SW THEORY IN DIMENSION 3 AND NORMS ON COHOMOLOGY

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Abstract. This is the text of a talk given at the *Séminaire Sud-Rhodanien de Géométrie* of Avignon, in March 26, 1999, on a SW proof of C.McMullen's inequalities between Alexander and Thurston norms in dimension 3 and some properties of these norm for those three manifolds whose product with S^1 admits a symplectic structure.

1. Introduction and statement of the results

Three dimensional SW theory appears as a natural dimensional reduction of the four dimensional version, and the need of its comprehension appears as a basic tool for solving many of the classical problems of four dimensional smooth topology, e.g. decomposition theorems and adjunction formulae. Apart from this auxiliary role, related to its appearance as theory of critical points of the Chern-Simons-Dirac functional, the theory has an interest on its own. One of the reasons of this interest is the relation it has with some "standard" theory in three dimensional topology, like Alexander invariants, theory of foliations, contact geometry.
My aim is to illustrate some examples of these facts and use them to give, via SW methods, a proof of a theorem, recently stated by McMullen and proven by him with purely classical methods (see [Mc]). I will then quote the results of some further investigations in this direction. The SW proof of the theorem, together with the results on the symplectic case, appear in [V2].

We start with a brief review of some classical results in three dimensional topology, in order to be able to define two norms on cohomology. First, we assume N to be a smooth, oriented, closed, connected 3-manifold. We denote by F the free abelian group $H_1(N, \mathbb{Z})/Tor$, whose rank is given of course by the first Betti number of N. Elements of F can be written, choosing an identification of F with $\mathbb{Z}^{b_1(N)}$, as t^I, where I is a multi-index of cardinality $b_1(N)$ and the t_i are generators of $\mathbb{Z}$. For reasons that will appear clearly in the sequel we concentrate to the $b_1(N) > 0$ case. Denote by $\tilde{N}$ the free abelian covering of N, corresponding to the morphism $\pi_1(N) \to F$. The deck transformations of $\tilde{N}$ induce a natural structure of $\mathbb{Z}[F]$-module on

$H_1(\tilde{N}, \mathbb{Z})$. By virtue of this we can present it as the cokernel of a (non unique) morphism A

$$\mathbb{Z}[F]^r \xrightarrow{A} \mathbb{Z}[F]^s \longrightarrow H_1(\tilde{N}, \mathbb{Z}) \longrightarrow 0. \tag{1}$$

From these data it is possible to define a topological invariant of N in the following way. Consider the ideal of $\mathbb{Z}[F]$ generated by the $(s \times s)$-minors of the matrix (with entries in $\mathbb{Z}[F]$) A; it can be proven that this ideal is independent of the given resolution of $H_1(\tilde{N}, \mathbb{Z})$. The greatest common divisor of the elements of the ideal, denoted by Δ_N, is called Alexander polynomial of N, and is well defined up to multiplication by the units of $\mathbb{Z}[F]$, i.e. terms of the form $\pm t^I$. There is a canonical way to get rid of the sign, so that Δ_N is a well defined element of $\mathbb{Z}[F]/F$. Moreover, under the canonical involution of F (which sends t^I to t^{-I}), that we denote with an overbar, we have $\bar{\Delta}_N = t^{2J}\Delta_N$. This implies that there is a preferred representative of Δ_N in $\mathbb{Z}[F]$, given by the symmetric representative. In the sequel we will always assume that is the representative we consider, and we will write it as $\Delta_N = \sum_I a_I t^I$.
We are in position to introduce the following

Definition 1.1. *Let N be as above; for any $\varphi \in H^1(N;\mathbb{Z})$ we define*

$$\|\varphi\|_A := max_{IJ}\varphi(t^{I-J}), \tag{2}$$

where the multi-indexes run over all I, J such that a_I, a_J are non zero. This norm is linear on rays and satisfies the triangular equality. Note that this norm can be degenerate.

This definition can be extended without any difficulty to manifolds with boundary a union of tori. When N is the complement of a knot K, the Alexander polynomial of N is the Alexander polynomial of K; the first homology group of N can be identified with $\mathbb{Z}$ so that Δ_N takes the familiar form $\Delta_N = \sum_k a_k t^k$ and is well defined up to the multiplication by $\pm t^m$. If we compute the Alexander norm of ϕ, the generator of $H^1(N;\mathbb{Z})$ such that $\phi(t) = 1$, we have $\|\phi\|_A = 2r(N)$ where $2r(N)$ is the difference between the highest and lowest power of t appearing in a given presentation of Δ_N. This is, as notation suggests, an even number by the symmetry properties of the Alexander polynomial.

For what concerns the Thurston norm, it is usually defined on the second homology group of N; we will define it on cohomology as follows: if σ is a class in $H_2(N;\mathbb{Z})$ we consider embedded, eventually disconnected, Riemann surfaces Σ that represent σ and we define their Euler characteristic as the sum of the Euler characteristic over all components of positive genus. Denoting $\chi_-(\Sigma) = -\chi(\Sigma)$, minus the Euler characteristic of Σ, we have the

Definition 1.2. *Let N be as above; for any $\phi \in H^1(N,\mathbb{Z})$ we define*

$$\|\phi\|_T = min\{\chi_-(\Sigma)|\Sigma \hookrightarrow N, [\Sigma]^{PD} = \phi\}, \tag{3}$$

where PD denotes Poincareé duality. Also this norm is linear on rays, satisfies triangular inequality and can be degenerate.

The theorem of McMullen has then the following form:

Theorem 1.3. *(McMullen) Let N be a compact, connected, oriented three manifold; then the Alexander and Thurston norm satisfy*

$$\|\cdot\|_A \leq \|\cdot\|_T + \begin{cases} 2div(\cdot) & \text{if } b_1(N) = 1 \\ 0 & \text{if } b_1(N) > 1 \end{cases} \tag{4}$$

This inequality, suitably adapted to the case where the three manifold is the complement of a knot K, reduces to the well known fact that the degree of the Alexander polynomial of the knot (i.e. the difference between highest and lowest power) is bounded by above by twice the genus of the knot, i.e. the lowest value of the genus for the Seifert surfaces for the knot.
We will use SW theory to prove Theorem 1.3. We will then address to the question of characterizing the Alexander and Thurston norm of a particular class of three manifolds, namely those whose product with the circle admits a symplectic structure. Manifolds that fiber on the circle are an example and it is conjectured that these are the only ones. We can prove the following results:

Theorem 1.4. *Let N be a three manifold such that $S^1 \times N$ admits a symplectic structure; then a) N must be prime (up to Thurston' geometrisation conjecture); b) if $b_1(N) > 1$ there exist a cone in $H^1(N,\mathbb{Z})$ such that, for any φ in that cone, we have $\|\varphi\|_A = \|\varphi\|_T$; c) if $b_1(N) = 1$ and $r(N) > 0$ then, for any $\varphi \in H^1(N,\mathbb{Z})$ then $\|\varphi\|_A = \|\varphi\|_T + 2div\varphi$ and the leading coefficient of Alexander polynomial is ± 1; d) if $b_1(N) = 1$ and $r(N) = 0$ then $N = S^1 \times S^2$ and the norms vanish.*

Some of these results have been proven, in the case where N is 0-surgery of a knot, by Kronheimer (see [K2]).

2. Averaged basic classes and monopole classes

The idea to prove Theorems 1.3 and 1.4 is to construct a sort of dictionary which translates the information contained in the Alexander polynomial and in the Thurston norm in terms of

SW theory. Essentially the relation between Alexander norm and SW theory will be deduced from Meng-Taubes proof of the equivalence of a SW invariant of a three manifold and refined Reidemeister-Franz torsion (see [MT]) which is on his own related, by a result of Turaev ([Tu]), to the Alexander polynomial of the manifold. The relation of Thurston norm and SW theory, instead, is the content of [KM].

We start with a brief review of SW theory in dimension three, in order to have the formulation which is the more suitable for our purposes. Let (N, g) be a smooth, closed, oriented, riemannian three dimensional manifold. Once N is endowed of a spin^c structure P_N, i.e. a $U(1)$ lifting of the $SO(3)$ frame bundle, we can consider the three dimensional SW equations

$$F_A = q(\psi) - i\eta, \quad \partial\!\!\!/_A \psi = 0, \tag{5}$$

where A is a connection on the determinant bundle of the spin^c structure, $q(\cdot)$ is an $\Omega^2(N; i\mathbb{R})$-valued bilinear on the sections of the spinor bundle associated to P_N, η is a perturbation term that lives in $\Omega^2(N;\mathbb{R}) \cap ker\, d$, $\partial\!\!\!/_A$ is the Dirac operator that acts on the the spinors. These equations are invariants under the gauge group of those automorphisms of P_N that act trivially on the frame bundle. This group acts freely away from reducible couples. Using the same ideas and techniques that have been exposed by O.Biquard in the four dimensional cases it is possible to get, choosing generic suitably small perturbations, a 0-dimensional smooth compact oriented moduli space associated to each spin^c structure. With this moduli space we can define, counting with sign, an invariant

$$SW_N : Spin^c(N) \longrightarrow \mathbb{Z}. \tag{6}$$

We want to point out that the choice of a small perturbation, in the case of $b_1(N) = 1$ (which will be quite peculiar in all what follows) is not the usual one to define the invariant: in fact in this case the invariant could change, if we pass to another class of perturbations ("wall crossing").

We construct now a map γ from $Spin^c(N)$ to F; for each spin^c structure take its first Chern class, which is an even element in $H^2(N,\mathbb{Z})$, take its Poincare dual and forget torsion, to get an even element of F; divide by 2 and call γ this composed map. It is easy to verify that the fibre of this map has $ord(N) = |TorH_1(N,Z)|$ elements. We group together the spin^c structures which have the same images under γ: roughly speaking these are structures that differ by torsion. We can define then a function

$$SW_N(t^I) = \sum_{\gamma^{-1}(t^I)} SW_N(P_N). \tag{7}$$

With this definition we have a well defined polynomial $SW_N = \sum_I SW_N(t^I)t^I \in \mathbb{Z}[F]$. By some standard property of SW invariants, is symmetric under the involution of F. In correspondence of nonzero coefficients of this multivariable polynomial we introduce the following

Definition 2.1. *Let $c \in H^2(N,\mathbb{Z})/Tor$ be a cohomology class that arises, up to torsion, as Chern class of a spin^c structure P_N such that for $t^I = \gamma(P_N)$ we have $SW_N(t^I) \neq 0$: then c is called a-basic class (where the "a" stands for averaged). The set of a-basic classes is denoted by $\mathcal{A}(N)$.*

The basic fact is that the polynomial SW_N is strictly related to the Alexander polynomial. Adapting the results of [MT], [Tu] we obtain the

Theorem 2.2. *Let N be a closed three manifold with $b_1(N) > 0$; then we have*

$$SW_N = \Delta_N \quad if \;\; b_1(N) > 1, \tag{8}$$
$$(1-t)^2 SW_N + ord(N)t = t\Delta_N \quad if \;\; b_1(N) = 1.$$

The curious form of the second part of the statement is related to the presence of wall crossing terms. This result is very useful, because it allows to relate a-basic classes with elements of F with nonzero coefficient in the Alexander polynomial (this is particularly transparent in the $b_1(N) > 1$ case). With this identification we can rewrite the Alexander norm in terms of the a-basic classes. Leaving aside, for sake of simplicity, the case with $b_1(N) = 1$ and $r(N) = 0$ we have

$$\|\varphi\|_A = \begin{cases} max_{\mathcal{A}(N)}(c\cdot\varphi) & \text{if } b_1(N) > 1 \\ max_{\mathcal{A}(N)}(c\cdot\varphi) + 2div\varphi & \text{if } b_1(N) = 1 \end{cases} \tag{9}$$

This is the first part of our dictionary which translates Alexander and Thurston norms in term of SW theory. To write the second part, we introduce a definition, which is related to the presence of solution to *unperturbed* SW equations:

Definition 2.3. *Let $c \in H^2(N;\mathbb{Z})/Tor$ a cohomology class that arises, up to torsion, as Chern class of a spin^c structure P_N such that the unperturbed SW equations have solutions for any metric g. Then c is called a monopole class. The set of monopole classes is denoted by $\mathcal{C}(N)$.*

With the definition we gave to SW invariants (i.e. using "small" perturbations) a-basic classes are a particular examples of monopole classes: changing metric the moduli spaces corresponding to a-basic classes remain nonempty, even with perturbation equal to zero. We warn that, in the case of $b_1(N) = 1$, this is not true with the usual definition of SW invariants. For details we refer to [V2]. It is not true, instead, that all monopole classes are a-basic (nor basic): solutions can cancel each other. The relevance of monopole classes is contained in the following

Proposition 2.4. *Let N be a manifold as above: then the Thurston norm of a class $\varphi \in H^1(N,\mathbb{Z})$ satisfies*

$$\|\varphi\|_T \geq max_{\mathcal{C}(N)}(c \cdot \varphi). \tag{10}$$

It is not too difficult to prove this inequality; D.Auroux has given an example of this type of calculation in the four-dimensional case. We think it is interesting to say that, in particular cases, a stronger result holds:

Theorem 2.5. *(Kronheimer-Mrowka) Let N be an irreducible manifold: then the Thurston norm of a class $\varphi \in H^1(N,\mathbb{Z})$ is given by*

$$\|\varphi\|_T = max_{\mathcal{C}(N)}(c \cdot \varphi). \tag{11}$$

In any case, Proposition 2.4, together with the fact that a-basic classes are monopole, is enough to conclude the proof of Theorem 1.3.

3. A particular case: symplectic $S^1 \times N$

In this section we will at least illustrate the ingredients used in the proof of Theorem 1.4, omitting the primeness, which is quite independent from the rest of the topics and follows by vanishing theorem for generalized connected sum along $S^1 \times S^2$ (see [V1] and [V2] for details). Assume therefore that $S^1 \times N$ has a symplectic form ω.

We recall some general results which apply for the proof of Theorem 1.4. The first is the Donaldson theorem on the existence of symplectic submanifolds ([D]); we can assume, eventually passing to multiples, that there exist a connected symplectic submanifold $H \subset S^1 \times N$ such that

$$[H] = [\omega]^{PD} = [S^1] \times \gamma + \tau \in H_2(S^1 \times N, \mathbb{Z}) \tag{12}$$

where $\gamma \in H_1(N,\mathbb{Z})$, $\tau \in H_2(N,\mathbb{Z})$ and $\gamma \cdot \tau > 0$ (for sake of notation we will denote all products, both on N and on M, with a dot, the distinction being clear from the context).

The second result is that it is possible relate the, by pull back, SW theories on $S^1 \times N$ and on N (roughly speaking, they "coincide" up to sign).

The third point, discussed in the talks of D.Auroux, concerns spin^c structures on a symplectic four manifold (M,ω) with canonical bundle K_M. There exist, in that case, a canonical spin^c structure that decomposes as $\mathbb{C} \oplus K_M^{-1}$ (and has first Chern class equal to $-K_M$). Any other spin^c structure can be written as $E \oplus (K_M^{-1} \otimes E)$ for an $E \in H^2(M, \mathbb{Z})$. There are some constraints on spin^c structures with nonvanishing invariants that arises from Taubes' work (see [Ta1],[Ta2]). In the case of $b_+(M) > 1$ the canonical spin^c structure has SW invariant ± 1 and for any other structure $E \oplus (K_M^{-1} \otimes E)$ with nonzero invariants we have $0 \leq E \cdot \omega \leq K_M \cdot \omega$. Equality implies, respectively, $E = \mathbb{C}$ or $E = K_M$. The case of $b_+(M) = 1$, which corresponds to having $b_1(N) = 1$, is more subtle, because of the dependence on the chambers for the SW invariant, and for sake of simplicity we will not deal with it here, limiting ourselves to give the results.

We want apply these results to $S^1 \times N$. We proceed with the easier case, namely $b_1(N) > 1$; in this case if, for some spin^c structure $P_{S^1 \times N}$ on $S^1 \times N$, we have $SW(P_{S^1 \times N}) \neq 0$ then $P_{S^1 \times N}$ is a pull back structure. It follows that the canonical class and all the other basic classes are pull back: there exists a preferred line bundle $K_N \in H^2(N, \mathbb{Z})$ and a preferred spin^c structure on N of the form $\mathbb{C} \oplus K_N^{-1}$ with SW invariant ± 1 such that any other spin^c structure on N appears as $E \oplus (K_N^{-1} \otimes E)$ for $E \in H^2(N, \mathbb{Z})$ and if it has nonzero invariant must satisfy $0 \leq E \cdot \tau^{PD} \leq K_N \cdot \tau^{PD}$, the equalities implying respectively $E = \mathbb{C}$ or $E = K_N$. We will call the line $\mathcal{L} = \{\lambda \tau^{PD} \cap H^1(N, \mathbb{Z}), \lambda \in \mathbb{Q}\}$ a *symplectic line.*

The case of $b_1(N) = 1$ is not immediate but, leaving aside the case of $S^1 \times S^2$, it is possible to prove again that the three dimensional spin^c structures $E \oplus (K_N^{-1} \otimes E)$ with nontrivial SW_N invariants must satisfy again $0 \leq E \cdot \tau^{PD} \leq K_N \cdot \tau^{PD}$, the equalities implying respectively $E = \mathbb{C}$ or $E = K_N$. The values of SW_N invariants on the extremal classes correspond to the basic classes $\pm K_N$ and, in absolute value, are equal to 1.

This result reproduces exactly the same features of the $b_1 > 1$ case. Using this it is straightforward to prove the following

Proposition 3.1. *Let N be irreducible and take a $\varphi \in H^1(N, \mathbb{Z})$ belonging to a symplectic line $\mathcal{L}$; if $b_1(N) > 1$ then $\|\varphi\|_A = |K_N \cdot \varphi|$; if $b_1(N) = 1$ then $\|\varphi\|_A = |K_N \cdot \varphi| + 2div\varphi$, $r(N) > 0$ and the Alexander polynomial of N has leading term equal to ± 1.*

We recall that a prime three manifold is either irreducible or $S^1 \times S^2$. This Proposition tells us that, for elements of the symplectic line, the maximum of equation 9 is attained in correspondence of the basic classes $\pm K_N$, which are also a-basic (fact not obvious a priori). We will use now Proposition 3.1 to write the genus of the symplectic submanifold H of equation 12:

according to the adjunction formula for symplectic curves we have

$$\chi_-(H) = H \cdot H + K_M \cdot H = \begin{cases} 2\gamma \cdot \tau + \|\tau^{PD}\|_A & \text{if } b_1(N) > 1 \\ 2\gamma \cdot \tau + \|\tau^{PD}\|_A - 2div\tau^{PD} & \text{if } b_1(N) = 1 \end{cases} \tag{13}$$

To obtain the promised equality of norms on the symplectic line we would like to apply the adjunction inequalities for manifolds of the type $S^1 \times N$ that are contained in [K1], slightly stronger than the ones discussed by D.Auroux. These can be written in the form

$$\chi_-(H) \geq 2\gamma \cdot \tau + \|\tau^{PD}\|_T. \tag{14}$$

This inequality is compatible with the content of equation 13 only if all the elements of the symplectic line $\mathcal{L}$ attain the maximal value allowed in Theorem 1.3. The results of [D] guarantee in fact, if $b_1(N) > 1$, a result slightly stronger than this, as there is a neighborhood of the symplectic form ω (whose size in fact is not reasonably determinable) for which we can repeat the argument, i.e. we have a symplectic cone in $H^1(N, \mathbb{Z})$ for which the result holds.

References

[D] **S.Donaldson**, *Symplectic Submanifolds and Almost-Complex Geometry*, J.Differ.Geom. 44, 666-705 (1996).

[K1] **P.Kronheimer**, *Minimal Genus in $S^1 \times M^3$*, Invent.Math. 135, 45-61 (1999).

[K2] **P.Kronheimer**, *Embedded Surfaces and Gauge Theory in Three and Four Dimensions*, to appear in Surv.Differ.Geom.

[KM] **P.Kronheimer, T.Mrowka**, *Scalar Curvature and Thurston Norm*, Math.Res.Lett. 4 931-937 (1997).

[Mc] **C.McMullen**, *The Alexander Polynomial of a Three Manifold and the Thurston Norm on Cohomology*, preprint (1998).

[MT] **G.Meng, C.H.Taubes**, *SW = Milnor Torsion*, Math.Res.Lett. 3, 661-674 (1996).

[Ta1] **C.Taubes**, *The Seiberg-Witten Invariants and Symplectic Forms*, Math.Res.Lett. 1, 809-822 (1994).

[Ta2] **C.Taubes**, *More Constraints on Symplectic Forms from Seiberg-Witten Invariants*, Math.Res.Lett. 2, 9-13 (1995).

[Tu] **V.Turaev**, *Reidemeister Torsion in Knot Theory*, Russian Math.Surveys 41, 119-182 (1986).

[V1] **S.Vidussi**, *Seiberg-Witten Theory for 4-manifolds Decomposed Along 3-manifolds of Positive Scalar Curvature*, preprint (1999).

[V2] **S.Vidussi**, *The Alexander Norm is Smaller than the Thurston Norm: a Seiberg-Witten Proof*, preprint (1999).

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