THE SLOPE OF SURFACES WITH ALBANESE DIMENSION ONE

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Abstract. Mendes Lopes and Pardini showed that minimal general type surfaces of Albanese dimension one have slopes K^2/χ dense in the interval $[2, 8]$. This result was completed to cover the admissible interval $[2, 9]$ by Roulleau and Urzua, who proved that surfaces with fundamental group equal to that of any curve of genus $g \geq 1$ (in particular, having Albanese dimension one) give a set of slopes dense in $[6, 9]$. In this note we provide a second construction that complements that of Mendes Lopes–Pardini, to recast a dense set of slopes in $[8, 9]$ for surfaces of Albanese dimension one. These surfaces arise as ramified double coverings of cyclic covers of the Cartwright–Steger surface.

The basic topological invariants of a minimal irregular surface of general type, K^2 and χ, lie in the forward cone in the (K^2, χ)–plane delimited by the Bogomolov–Miyaoka–Yau line $K^2 = 9\chi$ and the Noether line $K^2 = 2\chi$. Phrased otherwise, the slope K^2/χ is contained in the interval $[2, 9]$. In [So84] Sommese proved that the set of attainable slopes is dense in $[2, 9]$. Several refinements of that result, tied to specific classes of surfaces, have since appeared. In particular Mendes Lopes and Pardini showed in [MLP11] that the slope of surfaces of Albanese dimension one is dense in $[2, 8]$. Their examples are obtained by taking the double coverings of hyperelliptic genus–2 fibrations over a torus, which lie on the line $K^2 = 2\chi$ and have irregularity $q = 1$, ramified over a collection of general fibers. The proof that slopes of general type surfaces of Albanese dimension one are dense in the entire interval $[2, 9]$ was then completed by Roulleau and Urzua in [RU15] (as a corollary of their main result) by showing that surfaces whose fundamental group is equal to that of a curve of genus $g \geq 1$ have slopes which are dense in $[6, 9]$.

In this note we will provide an alternative construction of minimal surfaces of general type and Albanese dimension one with slopes dense in the interval $[8, 9]$. Our construction follows the strategy of [So84, MLP11] (to whom the template should be credited, even if we will be a bit cavalier in punctually referring to), using as building block cyclic covers of the Cartwright–Steger surface and adapting the template to reflect the specifics of this case. The main adjustment involves the identification of an infinite family of unramified cyclic covers of the Cartwright–Steger surface that retain irregularity $q = 1$.

We summarize some properties of the Cartwright–Steger surface in the following proposition, referring to [CS10, CKY17] for proof and further details:

Proposition 1.1. *The Cartwright–Steger surface is a minimal surface of general type with* $(K^2, \chi) = (9, 1)$*, irregularity* $q = 1$ *and* $H_1(X, \mathbb{Z}) = \mathbb{Z}^2$*. The Albanese fibration* $f \colon X \to T$ *has generic fiber* F *of genus* $g(F) = 19$ *and no multiple fibers.*

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Our construction will start by taking unramified cyclic covers of the Cartwright–Steger surface. We will need to determine the irregularities (or equivalently the first Betti numbers) of those covers. In order to do so, we will use information on the Green–Lazarsfeld sets of the fundamental group of the Cartwright–Steger surface. Recall that the Green–Lazarsfeld sets of the fundamental group $G := \pi_1(X)$ of a Kähler variety X are subsets of the character variety $\widehat{G} := Hom(G; \mathbb{C}^*)$ defined as the collection of the cohomology jumping loci, namely

$$W_i(G) = \{\xi \in \widehat{G} \mid h^1(G; \mathbb{C}_\xi) \geq i\},$$

nested by the *depth* i: $W_i(G) \subset W_{i-1}(G) \subset \ldots \subset W_0(G) = \widehat{G}$. Moreover, the trivial character $\hat{1} \in \widehat{G}$ is contained in $W_i(G)$ if and only if $i \leq b_1(G)$.

The next proposition is a consequence of the previous, and of structure theorems on the Green–Lazarsfeld sets of a Kähler group.

Proposition 1.2. *The Green–Lazarsfeld sets $W_i(G)$ for the Cartwright–Steger surface X are composed of finitely many torsion characters.*

Proof. By the work of [Bea92, Si93] (that refined previous results of [GL87, GL91]; see also [De08]), $W_1(G)$ is the union of a finite set of torsion characters and the inverse image of the Green–Lazarsfeld sets of the bases Σ_α under the finite collection of fibrations $g_\alpha \colon X \to \Sigma_\alpha$ with base given by a hyperbolic orbisurface Σ_a of genus $g \geq 1$. If X did admit such fibration, the commutator subgroup of G would be infinitely generated, by [De10]. However, as $H_1(G) = \mathbb{Z}^2$, the commutator subgroup $[G, G]$ of G enters in the short exact sequence

$$1 \longrightarrow [G, G] \longrightarrow G \xrightarrow{f_*} \mathbb{Z}^2 \longrightarrow 1 \tag{1}$$

where f_* is the map induced, in homotopy, by the Albanese fibration $f \colon X \to T$. Given any fibration $g \colon X \to \Sigma$, the kernel of the induced map in homotopy $g_* \colon \pi_1(X) \to \pi_1(\Sigma)$ is finitely generated if and only if the fibration has no multiple fibers (see e.g. [Cat03]). By Proposition 1.1, the Albanese fibration has no multiple fibers, hence $[G, G]$ is finitely generated. It follows that $W_1(G)$ is composed of finitely many torsion characters, hence so do the Green–Lazarsfeld sets of higher depth. (In fact, as X has Albanese dimension one, the Albanese fibration is the unique fibration of X having base with positive genus, up to holomorphic automorphism of the base. As this fibration does not have multiple fibers, its base is not hyperbolic as orbifold.) □

Proposition 1.2 allows one, in principle, to determine the first Betti number of any abelian cover of X: Given an epimorphism $\alpha \colon G \to S$ to a finite abelian group S, denote by $\widehat{\alpha} \colon \widehat{S} \to \widehat{G}$ the induced (injective) map of character varieties. Denote $H := ker\, \alpha \leq_f G$; Hironaka proved, in [Hi97], that we have the equation

$$b_1(H) = \sum_{i \geq 1} |W_i(G) \cap \widehat{\alpha}(\widehat{S})| = b_1(G) + \sum_{i \geq 1} |W_i(G) \cap \widehat{\alpha}(\widehat{S} \setminus \hat{1})|. \tag{2}$$

This formula asserts that a character $\xi \colon G \to \mathbb{C}^*$ such that $\xi \in W_i(G)$ contributes with multiplicity equal to its depth to the Betti number of the cover defined by $\alpha \colon G \to S$ whenever it factorizes via α.

In principle, following [Hi97], we could explicitly determine the jumping loci $W_i(G)$ for the Cartwright–Steger surface out of the Alexander module of its fundamental group using an explicit presentation and Fox calculus. There is no conceptual difficulty in doing so, but it is a somewhat daunting task. We will weasel our way out of this undertaking with the following approach.

First, we will denote by $e(G) \in \mathbb{N}$ the least common multiple of the order of the elements of $W_1(G)$, thought of as elements of the group $\widehat{G}$, and we will refer to it as the *exponent* of $W_1(G)$ (or, by extension, of the Cartwright–Steger surface). This is a positive integer, well defined by virtue of the fact that $W_1(G)$ is a finite set composed of torsion elements of $\widehat{G}$. (This notion is borrowed from the notion of exponent of a finite group, and would coincide with it if $W_1(G)$ were a subgroup of $\widehat{G}$. We don't know whether this is the case.)

Lemma 1.3. *Let $e(G)$ be the exponent of the Cartwright–Steger surface. Then for every integer $\lambda \geq 0$ all cyclic covers X_d of X of order $d := \lambda e(G) + 1$ have $q(X_d) = q(X) = 1$. The fibration $f_d \colon X_d \to T$ induced by the Albanese fibration of X is therefore Albanese.*

Proof. Let $S := \mathbb{Z}_d$ and denote by $\alpha \colon G \to S$ a cyclic quotient of G of order d. Denote by X_d the corresponding cyclic cover, so that $\pi_1(X_d) = H = ker\ \alpha \leq_f G$. As $H_1(G) = \mathbb{Z}^2$, the quotient map $\alpha \colon G \to S$ factors through the maximal free abelian quotient of G, i.e. the homotopy Albanese map $f_* \colon G \to \mathbb{Z}^2$ of (1) and we have the following commutative diagram of fundamental groups

(3)
$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta & \longrightarrow & H & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 1 \\
 & & \downarrow \cong & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & [G,G] & \longrightarrow & G & \xrightarrow{f_*} & \mathbb{Z}^2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \alpha & & \downarrow & & \\
 & & 1 & \longrightarrow & S & \longrightarrow & S & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

By Equation (2) any contribution to $b_1(X_d)$ – hence to $q(X_d)$ – in excess of $b_1(X) = 2$ comes from nontrivial characters in $W_1(G)$ factoring through $\alpha \colon G \to S$. By definition, the order of the characters in $W_1(G)$ divides $e(G)$, while the order of characters in $\widehat{\alpha}(\widehat{S})$ divides d. By choosing d to have the form $d = \lambda e(G) + 1$ for $\lambda \geq 0$, $e(G)$ and d are coprime. It follows that $W_i(G) \cap \widehat{\alpha}(\widehat{S}) = \hat{1}$, and the first part of the statement follows. (Note that, if $H_1(X_d)$ has a nontrivial torsion subgroup, $[H,H] \leq_f \Delta \cong [G,G]$ is a *proper* subgroup. This has no relevance for us.) By pull–back, the Albanese fibration of X induces a fibration $f_d \colon X_d \to T$ (whose induced map in homotopy appears in the top row of the diagram in (3)), where T is an S–cover of itself. By the above, this fibration is Albanese. □

Remark. It would be interesting to know the virtual first Betti number of X or, more modestly, to know the largest Betti number of its finite abelian covers (Proposition 1.2 guarantees that the latter number is bounded above). It has been pointed out to us that in [St14] Stover describes a (nonabelian) cover of X with $b_1 = 14$. Note that the condition $vb_1(X) > 2$ entails that the virtual Albanese dimension of X is 2. (This latter result has been explicitly observed in [DR17].) For the record, using an explicit presentation of G concocted out of information provided in [CKY17] we verified, using GAP, that low index subgroups of G have $b_1 = 2$.

Now we have information on the first Betti number of "enough" covers of X, and we can proceed as in [So84]. Namely, take a cyclic cover X_d of the Cartwright–Steeger surface with $d = \lambda e(G)+1$ as in Lemma 1.3. Consider the pull–back fibration $f_d \colon X_d \to T$, whose generic fiber is isomorphic to F, the fiber of the Albanese fibration of X. Take the double cover of the base T, branched at $2k$ regular values of f_d. The branch locus uniquely determines a line bundle that we can write as $\mathcal{O}_T(p_1 + \ldots + p_k)$ where $\{p_1, \ldots, p_k\} \subset T$ are a collection of regular values. Denote by $\Sigma_{k+1} \to T$ the double cover, a surface of genus equal to the subscript. Next, define the fiber product $S_{d,k} := X_d \times_T \Sigma_{k+1}$. Phrased otherwise, $S_{d,k}$ is the double covering of X_d determined by the line bundle $f_d^* \mathcal{O}_T(p_1 + \ldots + p_k) = \mathcal{O}_{X_d}(F_1 + \ldots + F_k)$, where the $F_i = f_d^{-1}(p_i)$ are (generic) fibers of the fibration $f_d \colon X_d \to T$, with branch locus given by $2k$ copies of the generic fiber F. Note that $S_{d,k}$ fibers over Σ_{k+1}.

This construction gives the family of surfaces that we were looking for, as stated by the following theorem that embeds the results above in the template of [So84, MLP11].

Theorem 1.4. *The surfaces $S_{d,k}$, for $k > 0$ and $d := \lambda e(G) + 1$, $\lambda > 0$, are smooth minimal surfaces of general type with Albanese dimension one whose set of slopes is dense in the interval* $[8, 9]$.

Proof. The fact that the surfaces $S_{d,k}$ are minimal of general type with slope

$$\frac{K^2_{S_{d,k}}}{\chi(S_{d,k})} = 9 - \frac{k(g(F) - 1)}{2d + k(g(F) - 1)} \in [8, 9]$$

is discussed in [So84, Lemma 2.1] (see also [MLP11]) – using information on numerical invariants of branched double coverings that can be found e.g. in [BHPV04, Section V.22] – and is actually true for any choice for $d, k > 0$.

Next, let's show that for our choices of d the fibration $S_{d,k} \to \Sigma_{k+1}$ is Albanese. Using the formulae in [BHPV04, Section V.22] we have

$$\chi(S_{d,k}) = 2\chi(X_d) + \frac{1}{2}(\langle kF, K_{X_d}\rangle + \langle kF, kF\rangle) = 2\chi(X_d) + k(g(F) - 1) \tag{4}$$

where the latter equation follows from the adjunction equality, and

$$p_g(S_{d,k}) = p_g(X_d) + h^2(\mathcal{O}_{X_d}(-F_1 - \ldots - F_k)). \tag{5}$$

To determine this last term, consider the structure sequence of sheaves for $F_1 + \ldots + F_k$, namely

$$0 \longrightarrow \mathcal{O}_{X_d}(-F_1 - \ldots - F_k) \longrightarrow \mathcal{O}_{X_d} \longrightarrow \mathcal{O}_{F_1+\ldots+F_k} \longrightarrow 0.$$

From the induced long exact sequence in cohomology we can extract the following exact sequence:

$$\ldots \longrightarrow H^1(\mathcal{O}_{X_d}) \longrightarrow H^1(\mathcal{O}_{F_1+\ldots+F_k}) \longrightarrow H^2(\mathcal{O}_{X_d}(-F_1 - \ldots - F_k)) \longrightarrow H^2(\mathcal{O}_{X_d}) \longrightarrow 0.$$

As $f_d\colon X_d \to T$ is Albanese, the restriction map $H^1(X_d,\mathbb{Z}) \to H^1(F,\mathbb{Z})$ is trivial, hence the map $H^1(\mathcal{O}_{X_d}) \to H^1(\mathcal{O}_{F_1+\ldots+F_k})$ is the zero map (see e.g. [BHPV04, Lemma IV.12.7]). Moreover, we have $h^1(\mathcal{O}_{F_1+\ldots+F_k}) = kg(F)$ and $h^2(\mathcal{O}_{X_d}) = p_g(X_d)$. Applying this to Equation (5) we get

$$p_g(S_{d,k}) = 2p_g(X_d) + kg(F)$$

which, together with Equation (4) gives

$$q(S_{d,k}) = 2q(X_d) + k - 1 = k + 1 = q(\Sigma_{k+1})$$

which entails that the fibration $S_{d,k} \to \Sigma_{k+1}$ is Albanese.

In order to show that the set of slopes achieved by the surfaces $S_{d,k}$ is dense in $[8,9]$, we will need to tweak a bit the calculations of [So84] to take into account the fact that, under our assumptions, not all values of d are allowed. Let $p/q \in (0,1)$ be a rational number, where p,q are positive integers with $0 < p < q$. The choice of sequences

$$d_n = ne(G)(q-p)(g(F)-1)+1, \;\; k_n = 2ne(G)p$$

yields surfaces S_{d_n,k_n} where the first index has the form $d = \lambda e(G) + 1$, in particular of Albanese dimension one, and has the property that

$$\lim_n \frac{K^2_{S_{d_n,k_n}}}{\chi(S_{d_n,k_n})} = 9 - \lim_n \frac{2ne(G)p(g(F)-1)}{2ne(G)(q-p)(g(F)-1)+2+2ne(G)p(g(F)-1)} = 9 - \frac{p}{q}.$$

As the limit set of the slopes contains all rationals in $(8,9)$, it is dense in $[8,9]$ □

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Addendum. After completion of this manuscripts, Matthew Stover kindly informed us of his preprint titled *"On general type surfaces with $q = 1$ and $c_2 = 3p_g$"* ([St18]), which contains some results partly overlapping with ours. In particular, Stover shows that the Green–Lazarsfeld sets of the Cartwright–Steger surface are in fact trivial, i.e. $W_i(G) = \{\hat{1}\}$ for $i \leq 2$, strengthening Proposition 1.2 above. This entails that all abelian covers of that surface have irregularity 1. Moreover, he pointed out that as G is a congruence lattice of positive first Betti number, its virtual Betti number is infinite. Together, these answer the questions posed in the remark following Lemma 1.3 above.

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