Special Session on Knot Theory and the Topology of 3-manifolds

AMS Meeting

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SYMPLECTIC 4–MANIFOLDS WITH $\kappa = 0$

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1. General framework. (M^4, ω) *symplectic* when $\omega \in \Omega^2 M$ satisfies

$$d\omega = 0, \quad \omega \wedge \omega > 0.$$

Canonical examples: Kähler surfaces, highly "non–generic".

M symplectic $\Longrightarrow$ M admits almost complex structure $J \in End(TM)$.

Definition: $\kappa := c_1(J) \in H^2(M, \mathbb{Z})$.

Goal: Classify symplectic 4–manifolds with $\kappa = 0$.

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Examples of 4–manifolds with $\kappa = 0$:

1. $K3$ surface (Kähler);

2. T^4 (Kähler)

3. T^2–bundles over T^2 (some Kähler, some not, e.g. Kodaira-Thurston manifold).

Question: Other examples?

Potential new constructions:

1. Symplectic fiber sum: most likely fails (M. Usher)

2. Dimensional reduction:

   (a) Knot surgery construction of $K3_K$: to have $\kappa = 0$, need $\Delta_K = 1$;

   (b) S^1–bundles over N^3.

Main result:

Theorem: If $p : M \xrightarrow{S^1} N$ symplectic, $\kappa = 0$, M is a T^2–bundle over T^2.

Actually covers all symplectic manifolds M with $\text{Kod}(M) = 0$.

Relates with (and partially answers to)

Conjecture: If $M \xrightarrow{S^1} N$ symplectic, $N \xrightarrow{\Sigma} S^1$ with $< e(M), [\Sigma] >= 0$.

Easy to see (using geometrization): N prime.
To simplify presentation: $b_1(N) > 1$.

2. Proof for $M \xrightarrow{S^1} N$ with $e(M)$ torsion.

Lemma: Let $M = S^1 \times N$; $\kappa = 0 \Longleftrightarrow N$ has vanishing Thurston norm.

Proof: $\Longrightarrow$ wlog, can assume $[\omega] \in H^2(M, \mathbb{Z})$ and $H := PD[\omega]$ represented by symplectic surface (Donaldson), hence

$$\chi_-(H) = H \cdot H + \kappa \cdot H = H \cdot H.$$

Write $\phi = p_*[\omega] \in H^1(N, \mathbb{Z})$: by Kronheimer's refined adjunction,

$$\chi_-(H) \geq H \cdot H + \|\phi\|_T$$

hence $\|\phi\|_T = 0$; wiggle ω to get vanishing Thurston norm on N.

$\Longleftarrow$ $SW_{S^1 \times N}$ "=" Δ_N and $\kappa \in \text{supp } SW_{S^1 \times N}$, hence

$$0 \leq \kappa \cdot \phi \leq \|\phi\|_A \leq \|\phi\|_T = 0 \Longrightarrow \kappa = 0.$$

We have the following

Theorem: If $S^1 \times N$ is symplectic and N has vanishing Thurston norm, then $N \xrightarrow{T^2} S^1$, hence $S^1 \times N \xrightarrow{T^2} T^2$.

Proof: Can assume $\phi = p_*[\omega]$ primitive. Let $\Sigma \in PD[\phi]$ connected, Thurston minimizing. Denote $\pi = \pi_1(N)$, $A = \pi_1(\Sigma), B = \pi_1(N \setminus \nu\Sigma)$; $A \subset B \subset \pi$. By Stallings, need to show $A = B$.

Let $\alpha : \pi \to G$ epimorphism onto finite group, $N_G \xrightarrow{G} N$ regular G–cover.

$S^1 \times N$ symplectic, $\kappa = 0 \implies S^1 \times N_G$ symplectic, $\kappa_G = 0$.

$SW_{S^1 \times N_G} = 1 \implies \Delta_N^\alpha = 1$, twisted Alexander polynomial associated to representation $\alpha : \pi \to G$.

It follows: $\forall\ \alpha : \pi \to G, \quad \Delta^{\alpha}_{N,\phi} = \mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]} H_1(\pi; \mathbb{Z}[G][t^{\pm 1}]) \neq 0.$

We have a Mayer-Vietoris type sequence for HNN extensions

$$\begin{array}{ccccccc} & & & \dots & & \to & H_1(\pi;\mathbb{Z}[G][t^{\pm 1}]) \\ \to & H_0(A;\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] & \to & H_0(B;\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] & & \to & H_0(\pi;\mathbb{Z}[G][t^{\pm 1}]) \end{array} .$$

But $H_i(\pi; \mathbb{Z}[G][t^{\pm 1}])$ are $\mathbb{Z}[t^{\pm 1}]$–torsion, hence

$$\mathrm{rk}_{\mathbb{Z}} H_0(A;\mathbb{Z}[G]) = \mathrm{rk}_{\mathbb{Z}} H_0(B;\mathbb{Z}[G]) \iff |\mathrm{Im}(A \to G)| = |\mathrm{Im}(B \to G)|.$$

Now as $\Sigma = T^2$, $A \subset \pi$ is abelian, hence separable: if by contradiction $A \subsetneq B$ $\exists\ \alpha : \pi \to G$ s.t. $|\mathrm{Im}(A \to G)| < |\mathrm{Im}(B \to G)|$.

Corollary: By going to finite cover, easily obtain same result for $M \xrightarrow{S^1} N$ with $e(M)$ torsion.

3. Proof for $M \xrightarrow{S^1} N$ with $e(M)$ not torsion

Problem: as above, N has vanishing Thurston norm $\Longrightarrow \kappa = 0$, but can't decide if $\Longleftarrow$ *a priori* holds: no (known) refined adjunction inequality.

Solution: use more algebra & topology!

Lemma: $\kappa = 0 \Longrightarrow vb_1(N, \mathbb{F}_p) \leq 3$.

Proof: let $M_G \xrightarrow{S^1} N_G$ be obvious S^1–bundle over N_G. As for all $\alpha : \pi \to G$, $\kappa_G \in H^2(M_G, \mathbb{Z})$ is the sole basic class,

$$\text{aug } \Delta_N^\alpha = \text{aug } SW_{M_G} = 1.$$

But if $vb_1(N, \mathbb{F}_p) > 3$, $\exists\ \alpha : \pi \to G$ s.t. aug $\Delta_N^\alpha = 0(p)$ (Turaev).

Theorem: If $p : M \xrightarrow{S^1} N$ symplectic, $\kappa = 0$, M is a T^2–bundle over T^2.

Proof: If N is a T^2–bundle over S^1, as $b_1(N) > 1$ it is also an S^1–bundle over T^2, hence the statement follows. Otherwise it satisfies one of the following:

1. N has a nontrivial JSJ decomposition;
2. N is Seifert-fibered with an incompressible T^2 that is not a fiber;
3. N is hyperbolic.

If 1. or 2. hold we have incompressible tori that are not fibers.
Using abelian subgroup separability, can show that $vb_1(N) = \infty$ (Kojima, Luecke).

If 3. holds, π is f.g. linear group: by Lubotzky alternative, f.g. linear groups must be virtually solvable or $vb_1(N, \mathbb{F}_p) = \infty$; the first implies N covered by torus bundle, impossible.

New directions: Extend to other 4–manifolds.

Problems

1. Known that if M symplectic, $\kappa = 0$, $\Longrightarrow vb_1(M) \leq 4$ (T.J.Li, Bauer); not known if $vb_1(M, \mathbb{F}_p) \leq 4$ (within reach?);

2. Lubotzky alternative holds, but we don't have JSJ; are linear groups "interesting enough"?