SYMPLECTIC 4–MANIFOLDS WITH $\kappa = 0$

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(joint work with Stefan Friedl, UQAM & Warwick University)
1. **General framework.** \((M^4, \omega)\) *symplectic* when \(\omega \in \Omega^2 M\) satisfies

\[
d\omega = 0, \quad \omega \wedge \omega > 0.
\]

Canonical examples: Kähler surfaces, highly “non–generic”.

\(M\) symplectic \(\implies M\) admits almost complex structure \(J \in \text{End}(TM)\).

**Definition:** \(\kappa := c_1(J) \in H^2(M, \mathbb{Z})\).

**Goal:** Classify symplectic 4–manifolds with \(\kappa = 0\).
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Examples of 4–manifolds with $\kappa = 0$:

1. $K3$ surface (Kähler);

2. $T^4$ (Kähler)

3. $T^2$–bundles over $T^2$ (some Kähler, some not, e.g. Kodaira-Thurston manifold).

Question: Other examples?
Potential new constructions:

1. **Symplectic fiber sum**: most likely fails (M. Usher)

2. **Dimensional reduction**:
   
   (a) Knot surgery construction of $K3_K$: to have $\kappa = 0$, need $\Delta_K = 1$;

   (b) $S^1$–bundles over $N^3$.  

Main result:

**Theorem:** If \( p : M \xrightarrow{S^1} N \) is symplectic, \( \kappa = 0 \), \( M \) is a \( T^2 \)-bundle over \( T^2 \).

Actually covers all symplectic manifolds \( M \) with \( \text{Kod}(M) = 0 \).

Relates with (and partially answers to)

**Conjecture:** If \( M \xrightarrow{S^1} N \) is symplectic, \( N \xrightarrow{\Sigma} S^1 \) with \( \langle e(M), [\Sigma] \rangle \geq 0 \).

Easy to see (using geometrization): \( N \) prime.
To simplify presentation: \( b_1(N) > 1 \).
2. Proof for $M \xrightarrow{S^1} N$ with $e(M)$ torsion.

**Lemma:** Let $M = S^1 \times N$; $\kappa = 0 \iff N$ has vanishing Thurston norm.

**Proof:** $\implies$ wlog, can assume $[\omega] \in H^2(M, \mathbb{Z})$ and $H := PD[\omega]$ represented by symplectic surface (Donaldson), hence
\[
\chi_-(H) = H \cdot H + \kappa \cdot H = H \cdot H.
\]
Write $\phi = p_*[\omega] \in H^1(N, \mathbb{Z})$: by Kronheimer’s refined adjunction,
\[
\chi_-(H) \geq H \cdot H + \|\phi\|_T
\]
hence $\|\phi\|_T = 0$; wiggle $\omega$ to get vanishing Thurston norm on $N$.

$\iff$ $SW_{S^1 \times N}$ “$=$” $\Delta_N$ and $\kappa \in \text{supp } SW_{S^1 \times N}$, hence
\[
0 \leq \kappa \cdot \phi \leq \|\phi\|_A \leq \|\phi\|_T = 0 \implies \kappa = 0.
\]
We have the following

**Theorem:** If $S^1 \times N$ is symplectic and $N$ has vanishing Thurston norm, then $N \xrightarrow{T^2} S^1$, hence $S^1 \times N \xrightarrow{T^2} T^2$.

**Proof:** Can assume $\phi = p_*[\omega]$ primitive. Let $\Sigma \in PD[\phi]$ connected, Thurston minimizing. Denote $\pi = \pi_1(N), A = \pi_1(\Sigma), B = \pi_1(N \setminus \nu \Sigma); A \subset B \subset \pi$. By Stallings, need to show $A = B$.

Let $\alpha : \pi \to G$ epimorphism onto finite group, $N_G \xrightarrow{G} N$ regular $G$–cover.

$S^1 \times N$ symplectic, $\kappa = 0 \implies S^1 \times N_G$ symplectic, $\kappa_G = 0$.

$SW_{S^1 \times N_G} = 1 \implies \Delta^\alpha_N = 1$, twisted Alexander polynomial associated to representation $\alpha : \pi \to G$. 
It follows: \( \forall \alpha : \pi \to G, \quad \Delta^\alpha_{N,\phi} = \text{ord}_{\mathbb{Z}[t^{\pm 1}]} H_1(\pi; \mathbb{Z}[G][t^{\pm 1}]) \neq 0. \)

We have a Mayer-Vietoris type sequence for HNN extensions

\[
\cdots \to H_0(A; \mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] \to H_0(B; \mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] \to H_0(\pi; \mathbb{Z}[G][t^{\pm 1}]).
\]

But \( H_i(\pi; \mathbb{Z}[G][t^{\pm 1}]) \) are \( \mathbb{Z}[t^{\pm 1}] \)-torsion, hence

\[
\text{rk}_{\mathbb{Z}} H_0(A; \mathbb{Z}[G]) = \text{rk}_{\mathbb{Z}} H_0(B; \mathbb{Z}[G]) \iff |\text{Im}(A \to G)| = |\text{Im}(B \to G)|.
\]

Now as \( \Sigma = T^2, A \subset \pi \) is abelian, hence separable: if by contradiction \( A \subsetneq B \exists \alpha : \pi \to G \) s.t. \( |\text{Im}(A \to G)| < |\text{Im}(B \to G)|. \)

**Corollary:** By going to finite cover, easily obtain same result for \( M \xrightarrow{S^1} N \) with \( e(M) \) torsion.
3. Proof for $M \xrightarrow{S^1} N$ with $e(M)$ not torsion

Problem: as above, $N$ has vanishing Thurston norm $\implies \kappa = 0$, but can’t decide if $\iff$ a priori holds: no (known) refined adjunction inequality.

Solution: use more algebra & topology!

Lemma: $\kappa = 0 \implies vb_1(N, \mathbb{F}_p) \leq 3$.

Proof: let $M_G \xrightarrow{S^1} N_G$ be obvious $S^1$–bundle over $N_G$. As for all $\alpha : \pi \to G$, $\kappa_G \in H^2(M_G, \mathbb{Z})$ is the sole basic class,

$$\text{aug } \Delta^\alpha_N = \text{aug } SW_{M_G} = 1.$$ 

But if $vb_1(N, \mathbb{F}_p) > 3$, $\exists \alpha : \pi \to G$ s.t. $\text{aug } \Delta^\alpha_N = 0(p)$ (Turaev).
**Theorem:** If \( p : M \xrightarrow{S^1} N \) symplectic, \( \kappa = 0 \), \( M \) is a \( T^2 \)–bundle over \( T^2 \).

**Proof:** If \( N \) is a \( T^2 \)–bundle over \( S^1 \), as \( b_1(N) > 1 \) it is also an \( S^1 \)–bundle over \( T^2 \), hence the statement follows. Otherwise it satisfies one of the following:

1. \( N \) has a **nontrivial JSJ decomposition**;

2. \( N \) is Seifert-fibered with an **incompressible \( T^2 \) that is not a fiber**;

3. \( N \) is **hyperbolic**.
If 1. or 2. hold we have incompressible tori that are not fibers. Using abelian subgroup separability, can show that $vb_1(N) = \infty$ (Kojima, Luecke).

If 3. holds, $\pi$ is f.g. linear group: by Lubotzky alternative, f.g. linear groups must be virtually solvable or $vb_1(N, \mathbb{F}_p) = \infty$; the first implies $N$ covered by torus bundle, impossible.
New directions: Extend to other 4–manifolds.

Problems

1. Known that if \( M \) symplectic, \( \kappa = 0 \), \( \implies \) \( vb_1(M) \leq 4 \) (T.J.Li, Bauer); not known if \( vb_1(M, \mathbb{F}_p) \leq 4 \) (within reach?);

2. Lubotzky alternative holds, but we don’t have JSJ; are linear groups “interesting enough”? 