Symplectic 4–manifolds with $\kappa = 0$

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General framework

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Definition: $\kappa := c_1(J) \in H^2(M, \mathbb{Z})$.

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Question: Are these the only examples?

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It is possible to construct several manifolds, using 2., whose Seiberg–Witten invariants are compatible with the condition $\kappa = 0$

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Two cases to consider: $e(M) \in H^2(N, \mathbb{Z})$ torsion or not torsion.

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Write $\phi = p_*[\omega] \in H^1(N, \mathbb{Z})$: by Kronheimer's refined adjunction,

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$\Longleftarrow$ $SW_{S^1 \times N}$ "=" Δ_N and $\kappa \in \operatorname{supp} SW_{S^1 \times N}$, hence

$$0 \leq \kappa \cdot \phi \leq \|\phi\|_A \leq \|\phi\|_T = 0 \Longrightarrow \kappa = 0.$$

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Denote $\pi = \pi_1(N)$, $A = \pi_1(\Sigma)$, $B = \pi_1(N \setminus \nu\Sigma)$; it is well known that $A \subset B \subset \pi$. By Stallings, we need to show that $A = B$.

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Let $\alpha : \pi \to G$ be an epimorphism onto a finite group, $N_G \xrightarrow{G} N$ the regular G–cover;
if $S^1 \times N$ is symplectic, $\kappa = 0$, then $S^1 \times N_G$ is symplectic, $\kappa_G = 0$.

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The condition that $\kappa_G = 0$, together with Taubes' "more constraints" implies that $SW_{S^1 \times N_G} = 1$, hence the twisted Alexander polynomial Δ_N^α associated to the representation $\alpha : \pi \to G$ satisfies $\Delta_N^\alpha = 1$.

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It follows: $\forall\ \alpha : \pi \to G, \quad \Delta_{N,\phi}^\alpha = \mathrm{ord}_{\mathbb{Z}[t^{\pm 1}]} H_1(\pi; \mathbb{Z}[G][t^{\pm 1}]) \neq 0.$

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We have a Mayer-Vietoris type sequence for HNN extensions

$$\begin{aligned} H_1(\pi; \mathbb{Z}[G][t^{\pm 1}]) \quad &\to H_0(A; \mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] \quad \to \\ \to H_0(B; \mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] \to \quad &H_0(\pi; \mathbb{Z}[G][t^{\pm 1}]) \end{aligned}$$

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But $H_i(\pi; \mathbb{Z}[G][t^{\pm 1}])$ are $\mathbb{Z}[t^{\pm 1}]$–torsion, hence

$$\mathrm{rk}_{\mathbb{Z}} H_0(A; \mathbb{Z}[G]) = \mathrm{rk}_{\mathbb{Z}} H_0(B; \mathbb{Z}[G]) \iff |\mathrm{Im}(A \to G)| = |\mathrm{Im}(B \to G)|.$$

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Now as $\Sigma = T^2$, $A \subset \pi$ is abelian, hence separable: if by contradiction $A \subsetneq B$, then there exist an epimorphism $\alpha : \pi \to G$ s.t. $|\mathrm{Im}(A \to G)| < |\mathrm{Im}(B \to G)|$.

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Corollary: By going to a finite cover, we can easily obtain same result for nonzero torsion $e(M)$.

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Lemma: If $\kappa = 0$ then $vb_1(N, \mathbb{F}_p) \leq 3$.

Proof: let $M_G \xrightarrow{S^1} N_G$ be obvious S^1–bundle over N_G. As for all $\alpha : \pi \to G$, $\kappa_G \in H^2(M_G, \mathbb{Z})$ is the only basic class,

$$\text{aug } \Delta_N^\alpha = \text{aug } SW_{M_G} = 1.$$

But if $vb_1(N, \mathbb{F}_p) > 3$, $\exists\ \alpha : \pi \to G$ s.t. aug $\Delta_N^\alpha = 0(p)$ (Turaev).

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1. N has a nontrivial JSJ decomposition;
2. N is Seifert-fibered with an incompressible T^2 that is not a fiber;
3. N is hyperbolic.

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If 3. holds, π is f.g. linear group: by Lubotzky alternative, f.g. linear groups must be virtually solvable or $vb_1(N, \mathbb{F}_p) = \infty$; the first implies N covered by torus bundle, impossible.

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2. Lubotzky alternative holds, but we don't have JSJ; are linear groups "interesting enough"?
3. If M is hyperbolic, it has been conjectured that all SW invariants vanish.