of $N = S^1 \times S^2$; clearly $T^2 \times S^2$ is a symplectic manifold but the Alexander and Thurston norms of $S^1 \times S^2$ present no information. With this proviso, we know that all the remaining symplectic $S^1 \times N$ must satisfy both $K_M^2 = 2\chi + 3\sigma = 0$, $K_M \cdot \omega \geq 0$: otherwise, by Liu's theorems in [8], they would be rational or ruled surfaces (note that they are minimal), and this not possible by topological reasons: in fact, they should be ruled surfaces of genus 1, as $\chi = 0$, and have vanishing Stiefel-Withney class, which is possible only for $T^2 \times S^2$. We have now the following Lemma, which rules that K_M is the pull-back of a class $K_N \in H^2(N, \mathbb{Z})$:

Lemma 5.3. *The spin^c structure $\mathbb{C} \oplus K_M^{-1}$ is pull-back of a spin^c structure $\mathbb{C} \oplus K_N^{-1}$ on N.*

Proof: taking a suitable generator of $H_1(N, \mathbb{Z})$ we can identify a torus $T \subset S^1 \times N$ of zero self intersection and $\omega \cdot [T] > 0$. Recall the adjunction inequality of Li and Liu ([9]) which states that $K_M \cdot [T] \leq 0$. If K_M is zero or torsion the statement is immediate. Assume this does not happen: as $K_M^2 \geq 0$, $\omega^2 > 0$ and $K_M \cdot \omega \geq 0$ we know that K_M and ω must be on the same component of the closure of the positive cone in $H^2(S^1 \times N, \mathbb{R})$. As $\omega \cdot [T] > 0$ we must have $K_M \cdot [T] \geq 0$, which together with the adjunction inequality implies that K_M is a pull back class and then the statement. □

Now that we know about the canonical class of $S^1 \times N$ we can proceed very much as in the case of $b_1 > 1$. We write any spin^c structure on N as $E \oplus (K_N^{-1} \otimes E)$: we want to get information on the three dimensional invariant $SW_N(E \oplus (K_N^{-1} \otimes E))$. We can fix the generator ϕ of $H^1(N, \mathbb{Z})$ in such a way that $\tau^{PD} = \lambda\phi$ for some $\lambda > 0$. With this choice we have $c_1(\mathbb{C} \oplus K_N^{-1}) \cdot \phi \leq 0$. By symmetry of the invariant SW_N it is enough to consider the invariant for those spin^c structures such that $c_1(E \oplus (K_N^{-1} \otimes E)) \cdot \phi \leq 0$. It is just a matter of checking the sign to see that for those spin^c structures, on $S^1 \times N$, the value $-2\pi c_1(E \oplus (K_N^{-1} \otimes E))^+ \cdot [T]$ is nonnegative and it is zero if and only if the spin^c structure is torsion. This implies that passing from the perturbation $s\omega$ to a perturbation of type $d^+\epsilon$ (or a suitably small perturbation) no wall is crossed, so that the value of the invariant $SW_{s\omega}(E \oplus (K_N^{-1} \otimes E))$ coincides (eventually up to sign) with the three dimensional invariant $SW_N(E \oplus (K_N^{-1} \otimes E))$. Applying the constraints valid for the classes which are basic w.r.t. the invariant $SW_{s\omega}$ and using the symmetry of SW_N we obtain that the three dimensional spin^c structures $E \oplus (K_N^{-1} \otimes E)$ with nontrivial SW_N invariants must satisfy

$0 \leq E \cdot \phi \leq K_N \cdot \phi$. The values of SW_N invariants on the extremal classes correspond to the basic classes $\pm K_N$ and, in absolute value, are equal to 1. Note that in the case where K_N is a torsion bundle, it is the only basic class, and moreover it must be $K_N = 0$.
This result reproduces exactly the same features of the $b_1 > 1$ case. Using this it is straightforward to prove the following

Proposition 5.4. *Let $S^1 \times N$ be a symplectic manifold, $N \neq S^1 \times S^2$, and take a $\varphi \in H^1(N, \mathbb{Z})$ belonging to a symplectic line $\mathcal{L}_\omega$; if $b_1(N) > 1$ then $\|\varphi\|_A = |K_N \cdot \varphi|$; if $b_1(N) = 1$ then $\|\varphi\|_A = |K_N \cdot \varphi| + 2div\varphi$, the Alexander polynomial of N has leading term equal to ± 1 and $r(N) > 0$.*

Proof: the constraints on spin^c structures with nonvanishing invariant translate to the fact that the maximum of $|c \cdot \varphi|$ for c basic and $\varphi \in \mathcal{L}_\omega$ is attained for $c = \pm K_N$; moreover if c attains the maximal value then it coincides with one of $\pm K_N$. We want to use this property to evaluate the Alexander norm, in the form expressed in Proposition 4.1. To do this we have to prove that $\pm K_N$ are a-basic classes. The constraints on basic classes guarantee that the only spin^c structures P_N with $c_1(P_N)^F = \pm K_N^F$ and nonvanishing invariants are in fact respectively $K_N \oplus \mathbb{C}$ and $\mathbb{C} \oplus K_N^{-1}$, so that the sum of equation 11, namely $\sum_{c_1^F(P_N) = \pm K_N^F} SW(P_N)$, contains in both cases only one nonzero term, that term being equal to ± 1. This implies that the canonical and anticanonical classes are a-basic. In the case where $b_1(N) > 1$ we can conclude, following Proposition 4.1, that for $\varphi \in \mathcal{L}_\omega, \|\varphi\|_A = |K_N \cdot \varphi|$; in the case where $b_1(N) = 1$ and $r(N) > 0$, according to equation 19, the leading coefficient of the Alexander polynomial $a_{r(N)}$ must be equal to ± 1 and, using again Proposition 4.1, for any $\varphi \in H^1(N, \mathbb{Z})$, we have $\|\varphi\|_A = |K_N \cdot \varphi| + 2div\varphi$. The case of $b_1(N) = 1$ and $r(N) = 0$ is instead ruled out, in virtue of equation 19: for no manifold N satisfying these conditions (and different from $S^1 \times S^2$), $S^1 \times N$ admits a symplectic structure. □

Remarks: 1) The previous statement, specialized to the case of the 0-surgery of a nontrivial knot, which is different from $S^1 \times S^2$, by Property R ([3]), slightly refines Proposition 4 of [4] for what concerns genus 1 knots, as it rules they should have $r(N) = 1$. 2) According to [17], page 141, any Laurent polynomial $\Delta(t_1, t_2)$ which satisfies the relation $\Delta(t_1^{-1}, t_2^{-1}) = \Delta(t_1, t_2)$ (in $\mathbb{Z}[F]$) can be realized as Alexander polynomial of a closed three manifold N with $b_1(N) = 2$, $ord(N) = 1$. From the previous results we see that the choice, e.g., of $\Delta_N = k$ for $k \in \mathbb{Z}$, gives examples of three manifolds having just one basic classes, the trivial one. Whenever $k \neq \pm 1$, these manifolds allow to construct simple examples of nonsymplectic manifold with only one basic class. Many examples with only one basic class up to sign can be constructed in the same way.

We will use Proposition 5.4 to write the genus of the symplectic submanifold H_ω of equation 24, in conjunction with the adjunction inequalities for manifolds of type $S^1 \times N$ that are contained in [4]. These apply to irreducible manifolds N which have not a basis of $H_2(N, \mathbb{Z})$ composed of tori. Leaving aside the condition on the basis that, as we have seen, excludes case of completely degenerate Alexander and Thurston norms, we know that the irreducibility condition is supported by Theorem 5.1. We can state the following

Proposition 5.5. *Let N an irreducible three manifold of non completely degenerate Thurston norm, such that $S^1 \times N$ is symplectic: then, if $b_1(N) > 1$, for any φ contained in the cone $\mathcal{C}$, we have $\|\varphi\|_A = \|\varphi\|_T$; if $b_1(N) = 1$ then, for any $\varphi \in H^1(N, \mathbb{Z})$, we have $\|\varphi\|_A = \|\varphi\|_T + 2div\varphi$.*

Proof: we apply the adjunction formula for the symplectic submanifolds H_ω, which takes the form

$$\chi_-(H_\omega) = H_\omega \cdot H_\omega + K_M \cdot H_\omega = \begin{cases} 2\gamma_\omega \cdot \tau_\omega + \|\tau_\omega^{PD}\|_A & \text{if } b_1(N) > 1 \\ 2\gamma_\omega \cdot \tau_\omega + \|\tau_\omega^{PD}\|_A - 2div\tau_\omega^{PD} & \text{if } b_1(N) = 1 \end{cases} \tag{25}$$

The adjunction inequalities for embedded submanifolds of $S^1 \times N$ of [4] can be written in the form

$$\chi_-(H_\omega) \geq 2\gamma_\omega \cdot \tau_\omega + \|\tau_\omega^{PD}\|_T. \tag{26}$$

This inequality is compatible with the content of equation 25 if and only if all the elements of the symplectic lines $\mathcal{L}_\omega$ attain the maximal value allowed in Theorem 1.1, for any admissible ω. This proves the statement. □

In the case of $b_1(N) > 1$, we can somehow strengthen this result, but in order to do so, let us recall that the Thurston norm admits a unique continuous extension to $H^1(N, \mathbb{R})$. We can use as well the definition of equation 2 to extend the Alexander norm. With these definitions the unit ball B_T of the Thurston norm is a finite, convex, possibly non compact polyhedron ([16]). Theorem 1.1 guarantees that it is contained in the unit ball B_A of the Alexander norm, which is too, from its definition, a polyhedron with the same properties, dual (up to scale) to the Newton polyhedron of Δ_N, see [11]. It is clear that the statement of Proposition 5.5 continues to hold in a cone on $H^1(N, \mathbb{R})$ determined by $\mathcal{C}$, that we will denote in the same way.

We can now refine the result of Proposition 5.5 :

Proposition 5.6. *Under the same hypothesis of Proposition 5.5 and $b_1(N) > 1$, there exists a cone over a closed face of ∂B_T such that, for any φ in this cone, the Thurston and the Alexander norm coincide.*

Proof: the results of [1], in the form of Proposition 5.5, imply that the unit spheres ∂B_T, ∂B_A of the two norms coincide, at least, on a nonempty open set U, determined by the intersection of the cone $\mathcal{C}$ of Proposition 5.5 with the unit spheres (we assume that the Thurston norm is not completely degenerate). This set U must be contained in an open face of both ∂B_T and ∂B_A (the dual vertex K is the same for all points of U). We can apply now the convexity and inequality 1: ∂B_A can not disjoin from ∂B_T on the closed face of ∂B_T containing U; it follows that the closed face of ∂B_T is completely contained in the corresponding face of ∂B_A. Linearity on the rays completes the proof. □

Corollary 5.7. *If N as above has a basis for $H_2(N, \mathbb{Z})$ composed by all but one tori, the Alexander and the Thurston norm coincide.*

Corollary 5.8. *If there exist an element τ_ω^{PD} of vanishing Thurston norm, then the Thurston norm is completely degenerate (and $K_N = 0$).*

We resume our results, considering also the trivial case of completely degenerate Thurston norm, in the following form:

Theorem 5.9. *Let N be an prime three manifold such that $S^1 \times N$ admits a symplectic structure: if $b_1(N) > 1$ there exist a cone in $H^1(N, \mathbb{R})$ over a closed face of the Thurston unit sphere such that, for any φ in that cone, we have $\|\varphi\|_A = \|\varphi\|_T$; if $b_1(N) = 1$ and $r(N) > 0$ then, for any $\varphi \in H^1(N, \mathbb{R})$, $\|\varphi\|_A = \|\varphi\|_T + 2 div\varphi$ and the Alexander polynomial has leading coefficient ± 1; if $b_1(N) = 1$ and $r(N) = 0$ then $N = S^1 \times S^2$ and the norms vanish.*

It is interesting to note that, when N fibers over a circle, it is possible to prove by a direct topological argument that the Alexander and Thurston norm coincide on a cone over a closed face of the Thurston unit sphere determined by the fibration. The reason is the fact that the integral points in the cone over an open face all represent a fibration, or none does ([16]). In the first case, it is easy to verify that the Alexander and Thurston norm must coincide. With little more work it is possible to link directly this result to the presence of symplectic structures for $S^1 \times N$: each integral cohomology class corresponding to a fibration can be represented by a nonsingular closed 1-form ρ, harmonic w.r.t. some metric, and with the help of this we can construct a symplectic form $dt \wedge \rho + *\rho$ on $S^1 \times N$. The canonical class of this symplectic structure, restricted to a fiber, gives the canonical class of the fiber. Theorem 5.9, in the case of $b_1(N) > 1$, represents the analogue of the condition $degree = 2g$ of fibered knots: as all the other results we have discussed, it is in support of the hypothesis that all N must fiber over a circle, although it does not give way to control the nondegeneracy of the symplectic form outside the symplectic cone $\mathcal{C}$.

We finish this section with a remark concerning the constraints on the Alexander polynomial of N for $S^1 \times N$ to be symplectic. In [5] it is asked whether 0-surgery on knots which have monic Alexander polynomial with $degree = 2g$, and so satisfy the above constraints, have or not symplectic structure. We can give some answer to a related question, namely whether this holds or not for any three manifold. Considering for simplicity the case of $b_1(N) = 1$, we take the 0-surgery $N_K(0)$ of a fibered not $K \subset S^3$; $S^1 \times N_K(0)$ is symplectic. If we take the manifold $N = N_K(0)\#\Sigma$, for Σ any homology sphere excluded from Theorem 5.1 (e.g. the Poincaré sphere, or an hyperbolic homology sphere, will do the job), we see from gluing theory that N has the same SW invariants and Alexander polynomial as $N_K(0)$ (and $S^1 \times N$ has the same SW invariants of the symplectic $S^1 \times N_K(0)$), satisfying therefore the constraints imposed on Alexander polynomial. However, according to Theorem 5.1, $S^1 \times N$ can not admit a symplectic structure. Unfortunately this observation gives no criterion for manifolds which are 0-surgery of knots, as these are prime (as follows from the proof of Property R of [3]). Torus decomposition of three manifolds, in some cases, could shed some more light on this question.

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