GENERALIZED THOM CONJECTURE (AFTER OZSVÁTH AND SZABÓ)

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1. Introduction

The aim of this seminar is to discuss the proof of the generalized Thom conjecture, recently solved by Ozsváth and Szabó, building also on previous work by [MOY] and [FS]. I will take the opportunity also of making some remarks on the state of the art in Seiberg-Witten theory for what concerns several technical results which sometimes are given for granted and whose proof in fact is often available only in Yang-Mills form (if any).

For all this seminar I will suppose that M is a smooth, compact, connected, closed, oriented riemannian four manifold having the number of positive eigenvalues of the intersection form on the two cycles, $b_2^+(M)$, strictly greater than 1. Most of the result I will discuss remain in fact true also under the assumption that $b_2^+(M) = 1$ but, as usual in this case, that discussion would require some digressions on the wall crossing formulae for the invariants, and as these topics are not related with the very nature of the proof of the conjecture, I refer to the original paper for these details (although this excludes from my discussion, clearly, the case of $\mathbf{P}^2$, which is the original ambient of the conjecture).

The original conjecture, attributed to Thom, states that a (smooth) embedded algebraic curve which represents the class dH (where H is the hyperplane class) in the projective plane is genus minimizing among the embedded Riemann surfaces Σ which represent the same cohomology class; according to the adjunction formula for algebraic curves in an algebraic surface (or, more generally, symplectic curves in a four dimensional symplectic manifold), the following inequality should hold true:

$$K_M \cdot \Sigma + \chi(\Sigma) + \Sigma \cdot \Sigma \leq 0, \tag{1}$$

where K_M denotes the canonical class of the (almost) complex structure (in the case of $\mathbf{P}^2$, $K_{\mathbf{P}^2} = -3H$); we note that in the previous formula only the middle term on the l.h.s. depends

on Σ, while the other two are defined by the cohomology class. Note that, by fiat, there can be no classes, in $\mathbf{P}^2$, of negative selfintersection. The conjecture is completely classical and some classical bounds (quite far from being sharp) had been found with classical methods. With the arrival of gauge theory in four dimensional geometry, it was Donaldson that first suggested an application of his polynomial invariants to this problem, ameliorating the known bounds, but the case of $\mathbf{P}^2$ remained largely unsolved.
In the form above, the conjecture has been solved in the positive by [KM] and [MST], as first application of Seiberg-Witten theory.
It is easy to extend the conjecture to more general algebraic surfaces or symplectic four manifolds guessing the minimality of the genus for their holomorphic (symplectic) curves. Within the framework of Donaldson theory, Kronheimer and Mrowka solved the conjecture for several classes of algebraic surfaces, using their gauge theory for embedded surfaces, with the assumption that the curves have nonnegative selfintersection. But, again, it was Seiberg-Witten theory which provided the groundbreaking result, giving the instruments for solving the conjecture in the positive for all symplectic four manifolds, with the only restriction of considering classes of nonnegative selfintersection. The first complete proof is contained in [MST], but it worths to give a look to [K] for a proof of astonishing brevity and elegance.

As I will describe, the hypothesis of nonnegative self intersection can now be removed, so we can state the theorem in full generality:

Theorem 1.1. *(Ozsváth-Szabó) An embedded symplectic surface in a symplectic four manifold minimizes the genus among embedded representatives of its homology class.*

The proof of [OS] is based on the results of [MOY] concerning solutions of three dimensional Seiberg-Witten equations on Seifert fibered manifolds and an idea, similar to that contained in [FS], for comparing SW invariants of different spin^c structures. In the latter paper, in fact, most of the results concerning SW theory for embedded surfaces I will discuss are obtained for the case of embedded (or even immersed) spheres, and I will take the time to illustrate some of the results as a warm up for the higher genus case.

I finish this section pointing out that this is not the end of the story. In fact, the problem of determining condition of minimal genus representative for an homology class (which, for example, does not admit symplectic representatives, eventually in a non symplectic manifold), is a much general one. And there are cases where obvious guesses are not the correct ones: for example there exist almost complex surfaces in which pseudo holomorphic curves are not genus minimizing. Although the SW setting seems particularly suited for the symplectic case, it can provide answers also in the non symplectic world, and even when the theory is kind of trivial: to have a taste of this results, we quote (see [K]) the bounds on the genus of embedded surfaces in four manifolds of the shape $S^1 \times N$ where N is an irreducible three manifold admitting a taut foliation. These bounds are sharp whenever $b_1(N) = 1$.

2. SW Theory and Gluing Along Reducibles

I will now briefly recall the definition of Seiberg-Witten invariants, with the aim first of all of fixing the notation and moreover because we will need, in the case of manifolds having $H^1(M, \mathbb{Z}) \neq 0$, a slightly broader and unusual definition. Then I will pass to an analysis of some gluing tools which are necessary for the proof of the Theorem 1.1. On the manifold M, equipped with a spin^c structure $\tilde{Q}_M$, (which is by definition a $U(1)$ lifting of the frame bundle Q_M), we can define a system of equations, called SW, as follows. Consider the determinant line bundle $\mathcal{L}_M$ and the positive $U(2)$ spinor bundle $S^+(\tilde{Q}_M)$ associated to the spin^c structure: the SW equations are then the couple

$$F_A^+ = q(\psi) + \eta, \quad \partial\!\!\!/_A \psi = 0, \tag{2}$$

where $A \in \mathcal{A}(\mathcal{L}_M)$, F_A^+ is the selfdual part of the curvature, i.e. a purely imaginary selfdual two form, $\psi \in \Gamma(S^+(\tilde{Q}_M))$ a section of the bundle of spinors, $q(\psi)$ a bilinear in the argument with values in selfdual purely imaginary two forms, $\eta \in \Omega^2_+(M, i\mathbb{R})$ a perturbation term that is used to obtain genericity conditions and $\partial\!\!\!/_A$ the (positive) Dirac operator defined by the Levi-Civita connection on M and the $U(1)$ connection A. These equations are invariant under the action of the group $\mathcal{G}(\tilde{Q}_M)$ of those vertical automorphisms of $\tilde{Q}_M$ which project to the identity as automorphism of the frame bundle Q_M; the moduli space of solutions $\mathcal{M}(\tilde{Q}_M)$ has many interesting properties: first, it is non empty (for some spin^c structure) in several cases of interest (e.g. algebraic surfaces), oriented, compact, generically (in η) smooth away from reducible solutions $(A, 0)$ which can be removed on their own by a generic perturbation if $b_2^+(M)$ is positive. Moreover it has finite dimension, and Atiyah-Singer theorem provides the dimension formula

$$d(\tilde{P}_M) = \frac{1}{4}(c_1^2(\mathcal{L}_M) - 2\chi(M) - 3\sigma(M)). \tag{3}$$

Now we need some information on the topology of the space where this moduli space lives, i.e. the (irreducible) gauge orbit space. We have a weak homotopy equivalence

$$\bar{\mathcal{B}}(\tilde{Q}_M) = K(H^1(M,\mathbb{Z}),1) \times K(\mathbb{Z},2); \tag{4}$$

if we set $n = rkH^1(M,\mathbb{Z})$ and we let $e_1,...,e_n$ be the generators of $H^1(\bar{\mathcal{B}}(\tilde{Q}_M),\mathbb{Z})$ and μ the generator of $H^2(\bar{\mathcal{B}}(\tilde{Q}_M),\mathbb{Z})$ the cohomology ring of $\bar{\mathcal{B}}(\tilde{Q}_M)$ is given by

$$H^*(\bar{\mathcal{B}}(\tilde{Q}_M),\mathbb{Z}) = \mathbb{Z}(e_1,...,e_n) \otimes \mathbb{Z}[\mu]. \tag{5}$$

These cohomology generators have a geometrical meaning, which naturally is very similar to the one that analogous cohomology generators have in the Yang-Mills case. In fact these can be realized as images, under the slant product on the generators of the homology classes of M, of the first Chern class of the universal line bundle $\mathcal{E}$ over $M \times \bar{\mathcal{B}}(\tilde{Q}_M)$; this is just the abelian case of Donaldson μ map, defined as

$$\begin{gathered}\mu_{c_1(\mathcal{E})} : H_i(M,\mathbb{Z}) \longrightarrow H^{2-i}(\bar{\mathcal{B}}(\tilde{Q}_M),\mathbb{Z}) \\ \mu(\gamma) := \mu_{c_1(\mathcal{E})}(\gamma) = c_1(\mathcal{E})/\gamma.\end{gathered} \tag{6}$$

We will denote now on by $\mathcal{H}(M)$ the graded $\mathbb{Z}_2$-commutative algebra generated by the free parts of $H_0(M,\mathbb{Z})$ and $H_1(M,\mathbb{Z})$, with product denoted by juxtaposition; with this definition we can extend the μ map on $\mathcal{H}(M)$ as the cup product of the μ maps over the simple elements in $\mathcal{H}(M)$. In detail, we have

$$\begin{gathered}\mu : \mathcal{H}(M) \longrightarrow H^*(\bar{\mathcal{B}}(\tilde{Q}_M),\mathbb{Z}) \\ \beta_1...\beta_r\gamma_1...\gamma_s \to \mu(\beta_1) \cup ... \cup \mu(\beta_r) \cup \mu(\gamma_1) \cup ... \cup \mu(\gamma_s),\end{gathered} \tag{7}$$

where $\beta_i \in H_0(M,\mathbb{Z})$ and $\gamma_i \in H_1(M,\mathbb{Z})$. It is natural although unconventional set the degree of an element of $H_0(M,\mathbb{Z})$ to be equal to 2 and the degree of an element of $H_1(M,\mathbb{Z})$ to be equal to 1 (it is the degree of their image under the μ-map). The element of eq. 7 has degree $2r+s$, and defines a cohomology class of the same degree. Restricting the cohomology classes of the orbit space to $\mathcal{M}(\tilde{Q}_M)$, we can define the Seiberg-Witten polynomial as

$$\begin{gathered}SW : Spin^c(M,\mathbb{Z}) \times \mathcal{H}(M) \longrightarrow \mathbb{Z} \\ SW(\tilde{Q}_M;h) :=< \mu(h),[\mathcal{M}(\tilde{Q}_M)] >\end{gathered} \tag{8}$$

with the obvious convention of vanishing whenever $deg(h) \neq d(\tilde{Q}_M)$.

For sake of comprehension, let's specify how the "standard" SW invariants appear. If we fix a point p in the manifold M we can consider the subgroup $\mathcal{G}^o(\tilde{Q}_M) \subset \mathcal{G}(\tilde{Q}_M)$ of automorphisms based at that point; the quotient of the solution spaces by this based gauge group, called the based moduli space, has a natural $U(1)$ fibration

$$\mathcal{M}^o(\tilde{Q}_M) \stackrel{U(1)}{\longrightarrow} \mathcal{M}(\tilde{Q}_M). \tag{9}$$

Then, the line bundle E associated to this fibration is the restriction to $\mathcal{M}(\tilde{Q}_M)$ of the universal line bundle $\mathcal{E}$; if we denote by μ the Chern class of the fibration, then it coincides with $\mu([p])$. The more usual SW invariants appear then as

$$SW(\cdot) = SW(\cdot;[p]^r) : Spin^c(M) \to \mathbb{Z}, \tag{10}$$

and we have

$$\begin{aligned} &SW(\tilde{Q}_M) = 0 \quad if \quad d(\tilde{Q}_M) = 1 \quad mod \ \ 2, \\ &SW(\tilde{Q}_M) = SW(\tilde{Q}_M;[p]^{\frac{d}{2}}) =< \mu^{\frac{d}{2}}, [\mathcal{M}_{\tilde{Q}_M}] > \quad if \quad d(\tilde{Q}_M) = 0 \quad mod \ \ 2. \end{aligned} \tag{11}$$

When there is not explicit mentioning of the element of $\mathcal{H}(M)$ in the argument, by SW we will always refer to the invariants of eq. 11.

These invariants, that we have defined with the aid of a perturbation η and the choice of a metric on M, necessary in order to introduce the equations, are in fact smooth invariants, i.e. depend only on the smooth structure of M, as long as $b_2^+(M) > 1$: it is easy verifying that moduli spaces defined for generic choices of perturbation and metric are in fact compactly cobordant, so the definition of SW is unaffected by these choices. The determinant bundles of the spin^c structures for which SW does not vanishes are called *basic classes*. A folklore conjecture ("simple type conjecture") suggests that basic classes always correspond to zero-dimensional moduli spaces. In particular, all SW invariants should be of the standard type. A fundamental result of [T] states that, for a symplectic manifold, the canonical and the anticanonical bundle are basic classes, and that these manifolds are simple type. In fact, it is generally true that conjugate spin^c structures have the same (up to a sign) SW invariant.

It is clearly interesting to know the behavior of SW solutions and invariants under the usual transformations on a manifold, i.e. blow ups, connected sum, various surgeries, etc. In order to get this information, i.e. to understand the *gluing* and *ungluing* of moduli spaces under decomposition of M along a three manifold N, it proves to be useful to know a kind of relative SW theory, for manifolds with boundary, which provides a fair comprehension of the problem. This part of the theory, at least in the cases of interest for our discussion, is quite well established, as well as the ungluing procedure. The gluing part, instead, as not yet received, in the SW setup, a detailed treatment, and in different aspects, it is based on results which are parlayed from

Yang-Mills theory or, in the worst case, are a kind of "common believe".
In discussing this point I will keep an eye to the specific problem we are addressing to and therefore I shall limit myself to a specific situation. Therefore, I will assume that M is decomposed along a three manifold N in two factors $M = M^+ \cup_N M^-$ and I will study spin^c structures on M, say $\tilde{Q}_M$, such that the three dimensional SW equations induced by restriction on N (that I will describe in detail later) have just reducible solutions which form a nondegenerate moduli space. During the discussion I will add some extra hypothesis which will be the suitable ones for us. The first step to do is to understand the link between the moduli space for $\tilde{Q}_M$ and the finite energy moduli spaces that are constructed on the cylindrical end manifolds $\hat{M}^{\pm} := M^{\pm} \cup_N N \times [0, \infty)$, which arise naturally taking the geometric limit of M with a metric, on a tubular neighborhood $(-\epsilon, \epsilon) \times N$, of length $t \mapsto \infty$. There are some results, concerning these finite energy moduli spaces, that are somehow the building blocks of the gluing construction.
Let $\hat{X}$ a cylindrical end manifold, with a spin^c structure $\tilde{Q}_{\hat{X}}$, $det\tilde{Q}_{\hat{X}} =: \mathcal{L}_{\hat{X}}$, on it; as well known, once the connection is put in temporal gauge, the SW equations on the cylinder assume the form

$$\left\{ \begin{array}{c} \frac{\partial}{\partial t}\psi(t) = \partial\!\!\!/_{A(t)}\psi(t), \\ \\ \frac{\partial}{\partial t}A(t) = *(q(\psi(t)) - F_{A(t)}). \end{array} \right. \tag{12}$$

These equations are gradient flow equations for the Chern-Simons-Dirac functional

$$C(A, \psi) = \frac{1}{2}\int_N (F_A + F_{A_0}) \wedge (A - A_0) + \int_N < \psi, \partial\!\!\!/_A \psi >, \tag{13}$$

where A_0 is a fixed reference connection (we use now paths of connections and spinors on $\mathcal{L}_N$ and $S(\tilde{Q}_N)$, induced on N by the projection map of the cylinder). We call *finite energy* the couples which have finite variations of CSD on the cylindrical end and, among these, we define the (irreducible) finite energy moduli space

$$\mathcal{M}(\tilde{Q}_{\hat{X}}) := \{(A, \psi) \in \mathcal{A}(\mathcal{L}_{\hat{X}}) \times \Gamma(S^+(\tilde{Q}_{\hat{X}})) | F_A^+ = q(\psi), \partial\!\!\!/_A \psi = 0\} / \mathcal{G}(\tilde{Q}_{\hat{X}}). \tag{14}$$

The following results hold true:

- Up to gauge transformation, any finite energy solution decays, with exponential rate, to a static solution on the cylinder, which represents a solution of the three dimensional SW equations

$$F_A = q(\psi), \quad \partial\!\!\!/_A \psi = 0. \tag{15}$$

This space is the critical set of the CSD functional $C(A(t), \psi(t))$. According to our assumptions, the space of such solutions is identified, as smooth manifold, to the space of flat connections on the line bundle $\mathcal{L}_N$ (which is therefore torsion), a torus $\chi(N) := T^{b_1}(N)$. The condition of nondegeneracy has the following meaning: the three dimensional Kuranishi model of a neighborhood of a solution of eq. 15 is given by

(16) $$H^1(N, i\mathbb{R}) \times ker\not{\partial}_A \longrightarrow H^2(N, i\mathbb{R}) \times coker\not{\partial}_A;$$

nondegeneracy means that the tangent space of the space of solutions is isomorphic to the Zariski tangent space, i.e. $\forall A \in \chi(N)$ we have $ker\not{\partial}_A = 0$. This definition is equivalent to saying that the critical set of the CSD functional is (Bott) nondegenerate. This manifold has an obstruction, at each point, which is identified with $H^2(N, i\mathbb{R})$. This means that the critical set, although being a smooth manifold, is not cut out transversely by the SW equations.
We want to stress a point concerning this kind of solution spaces: as an obvious dimension counting shows, if N is not a rational homology sphere, this space of solutions is not a generic one, and it is quite clear how to make it generic (in this case empty) by adding a cokernel perturbation to the eq. 15. But there are two unrelated reasons why, in some situations, we have not interest in doing this: the first one can be that we simply cannot perturb the three dimensional equations without loosing the link with their four dimensional counterpart (and this happens whenever $H^2(M, \mathbb{Q}) \to H^2(N, \mathbb{Q})$ is the zero map, and this will be our case); the second is that perturbing the equation why might eventually lose any information on the space of solutions: it becomes generic but we don't know what it is.
We have a limit map (and its equivariant version for spaces based at infinity)

(17) $$\partial_\infty : \mathcal{M}(\tilde{Q}_{\hat{X}}) \longrightarrow \hat{\chi}(N),$$

which sends the $\mathcal{G}(\tilde{Q}_{\hat{X}})$-gauge equivalence class of finite energy solution $(A(t), \psi(t))$ on the end to the $\mathcal{G}(\tilde{Q}_{\hat{X}})$-gauge equivalence class of its limit flat connection A_0 on N. The space $\hat{\chi}(N)$ is, by definition, the space of equivalence classes of flat connections on N under the action of those gauge transformations of $\mathcal{G}(\tilde{Q}_N)$ which extend to $\tilde{Q}_{\hat{X}}$. Concerning this space, the following holds: $\hat{\chi}(N)$ is a covering of $\chi(N)$ with fiber $H^1(N, \mathbb{Z})/H^1(X, \mathbb{Z})$. We keep denoting by ∂_∞ the projection of the limit map to $\chi(N)$.
- The moduli space $\mathcal{M}(\tilde{Q}_{\hat{X}})$ is a compact, smooth (after generic compactly supported perturbation), oriented manifold of dimension

(18)
$$\hat{d} = dim\mathcal{M}(\tilde{Q}_{\hat{X}}) = 2ind_{\mathbb{C}}\not{\partial}_A - 1 + b^1(X) - b^2_+(X) = \frac{1}{4}c_X - \frac{1}{4}\sigma(X) - \eta_{A_0}(0) - 1 + b^1(X) - b^2_+(X)$$

where

$$c_X(A,\psi) := -\frac{1}{4\pi^2}\int_{\hat{X}} F_A \wedge F_A \tag{19}$$

is a well defined locally constant term, which depends only on the spin^c structure, and the Dirac η invariant is constant on the space of flat connections. Note that the compactness property of the moduli space is a consequence of the connectedness of the critical set of the CSD functional.

- The limit map ∂_∞ is smooth and transversal.

- If $SW(\tilde{Q}_M) \neq 0$, a solution on a spin^c structure $\tilde{Q}_M$ decomposes, on the geometric limit of M, in a couple of solutions on $\hat{M}^\pm$, in such a way that on one of the sides, which must have negative definite intersection form, the solution is reducible; the map $H^2(M,\mathbb{Q}) \to H^2(N,\mathbb{Q})$ must be the zero map, and therefore there can be only one factor of negative definite intersection form, say M^-.

We will denote by $\chi(\hat{M}^-)$ the reducible solutions for the spin^c structure $\tilde{Q}_{\hat{M}^-}$; this is identified (as a set) with the Jacobian torus of $\hat{M}^-$, i.e. $T^{b_1(M^-)}$ (see e.g. [DK]) but it is generally not generic as (based) moduli space; a Kuranishi model describes in fact the based moduli space around a solution as zero set of a map that reads as

(20)
$$E_A := H^1(M^-,N,i\mathbb{R}) \times H^1(N,i\mathbb{R}) \times ker\not{\partial}_A \longrightarrow H^1(N,i\mathbb{R})/H^1(M^-,i\mathbb{R}) \times coker\not{\partial}_A =: F_A$$

and there is no reason why reducible solutions should have unobstructed Zariski tangent space (a crucial point in the proof of transversality, for the irreducible moduli space, is the nonvanishing of the spinor part). In the cases that will interest us, the previous picture will appear slightly simplified, in the sense that we will have only "Dirac" obstruction, i.e. $H^1(M^-,i\mathbb{R}) \to H^1(N,i\mathbb{R})$ is an isomorphism (keep in mind that this implies, by Maier-Vietoris, that $H_1(M^+,\mathbb{R}) = H_1(M,\mathbb{R})$), and moreover $ker\not{\partial}_A = 0$. Recalling the definitions of the three dimensional case, these conditions guarantee us that this space is nondegerate, but has an obstruction bundle with fiber F_A. I will continue the discussion keeping in mind these assumptions. We will denote now on $r = -ind_{\mathbb{C}}\not{\partial}_A = dim_{\mathbb{C}} coker\not{\partial}_A$. The obstructions fit together over $\chi(\hat{M}^-)$ to define the obstruction bundle F of complex rank r. This bundle is by construction acted by the $U(1)$ base point action, via the natural multiplication action on spinors.

Starting with these results, which can be given a detailed proof (an insomniac reader might want to consult my Ph.D. thesis, for a part of them), it is reasonable to conceive a gluing theory which parallels the one of Yang-Mills theory. We make the following

Claim 2.1. *Let $M = M^+ \cup_N M^-$ be a four manifold endowed with a spin^c structure $\tilde{Q}_M$ having $d(\tilde{Q}_M) \geq 0$ such that both the $\chi(N)$ and $\chi(\hat{M}^-)$ are nondegerate (with their respective definitions above); then*

$$SW(\tilde{Q}_M; h) =< \mu(h) \cup e(\mathcal{O}), [\mathcal{M}(\tilde{Q}_{\hat{M}^+})] >, \tag{21}$$

where all the homology class on the r.h.s. are interpreted in terms of the corresponding classes in $\mathcal{H}(\hat{M}^+)$, and $e(\mathcal{O})$ is the Euler class of the bundle

$$\mathcal{O} := \mathcal{M}^o(\tilde{Q}_{\hat{M}^+}) \times_{U(1)} \partial_\infty^* F. \tag{22}$$

"Proof": on $M = M^+ \cup_N M^-$ with spin^c structure $\tilde{Q}_M$, the dimension formula of eq. 18 provides us the relation between the dimension of the moduli space on M and on the two factors, which reads as

$$dim\mathcal{M}(\tilde{Q}_M) = dim\mathcal{M}(\tilde{Q}_{\hat{M}^+}) + 2ind_{\mathbb{C}}\not{\partial}_A \tag{23}$$

where A is a connection on the bundle $det\tilde{Q}_{\hat{M}^-}$. The gluing formula (but a more honest name should be "gluing principle") states then that the based moduli space $\mathcal{M}^o(\tilde{Q}_M)$ appears as a generic equivariant section of the obstruction bundle over the fibered product of $\mathcal{M}^o(\tilde{Q}_{\hat{M}^+})$ and $\chi(\hat{M}^-)$; let's explain what does this mean: the fibered product has to be interpreted in terms of the (based) limit map of eq. 17, for $\mathcal{M}^o(\tilde{Q}_{\hat{M}^+})$, and the natural map $\chi(\hat{M}^-) \to \chi(N)$. The $U(1)$ action is the base point action on $\mathcal{M}^o(\tilde{Q}_{\hat{M}^+})$ and the trivial one on $\chi(\hat{M}^-)$. The obstruction bundle over this fibered product is the fibered product of the obstruction bundles: the obstruction vanishes for $\mathcal{M}(\tilde{Q}_{\hat{M}^+})$, is purely DeRahm on $\chi(N)$ and purely Dirac on $\chi(\hat{M}^-)$, and these are unrelated; this tells that, over $\mathcal{M}^o(\tilde{Q}_{\hat{M}^+}) \times_{\chi(N)} \chi(\hat{M}^-) = \mathcal{M}^o(\tilde{Q}_{\hat{M}^+})$ it appears as pull-back w.r.t. ∂_∞ of the bundle F over $\chi(\hat{M}^-)$, acted by $U(1)$ with the obvious multiplication action. This bundle could be trivial, but not equivariantly; its quotient bundle, under the $U(1)$ action, appears as

$$\mathcal{O} = \mathcal{M}^o(\tilde{Q}_{\hat{M}^+}) \times_{U(1)} \partial_\infty^* F \tag{24}$$

and as such is naturally associated to the base point fibration. In particular whenever F is trivial we have $e(\mathcal{O}) = \mu^r$. The moduli space $\mathcal{M}(\tilde{Q}_M)$ finally appears as the zero set of a section $\Phi \in \Gamma(\mathcal{O})$ and a necessary condition for being nonempty is that

$$dim\mathcal{M}(\tilde{Q}_{\hat{M}^+}) - rk\mathcal{O} \geq 0, \tag{25}$$

in agreement with eq. 23. We can use now the definition of Seiberg-Witten invariants given in eq. 8 to compute the SW invariants: we have

$$SW(\tilde{Q}_M;h) =< \mu(h), [\mathcal{M}(\tilde{Q}_M)] >=< \mu(h), [\Phi^{-1}(0)] >=< \mu(h) \cup e(\mathcal{O}), [\mathcal{M}(\tilde{Q}_{\hat{M}^+})] >, \tag{26}$$

which is what we wanted to prove. □

3. Adjunction Inequalities: the Case $g = 0$

In this section I will describe the first adjunction inequality, concerning embedded spheres of negative selfintersection in manifolds of simple type, in particular symplectic ones. First, by sake of completeness, I quote the result which covers the case of spheres of nonnegative selfintersection:

Theorem 3.1. *Let M be a simple type manifold with nonvanishing SW invariants; then M does not contain embedded spheres of positive selfintersection and any sphere of zero selfintersection is rationally trivial.*

It is not difficult to obtain this theorem from the results on moduli spaces for cylindrical end manifolds quoted in the previous Section.
Now we pass to the case of negative selfintersection: we have the following

Theorem 3.2. *(Fintushel-Stern) Let M be a simple type manifold with nonvanishing SW invariants, and let S be a sphere of negative selfintersection $-n$; then for any basic class $\mathcal{L}_i$ we have*

$$| < c_1(\mathcal{L}_i), S > | + S \cdot S \leq 0. \tag{27}$$

Proof: the proof of this theorem is based on an equality between the SW invariants of two spin^c structures, established with the aid of the gluing relation we obtained in eq. 26. We start with the proof of the

Lemma 3.3. *For each spin^c structure $\tilde{P}_M$ having $d(\tilde{P}_M) \geq 0$ and $| < c_1(\mathcal{L}_M), S > | \geq n$ we have*

$$SW(\tilde{P}_M;h) = SW(\tilde{P}_M \otimes PD(S)^{\epsilon}; h[p]^m). \tag{28}$$

In the previous equations, $\epsilon = sgn(< c_1(\mathcal{L}_M), S >)$, $2m$ denotes the nonnegative (even) quantity $| < c_1(\mathcal{L}), S > | - n$, PD denotes the Poincaré dual of a two cycle and the tensor product on the spin^c structure describes the affine action of $H^2(M,\mathbb{Z})$ on $Spin^c(M)$; in particular, identifying line bundles with their Chern class, we have $det(\tilde{P}_M \otimes PD(S)^\epsilon) = c_1(\mathcal{L}_M)+2\epsilon PD(S)$ (I use a product notation on $Spin^c(M)$ but an additive one on line bundles: this is slightly schizophrenic but often more clear). Note in particular that eq. 28 implies $SW(\tilde{P}_M) = SW(\tilde{P}_M \otimes PD(S)^\epsilon)$.

Proof: the proof is based on the comparison of the moduli spaces corresponding to the two spin^c structures, in order to show that the invariant coincide. We analyse the $\epsilon = 1$ case, the other one being analogous. We start with the observation that M contains a tubular neighborhood of S which is a degree $-n$ disk bundle $M^- = \nu S$ over the sphere; its boundary is a lens space $L(1,-n)$, a rational homology sphere which admits a metric of positive scalar curvature. We get therefore a decomposition $M = M^+ \cup_N M^-$ where the first factor has $b_2^+(M^+) = b_2^+(M)$ and the second one has negative definite intersection form $(-n)$ (this is not unimodular, for $n \neq 1$, but the boundary ∂M^- is not an integral homology sphere). It is clear from the definition that the two structures of eq. 28 agree on $\hat{M}^+$, while on $\hat{M}^-$ they differ by the twist by $PD(S)$. We want to use Claim 2.1 to relate the SW invariants for these two spin^c structures. In order to do this we have to verify that the conditions of the Claim are satisfied: first, concerning the properties of $\chi(N)$, it is a straightforward consequence of the Weitzenböck formula that $ker\partial\!\!\!/_A$ vanishes for any solution of eq. 15, and this implies both the fact that the only solution on $\tilde{P}_N$ is the flat connection on $\mathcal{L}_N$ and that this solution is nondegenerate (it is in fact also generic, as the obstruction vanishes). for what concerns the nondegeneracy of $\chi(\hat{M}^-)$, which is given by a reducible connection $A \in \mathcal{A}(det\tilde{P}_{\hat{M}^-})$ or $A \in \mathcal{A}(det\tilde{P}_{\hat{M}^-} + 2PD(S))$ according to the structure we consider, the curvature conditions on M^- guarantee that the kernel of the Dirac operator has to vanish as well. The obstruction bundle is just a complex vector space. Its dimension is anyhow different for the two spin^c structures, as tells the formula of eq. 23: if we denote by $r = rk\mathcal{O}(\tilde{P}_M \otimes PD(S))$ we have $rk\mathcal{O}(\tilde{P}_M) = r + m$.
Let's compute now $SW(\tilde{P}_M, h)$; we have, by Claim 2.1,

$$SW(\tilde{P}_M; h) =< \mu(h) \cup e(\mathcal{O}(\tilde{P}_M)), [\mathcal{M}(\tilde{P}_{\hat{M}^+})] >=< \mu(h) \cup \mu^{r+m}, [\mathcal{M}(\tilde{P}_{\hat{M}^+})] > . \tag{29}$$

For what concerns $\tilde{P}_M \otimes PD(S)$, the application of Claim 2.1 requires just a check of the dimension: application of eq. 3 shows that we have

$$d(\tilde{P}_M \otimes PD(S)) = d(\tilde{P}_M)+ < c_1(\mathcal{L}), S > +S \cdot S = d(\tilde{P}_M) + 2m \tag{30}$$

and this is a nonnegative quantity according to our hypothesis. We can apply again Claim 2.1, therefore, and we get

$$SW(\tilde{P}_M \otimes PD(S); h[p]^m) =< \mu(h[p]^m) \cup e(\mathcal{O}(\tilde{P}_M \otimes PD(S))), [\mathcal{M}(\tilde{P}_{\hat{M}+})] >= \tag{31}$$
$$=< \mu(h) \cup \mu^m \cup e(\mathcal{O}(\tilde{P}_M \otimes PD(S))), [\mathcal{M}(\tilde{P}_{\hat{M}+})] >=< \mu(h) \cup \mu^{r+m}, [\mathcal{M}(\tilde{P}_{\hat{M}+})] >,$$

as both the spin^c structures agree on $\hat{M}^+$. This proves the Lemma. □

We are in position now to prove the Theorem 3.2: let $\mathcal{L}$ be a counterexample to the statement; there exist at least one spin^c structure $\tilde{P}_M$ which has determinant $\mathcal{L}$ and nonvanishing SW invariants; as such, it satisfies the hypothesis of the previous Lemma, and therefore $\tilde{P}_M \otimes PD(S)^\epsilon$ defines a basic class with associated moduli space of dimension

$$d(\tilde{P}_M \otimes PD(S)^\epsilon) = d(\tilde{P}_M) + \epsilon < c_1(\mathcal{L}), S > +S \cdot S > 0. \tag{32}$$

This violates the simple type assumption. With this observation we have completed the proof of Theorem 3.2. □

I finish this section discussing the blow up formula, that is interesting on its own and that will be used in the next section. In fact, from the gluing theory illustrated in the previous section, it is not difficult to obtain it, the only difference w.r.t. to the case discussed now consisting in the fact that we have no more to compare different spin^c structure on the same manifold but on two different manifolds M and $M\#\bar{\mathbf{P}}^2$. In fact, the basic tool for the proof remains in the content of Claim 2.1, which allow to relate the value of the SW invariants for a spin^c structure on these two manifold to the value of the μ map for the *positive* part $M^+ = M \setminus D^4$, which coincides in the two cases. Note, moreover, that in correspondence of the above splitting, which is over S^3, we can easily compute explicitly the rank of the obstruction bundle for a spin^c structure, as all the terms of the dimension formula (in particular the η invariant, which vanishes) can be expressed in term of the structure.

I omit therefore the details of the proof, which is anyhow sketched in [FS], and I limit myself to quote the result:

Proposition 3.4. *Let M be a four manifold endowed with a spin^c structure $\tilde{P}_M$ and denote by E the exceptional class of its blow up $M\#\bar{\mathbf{P}}^2$; there exist on $M\#\bar{\mathbf{P}}^2$ a canonical spin^c structure $\tilde{P}_{M\#\bar{\mathbf{P}}^2}$ having determinant equal to det$\tilde{P}_M - E$; then for any for any spin^c structure $\tilde{P}_{M\#\bar{\mathbf{P}}^2} \otimes E^k$ such that $d(\tilde{P}_{M\#\bar{\mathbf{P}}^2} \otimes E^k) \geq 0$ we have*

$$SW_{M\#\bar{\mathbf{P}}^2}(\tilde{P}_{M\#\bar{\mathbf{P}}^2} \otimes E^k; h) = SW_M(\tilde{P}_M, h[p]^r) \tag{33}$$

where $2r = d(\tilde{P}_M) - d(\tilde{P}_{M\#\bar{\mathbf{P}}^2} \otimes E^k) = k(k-1) \geq 0$.

4. Adjunction Inequalities: the Case $g > 0$

I address now to the problem of higher genus embedded surfaces, which constitutes the new part of the story. The idea of the proof, in principle, should by now be quite clear, i.e. to imitate the work of [FS] producing a gluing formula which provides the same line of results that eq. 28 gives. The problem is that in order to use a gluing formula we have to know that the hypothesis of Claim 2.1 are satisfied, and then eventually get information on the obstruction bundle. In the previous case all these information where provided, for any spin^c structure, by the curvature condition on $M^- = \nu S$ and on $N = \partial \nu S$. In the case of the tubular neighborhood of an embedded surface of higher genus, this point is far from being trivial. And, in fact, it requires a little detour in order to get the information we look for.

We start by a general consideration. As I discussed before, we can get a good deal of information by perturbing the SW equations; we are allowed to do this, at least when $b_2^+(M) > 1$, because we can easily produce compact oriented cobordisms between moduli spaces of solutions corresponding to different perturbations. But there is a fruitful idea of [MOY] which pushes further this, starting from the fact that it can be an advantage to work with a connection, on the spin^c structure, which is not induced from the Levi-Civita on the frame bundle, but from some more tamed one, which differs from it in a "controlled" way. In fact we are interested precisely in working with the Dirac operator defined by this connection.

Before doing this, in any case, we must show that the choice of working with a Dirac operator which is not induced by the Levi-Civita connection is allowed. Concerning this point, we can note that any Dirac operator $\partial\!\!\!/_A$, acting on positive spinors, can be written in term of the standard Dirac operator $\hat{\partial\!\!\!/}_A$ (we use now on the hat to denote any operator defined via Levi-Civita connection) as

$$\partial\!\!\!/_A \psi = \hat{\partial\!\!\!/}_A \psi - \alpha \cdot \psi \tag{34}$$

where α is a one form acting by Clifford multiplication on spinors, extracted from the difference of the chosen connection and the Levi Civita. Suppose now that instead of studying the ordinary SW equations we consider the modified equation

$$F_A^+ = q(\psi), \quad \hat{\partial\!\!\!/}_A \psi = \alpha \psi, \tag{35}$$

equivalent to the ones with standard Dirac equation for the new operator, defined with some connection on the frame bundle. We want to show that the SW invariants defined with the new SW equations are the same of the usual ones: the point, as usual, is to show compact cobordism of the moduli spaces connected by a path (obtained, say, using a path which goes from the vanishing one form to the one form α of eq. 35). The only points which needs a check is the fact that compactness is not lost and unique continuation theorems hold; concerning these points we have the

Proposition 4.1. *Let (A, ψ) be a solution of eq. 35; the spinor part ψ has a bound on the $\mathcal{C}^0$ norm and satisfies unique continuation theorem.*

Proof: for what concerns the bound, we show how to modify the standard Kronheimer-Mrowka proof: first we observe that for any solution of $\hat{\not{\partial}}_A\psi = \alpha\psi$ we have

$$(\hat{\not{\partial}}_A)^*\hat{\not{\partial}}_A\psi = (d\alpha + d^*\alpha)\psi + |\alpha|^2\psi - 2\hat{\nabla}_{A,\alpha}\psi = -2\hat{\nabla}_{A,\alpha}\psi + (0^{th} \ \ order \ \ terms \ \ in \ \ \psi). \tag{36}$$

We use now the usual approach of evaluation the laplacian at a point of maximum:

$$\begin{aligned} 0 \leq \tfrac{1}{2}d^*d|\psi|^2 =&< \hat{\nabla}^*_A\hat{\nabla}_A\psi, \psi > -|\hat{\nabla}_A\psi|^2 =< (\hat{\not{\partial}}_A)^*\hat{\not{\partial}}_A\psi - \tfrac{1}{2}F^+_A\psi - \tfrac{s}{4}\psi, \psi > -|\hat{\nabla}_A\psi|^2 \leq \\ &\leq -2 < \hat{\nabla}_{A,\alpha}\psi, \psi > -\tfrac{1}{4}|\psi|^4 + c|\psi|^2 - |\hat{\nabla}_A\psi|^2 \leq -\tfrac{1}{4}|\psi|^4 + c|\psi|^2 \end{aligned} \tag{37}$$

where we used the inequality

$$2| < \hat{\nabla}_{A,\alpha}\psi, \psi > | \leq |\hat{\nabla}_A\psi|^2 + |\alpha|^2|\psi|^2 \tag{38}$$

(c is a constant which varies from line to line). From this formula, the bound and the consequent compactness follow in the usual way (in [OS] the authors proceed using the Weitzenböck formula for the operator $\not{\partial}_A$, a bit more involved to derive; here we have followed an approach more similar to the one in [B]). For what concerns the unique continuation property, we proceed more or less in the same way: we have

$$\begin{aligned} |\hat{\nabla}^*_A\hat{\nabla}_A\psi|^2 &= |(\hat{\not{\partial}}_A)^*\hat{\not{\partial}}_A\psi - \tfrac{1}{2}F^+_A\psi - \tfrac{s}{4}\psi|^2 \leq \\ &\leq | - 2\hat{\nabla}_{A,\alpha}\psi - \tfrac{1}{2}F^+_A\psi + c\psi|^2 + c'|\psi|^2 \leq c(|\hat{\nabla}_A\psi|^2 + |\psi|^2) \end{aligned} \tag{39}$$

where we used again eq. 38 and the $\mathcal{C}^0$ bound previously obtained. The proposition now follows from Aronszajin theorem. □

We have not yet made, in fact, any explicit choice of a Dirac operator, but whenever we work with an almost complex manifold it is quite clear that we would like to work with Dolbeault operator, very much like in the Kähler or symplectic case. In fact in these latter cases the Dolbeault operator coincides with the Dirac, up to a factor, while in general this does not hold true: in reference [G] it is proven that, on an almost hermitian manifold with $(1,1)$ form ω the

Dirac operator $\hat{\not{\partial}}_A$ can be written as

$$\hat{\not{\partial}}_A = \sqrt{2}(\bar{\partial}_A + \bar{\partial}_A^*) + \frac{1}{4}\theta \tag{40}$$

where the Lee form θ can be defined from $d\omega = \theta \wedge \omega$. When we work on the cylinder $N \times \mathbb{R}$ with N a Seifert manifold (roughly speaking an S^1-bundle over a Riemann orbifold, in our case an honest surface) there are natural ways to define an almost hermitian structure: given a constant curvature riemannian structure g_Σ to the Riemann surface Σ, with volume form μ_Σ, we choose a constant curvature connection form η on the S^1-bundle, which will satisfy the equation

$$d\eta = -\frac{2\pi deg(N)}{Vol(\Sigma)}\mu_\Sigma =: 2\xi\mu_\Sigma; \tag{41}$$

with this choice made the (cylindrical) metric is given by $g_{N\times\mathbb{R}} = dt^2 + \eta^2 + g_\Sigma$ and the $(1,1)$ form on the cylinder appears as

$$\omega = dt \wedge \eta + \mu_\Sigma \tag{42}$$

and we can easily verify by direct calculation that

$$\theta = 2\xi dt. \tag{43}$$

So we proceed by endowing $\hat{M}^-$ of a metric which coincides, outside a compact subset, with the almost hermitian defined above; with these choice, on the cylinder, solving the modified Dirac equation

$$\hat{\not{\partial}}_A\psi = \frac{1}{2}\xi dt \cdot \psi \tag{44}$$

is the same as solving the equation

$$\sqrt{2}(\bar{\partial}_A + \bar{\partial}_A^*)\psi = 0. \tag{45}$$

We want to know, now, how this equation appears on the cylinder once the correspondence between spin^c structures on N and $N \times \mathbb{R}$ is made explicit: for this recall that the one form dt defines exactly the isomorphism between positive and negative spinors on the cylinder, and the eq. 44 is written, for a path of spinors in $\tilde{Q}_N$, as

$$dt \cdot (\frac{\partial}{\partial t}\psi - \hat{\not{\partial}}_A^3\psi) = dt \cdot \frac{1}{2}\xi\psi, \tag{46}$$

where $\hat{\not{\partial}}_A^3$ is the standard Levi-Civita Dirac operator in three dimensions. So, very much like in the first Section, the gradient flow equations which correspond to SW equations become, as the curvature part is unaffected by our changes,

$$\begin{cases} \frac{\partial}{\partial t}\psi = \hat{\partial\!\!\!/}^3_A\psi + \frac{1}{2}\xi\psi, \\ \frac{\partial}{\partial t}A = *(q(\psi) - F_A). \end{cases} \tag{47}$$

The good news, concerning this equation, is that we have a good deal of knowledge about the operator $\partial\!\!\!/^3_A = \hat{\partial\!\!\!/}^3_A + \frac{1}{2}\xi$, which has been extensively studied in reference [MOY]. In [MOY] the authors discuss how to get a complete knowledge of the space of solutions of three dimensional SW equations on a Seifert manifold by using a connection on the frame bundle which respects the fibered structure, being the sum of the trivial connection on the S^1 direction plus the pull-back of the Levi-Civita connection on Σ, i.e. $\nabla = d \oplus \nabla_\Sigma$. The Dirac operator associated to this connection is given exactly by $\hat{\partial\!\!\!/}^3_A + \frac{1}{2}\xi$. The use of this operator allows to relate the SW equations 15 to the vortex equations on Σ, and obtain from this knowledge of three dimensional solutions.

Now I come back to the whole M; we can endow M^- of a metric of the shape $g_{M^-} = g_\Sigma + f(t)^2\eta^2 + dt^2$ where t is the radial coordinate along the disk and $f(t)$ is a smooth function equal to t on the $\frac{1}{2}$-disk and equal to 1 outside the 1-disk; this metric is defined on all M^- and is cylindrical outside a "small" tubular neighborhood of Σ. We can choose an almost complex structure on M^- which is orthogonal w.r.t. the previous metric. The result of the previous discussion is the following

Proposition 4.2. *Let M be a four manifold which contains an embedded Riemann surface Σ of selfintersection $-n$; stretching the neck along $\partial\nu\Sigma$ in such a way to get $M_T = M^- \cup_N [-T,T] \times N \cup_N M^+$ and endowing it with a metric which coincides, on $M^- \cup_N N \times [-T,T]$, with the extension of the metric described above, the standard SW moduli spaces are compactly cobordant with the moduli space of solutions of the equations*

$$F_A^+ = q(\psi) + \eta, \quad \hat{\partial\!\!\!/}_A\psi = \alpha \cdot \psi, \tag{48}$$

where α is a (smoothly cut off) one form which vanishes over M^+, is equal to $\frac{1}{4}\theta$ on $M^- \cup_N N \times [-T,T]$ where θ is the Lee form for the almost hermitian structure and in particular coincides with $\frac{1}{2}\xi dt$ on the cylinder.

Maybe it worths point out that such a "rigid" choice of metric on M does not interfere with genericity results w.r.t. the metric, as the latters can be obtained just by perturbing the metric on a small ball, that we can safely choose within M^+.

According to our previous discussion, therefore, the analysis of the SW equations 48 on $\hat{M}^-$

passes through the study of the solutions of holomorphicity equations that are usual in presence of almost hermitean structure. Decomposing forms and spinors in their (p,q) components we have in particular, when $det\tilde{Q}_M = 2E - K_N$, $S^+(\tilde{Q}_M) = \Lambda^{0,0}(E) \oplus \Lambda^{0,2}(E)$, $S^-(\tilde{Q}_M) = \Lambda^{0,1}(E)$, and the equations become

$$\Lambda Tr F_A = \frac{i}{2}(|\alpha|^2 - |\beta|^2), \quad Tr F_A^{0,2} = \bar{\alpha} \otimes \beta, \qquad (49)$$
$$\bar{\partial}_A \alpha + \bar{\partial}_A^* \beta = 0.$$

On the cylinder the SW equations 48 assume the form

$$\begin{cases} \frac{\partial}{\partial t}\psi = \not{\partial}_A^3 \psi, \\ \frac{\partial}{\partial t} A = *(q(\psi) - F_A), \end{cases} \qquad (50)$$

as discussed before, and concerning the static solutions of these equations the result of [MOY] can be written in the form:

Theorem 4.3. *(Mrowka-Ozsváth-Yu) Let N be a degree $-n$ circle bundle over a Riemann surface of genus g and let $\tilde{Q}_{M^-}$ a spin^c structure such that $< c_1(\mathcal{L}_{M^-}), \Sigma >= k$; then the moduli space of solutions of the three dimensional SW equations for $\tilde{Q}_N$ does not contain irreducible solutions as long as k is not congruent mod $2n$ to an integer in*

$$[-n - 2g + 2, -n - 2] \cup [n + 2, n + 2g - 2]. \qquad (51)$$

The reducible part $\chi(N)$ is isomorphic to the jacobian torus $Jac(\Sigma)$ and is nondegenerate as long as k is not congruent mod $2n$ to n.

This theorem provides us, under the stated conditions, half of the hypothesis of Claim 2.1. Moreover whenever this critical set is composed of reducible and nondegenerate solutions we can assume that the solutions of the four dimensional equations converge exponentially to the critical set (the proof of this fact requires just mild modifications of the proof with standard Dirac operator). This allows to make integration by part as in the closed case and show, with the same procedure, that solutions of equations 49, α or β have to vanish. The equations become therefore (generalized) vortex equations for E or $E \otimes K_N^{-1}$.

Let me describe now what will be the strategy: we want to obtain a relation for SW invariant, in the spirit of formula 28, for any value of n, g, k, and we will do it by reducing ourselves, by blowing up, to a cases where the hypothesis of Theorem 4.3 hold true. In that case we can prove the following

Lemma 4.4. *Let M be a manifold containing a genus g Riemann surface of selfintersection $-n$ satisfying $n \geq 2g$ and let $\tilde{P}_M$ be a spin^c structure with $d(\tilde{P}_M) \geq 0$ such that $< c_1(\mathcal{L}_M), \Sigma >= -2g - n$; then*

$$SW(\tilde{P}_M \otimes PD(\Sigma)^{-1}; \xi(-\Sigma)h) = SW(\tilde{P}_M; h) \tag{52}$$

where $\xi(\pm\Sigma) = \prod_1^g([p] \mp \alpha_i\beta_i) \in \mathcal{H}(M)$ for (α_i, β_i) in the image in $H_1(M, \mathbb{Z})$ of a symplectic basis of $H_1(\Sigma, \mathbb{Z})$.

Proof: it is easy verify that both the spin^c structures $\tilde{P}_M$ and $\tilde{P}_M \otimes PD(\Sigma)^{-1}$, restricted to M^-, satisfy the condition required in Theorem 4.3 in order to have critical set on N composed of reducible nondegenerate solutions, identified in both cases with the jacobian torus of Σ. By clear topological reasons, the reducible solutions on $\hat{M}^-$ are identified, as sets, with the jacobian torus $Jac(\Sigma)$. In order to prove the the Lemma we wish to apply Claim 2.1, but to do so we must verify that in both cases the reducible solutions on $\hat{M}^-$ are nondegenerate. To prove this, according to Proposition 4.2, amounts to prove the vanishing of the kernel of the Dolbeault operator $\bar{\partial}_A + \bar{\partial}_A^*$ where A is a reducible connection in $det\tilde{P}_{\hat{M}^-}$ or in $det\tilde{P}_{\hat{M}^-} - 2PD(\Sigma)$. In reference [MOY] it is analysed how to compute kernel and cokernel of this operator, in terms of the sheaf cohomology of a bundle, constructed from the spin^c structures above, on a ruled surface $\mathbf{R} = \mathbf{P}(\mathbb{C} \oplus L)$ where L is the degree $-n$ line bundle over Σ (if you want to, you can think to $\mathbf{R}$ as the natural compactification of M^- adding a curve Σ_+ of opposite self intersection). These cohomology groups, on their own, can be identified via Leray spectral sequence to cohomology groups on Σ. Without entering in detail, I limit myself to quote the fact that, for the case of $\tilde{P}_{\hat{M}^-} \otimes PD(\Sigma)^{-1}$, both the kernel and the cokernel vanish, guaranteeing therefore that the reducible solutions on $\hat{M}^-$ are nondegenerate and unobstructed, while for $\tilde{P}_{\hat{M}^-}$ the kernel vanishes but the cokernel (which, as can be verified just by use of dimension formulae, must have complex dimension g) is identified, over the point $A \in Jac(\Sigma)$, with $H^1(\Sigma, \bar{\partial}_A)$. It is a classical (but not trivial) result of algebraic geometry that these obstructions fit together to give a rank g vector bundle F over $Jac(\Sigma)$ whose Chern class are given by $c(F) = \prod_1^g(1 + \alpha_i\beta_i)$. For both spin^c structures we can in any case apply the gluing formula of eq. 2.1, for computing the SW invariants: for $\tilde{P}_M$ the Euler class of the obstruction bundle takes the form $e(\mathcal{O}) = \mu(\xi(-\Sigma))$ and so we have

$$SW(\tilde{P}_M; h) =< \mu(h) \cup \mu(\xi(-\Sigma)), [\mathcal{M}(\tilde{P}_{\hat{M}^+})] >=< \mu(\xi(-\Sigma)h), [\mathcal{M}(\tilde{P}_{\hat{M}^+})] > \tag{53}$$

while for $\tilde{P}_M \otimes PD(\Sigma)^{-1}$ the obstruction vanishes and we have, computing the invariant for the class $\xi(-\Sigma)h$,

$$SW(\tilde{P}_M \otimes PD(\Sigma)^{-1}; \xi(-\Sigma)h) =< \mu(\xi(-\Sigma)h), [\mathcal{M}(\tilde{P}_{\hat{M}^+})] > \tag{54}$$

and this is what we wanted to prove. □

Now that Lemma 4.4 is proven, it is quite easy, with the use of the blow up formula 33 to prove a general relation between SW invariants associated to different spin^c structures which generalizes Lemma 3.3 to higher genus case: we have the

Lemma 4.5. *Let M be a four manifold containing a genus g Riemann surface of selfintersection $-n$ and let $\tilde{P}_M$ be a spin^c structure with $d(\tilde{P}_M) \geq 0$ such that $| < c_1(\mathcal{L}_M), \Sigma > | \geq 2g + n$; then*

$$SW(\tilde{P}_M; h) = SW(\tilde{P}_M \otimes PD(\Sigma)^{\epsilon}, \xi(-\Sigma)[p]^m h) \tag{55}$$

where $2m = | < c_1(\mathcal{L}_M), \Sigma > | - 2g - n$.

Proof: we just need to show that, by blowing up, we can reduce ourselves to the case of Lemma 4.4: by symmetry reason we consider just the case of $\epsilon = -1$; start by blowing up M a number of times $l + m$ where l is chosen in such a way to satisfy the relation $n + l + m \geq 2g$; the proper transform $\hat{\Sigma}$ of Σ is and embedded Riemann surface of genus g and fundamental class $\Sigma - \sum_1^{l+m} E_i$ where the E_i are the exceptional classes of the blow up. It is immediate verifying that $\hat{\Sigma}$ has selfintersection $(-n - l - m)$. On $\hat{M} := M \#_{l+m} \bar{\mathbf{P}}^2$ there is a spin^c structure $\tilde{P}_{\hat{M}}$ having determinant bundle

$$det\tilde{P}_{\hat{M}} = det\tilde{P}_M - \sum_1^l E_i + \sum_{l+1}^{l+m} E_i \tag{56}$$

which satisfies the properties

$$< c_1(det\tilde{P}_{\hat{M}}), \hat{\Sigma} >= -2g - n - l - m, \quad d(det\tilde{P}_{\hat{M}}) = d(\tilde{P}_M). \tag{57}$$

The triple $\hat{M}$, $\tilde{P}_{\hat{M}}$, $\hat{\Sigma}$ clearly satisfies the hypothesis of Lemma 4.4, so we deduce that

$$SW_{\hat{M}}(\tilde{P}_{\hat{M}}; h) = SW_{\hat{M}}(\tilde{P}_{\hat{M}} \otimes PD(\hat{\Sigma})^{-1}; \xi(-\hat{\Sigma})h); \tag{58}$$

we can now use $m + l$ times the blow up formula 33, whose hypothesis can be easily verified from the previous data; the l.h.s. equals $SW_M(\tilde{P}_M; h)$, as each time we blow up we have $r = 0$; for the r.h.s. we have $r = 1$ for each of the m blow up labeled from $l + 1$ to $l + m$, and $r = 0$ for the others. So the r.h.s. equals $SW_M(\tilde{P}_M \otimes PD(\Sigma)^{-1}, \xi(-\Sigma)[p]^m h)$. From this, we deduce formula 55. □

With this Lemma we obtain, as in the genus $g = 0$ case, the adjunction formula for simple type manifolds:

Theorem 4.6. *Let M be a simple type manifold with nonvanishing SW invariants, and let Σ be a Riemann surface of negative selfintersection $-n$; then for any basic class $\mathcal{L}_i$ we have*

$$|< c_1(\mathcal{L}_i), \Sigma >| + \chi(\Sigma) + \Sigma \cdot \Sigma \leq 0. \tag{59}$$

Proof: the proof is much as the $g = 0$ case above: let $\mathcal{L}$ be a counterexample to the adjunction formula and $\tilde{P}_M$ a spin^c structure having determinant $\mathcal{L}$ and nonvanishing invariants. As basic classes are characteristic elements for the intersection form, the violation of the formula requires

$$|< c_1(\mathcal{L}), \Sigma >| \geq 2g(\Sigma) + n, \tag{60}$$

and this says, according to Lemma 4.5, that also $\tilde{P}_M \otimes PD(\Sigma)^\epsilon$ has nonvanishing SW invariants. Therefore it defines a basic class with associated moduli space of dimension

$$d(\tilde{P}_M \otimes PD(\Sigma)^\epsilon) = d(\tilde{P}_M) + \epsilon < c_1(\mathcal{L}), \Sigma > + \Sigma \cdot \Sigma > 0. \tag{61}$$

This violates the simple type assumption. □

With the adjunction formula we are finally home and we can state the

Theorem 4.7. *(Ozsváth-Szabó) An embedded symplectic surface in a symplectic four manifold minimizes the genus among embedded representatives of its homology class.*

Proof: by blowing up if necessary, we can suppose that the class represented by the symplectic surface C has negative square. The adjunction equality for embedded symplectic surfaces in a symplectic four manifold states that

$$< c_1(K_M), C > + \chi(C) + C \cdot C = 0. \tag{62}$$

Taubes theorem (ref. [T]) shows that the canonical bundle is a basic class, so for any Riemann surface Σ with $[\Sigma] = [C]$ we must have, according to the adjunction inequality of Theorem 4.6,

$$|< c_1(K_M), C >| + \chi(\Sigma) + C \cdot C \leq 0. \tag{63}$$

Putting everything together we get $g(C) \leq g(\Sigma)$. □

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