Conference on Geometry and Topology of Manifolds,
McMaster University, Hamilton - May 16, 2004

ISOTOPY PROBLEM IN SYMPLECTIC 4-MANIFOLDS

Let X be a closed simply connected 4-manifold, ω a symplectic form on X. A closed (oriented) embedded surface $\Sigma \hookrightarrow X$ is called

- *symplectic* if $\omega|_\Sigma$ is a symplectic form on Σ;
- *lagrangian* if $\omega|_\Sigma$ vanishes identically.

Given two submanifolds in the same homology class, we can ask whether they are isotopic (through embedded submanifolds).

Problem: Fix an homology class $\alpha \in H_2(X, \mathbb{Z})$; classify symplectic (lagrangian) representatives of α up to the equivalence relation above. (Arnold, Eliashberg-Polterovich, Siebert-Tian).

We can think of this problem in different ways:

- the study of symplectic (lagrangian) *surface-knots* in a given homotopy class of maps of $[\Sigma, X]$ (as $\pi_1(X) = 1$);
- a natural relative version of the study of exotic symplectic 4-manifolds.

Analogy: let X be a Kähler surface; by classical results, homologous smooth complex curves are unique up to isotopy (even deformation) $\Longrightarrow$ "complex knot theory" is trivial.

Emerging dichotomy: different behavior of “rigid” and “flexible” symplectic manifolds.

Uniqueness results

Eliashberg-Polterovich: a lagrangian $\mathbb{R}^2 \subset \mathbb{R}^4$, linear outside a ball, is isotopic to a standard one.

Siebert-Tian: a symplectic 2-manifold in $\mathbb{P}^2$ is isotopic to the complex curve of suitable degree (up to degree 17).

Non-uniqueness results

We can construct interesting classes of 4-manifolds, and study their submanifolds, starting from the following dictionary:

3-manifold	**4-manifold**
link $L \subset S^3$	link surgery manifold
fibered link	symplectic 4-manifold
fibration 1-form	symplectic 2-form
curve transverse to fibers	symplectic torus
curve on a fiber	lagrangian torus

Plan of the talk:

- summarize the definition of the ambient manifolds;
- outline three constructions of nonisotopic tori;
- state some results and discuss how to detect nonisotopy;
- discuss the higher genus case.

Constructions

- The ambient manifolds: link surgery manifolds (Fintushel-Stern).

Given a n-component link $L \subset S^3$ whose exterior fibers over S^1,

$$S^3 \setminus \nu L \xrightarrow{\Sigma} S^1$$

we can define the 4-manifold

$$X_L = \coprod_{i=1}^{n} (E(n_i) \setminus \nu F_i) \cup_{T_i^3} S^1 \times (S^3 \setminus \nu L).$$

The gluing map on the boundary 3-tori is defined in such a way to identify the meridian to the torus fiber $F_i \subset E(n_i)$ with a curve traced by Σ on the i-th component of $\partial(S^3 \setminus \nu L)$. We need a second form for X_L. Doing Dehn surgery on S^3 along L, with coefficients determined by Σ, we obtain a closed 3-manifold N that fibers over S^1. Let α be a closed form representing the fibration: $S^1 \times N$ admits a symplectic structure of the form

$$dt \wedge \alpha + \epsilon\beta$$

where β is a closed 2-from on N that restricts to a volume form on the fibers. Denote by C_i the cores of the Dehn filling: the tori $S^1 \times C_i$'s are symplectic.

Using canonical framings, we can rewrite X_L as fiber sum:

$$X_L = \coprod_{i=1}^{n} E(n_i)\#_{F_i=S^1\times C_i} S^1 \times N.$$

Gompf's theorem assures that this manifold admits a symplectic form that, away from the gluing locus, restricts to the one of the summands.

- Nonisotopic symplectic tori 1: braiding construction (Fintushel-Stern, V, Etgü-Park).

A curve $\gamma \subset S^3 \setminus \nu L$ defines a torus $S^1 \times \gamma \subset X_L$. If γ is transverse to the fibration of L, then

$$\alpha|_\gamma > 0 \Longrightarrow (dt \wedge \alpha + \epsilon\beta)|_{S^1\times\gamma} > 0.$$

Observation: the homology class of γ in $S^3 \setminus \nu L$ determines the homology class of $S^1 \times \gamma$ in X_L, as we have

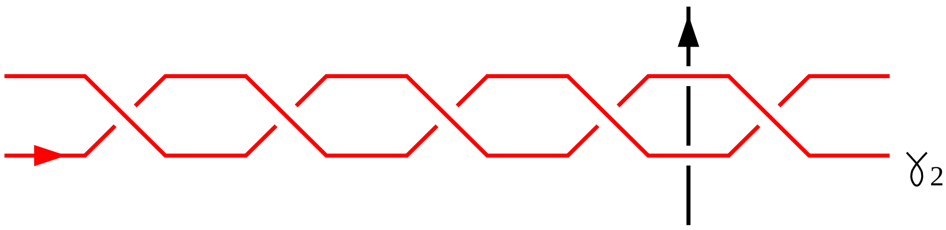
$$0 \to H_1(S^3 \setminus \nu L, \mathbb{Z}) \to H_2(S^1 \times (S^3 \setminus \nu L), \mathbb{Z}) \to H_2(X_L, \mathbb{Z}).$$

Therefore, if we find a family γ_p of homologous, nonisotopic curves transverse to a fibration, we can expect that $S^1 \times \gamma_p$ are symplectic, homologous, nonisotopic tori.

Taking L to be the trivial knot K, we can obtain the family $\gamma_p \subset S^3 \setminus \nu K$ as a closed braid with axis K; the homology class is given by

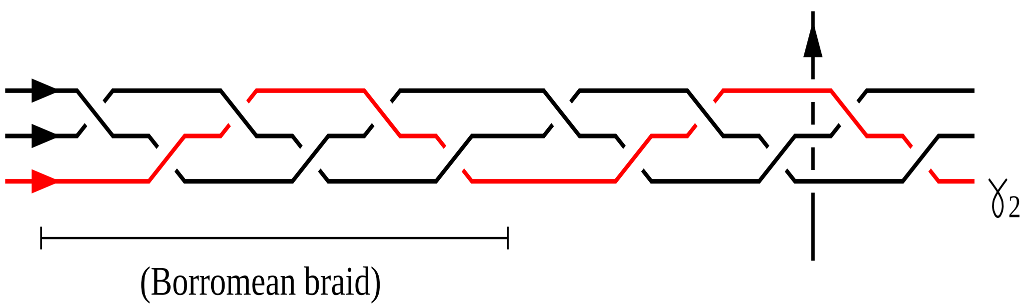
$$[\gamma_p] = lk(\gamma_p, K)[\mu(K)] \in H_1(S^3 \setminus \nu K, \mathbb{Z}).$$

An example of γ_p, with two strands, is presented here:



This construction provides homologous symplectic tori in $X_K = E(n)$ in a homology class with divisibility.

To get a primitive homology class, we can choose L the necklace with 3 components, and γ_p the transverse curve depicted below:



The homology class does not depend on p:

$$[\gamma_p] = \sum_{i=1}^{3} lk(\gamma_p, L_i)[\mu(L_i)] = [\mu(L_1)]. \quad \square$$

• Nonisotopic lagrangian tori

(V, Fintushel-Stern).

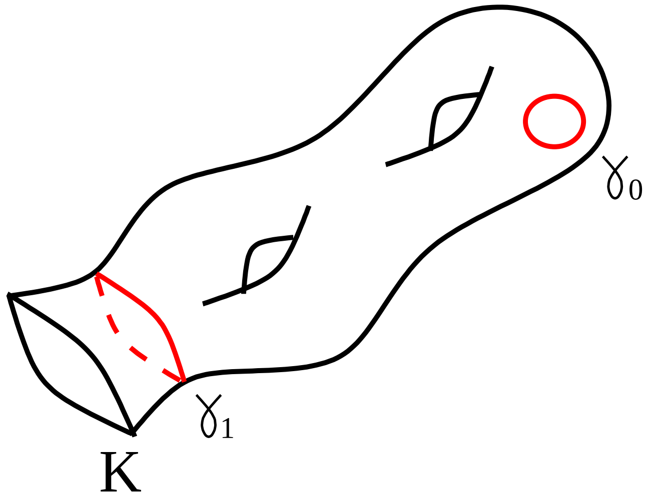
A curve $\gamma \subset S^3 \backslash \nu L$, laying on a fiber, defines a torus $S^1 \times \gamma \subset X_L$ that is lagrangian, as

$$\alpha|_\gamma = 0 \Longrightarrow (dt \wedge \alpha + \epsilon\beta)|_{S^1 \times \gamma} = 0.$$

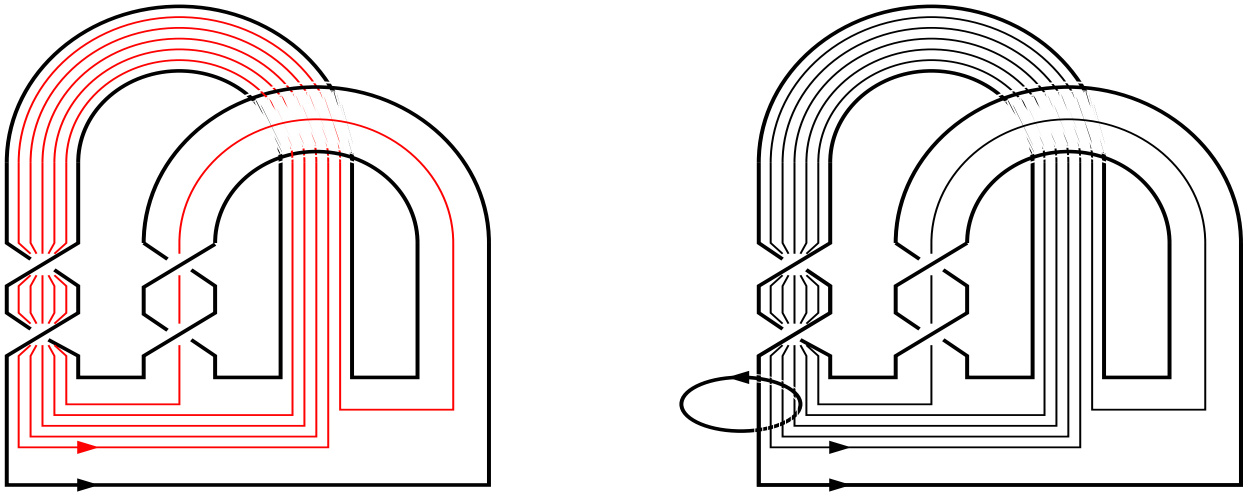
We look for a family γ_p of nonisotopic curves laying on a fiber: we need to consider a nontrivial K.

I will consider two examples, the first being a finite toy model.

Finite version: for K any nontrivial fibered knot, the curves are ∂D^2 and $\lambda(K)$:



Infinite version: K left-handed trefoil knot, γ_p the curve with p strands in the figure (note: γ_p is the torus knot $T_{(p,p+1)}$).



As $lk(\gamma_p, K) = 0$, the resulting lagrangian tori are nullhomologous in X_K.

To get essential tori, we can work with $L = K \cup M$, M being the meridian of K, and "circle sum" $\mu(M)\#\gamma_p$.

This construction provides homologous lagrangian tori in $X_L = E(n)_K$, both nullhomologous and essential. In the latter case, the tori can be made symplectic by perturbing the symplectic form. $\square$

• Nonisotopic symplectic tori 2: circle sum construction (V, Etgü-Park).

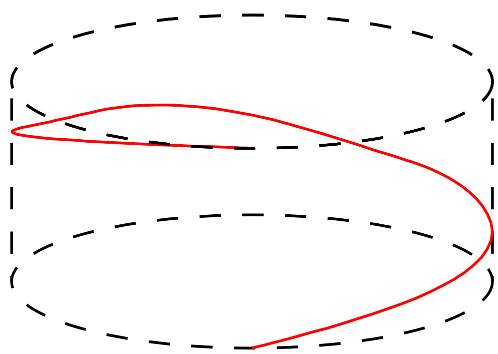
We can take advantage of the existence of different nullhomologous lagrangian tori to build different homologous symplectic tori, by starting with a fixed symplectic torus.

$$E(n)_K = (E(n) \setminus F) \cup_{T^3} S^1 \times (S^3 \setminus \nu K)$$

contains the preferred symplectic torus $F = S^1 \times \mu(K)$ (obtained from the transverse curve $\mu(K)$).

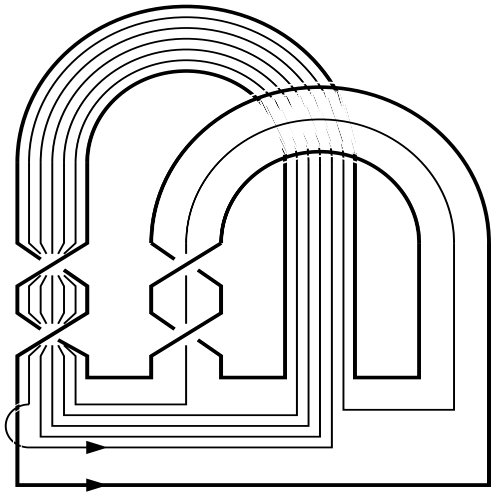
Idea: circle sum $\mu(K)$ with one (or more) curves γ_p on the fiber, to obtain a curve transverse to the fibration. To do so, consider the knotted ribbons $\gamma_p \times [-1, 1]$ transverse to the fibration and proceed in one of the following ways (or combinations of):

- draw a helix along each ribbon and glue the endpoints to a copy of $\mu(K)$ cut open, to get a family of transverse curves $\mu(K)\#\gamma_p$;



- draw a p-sheeted helix along a fixed ribbon $\gamma \times [-1, 1]$ and glue the endpoints to a copy of $\mu(K)$ cut open, to get a family of transverse curves $\mu(K)\#_p\gamma$. (If we take $\gamma = \lambda(K)$ the resulting curve is exactly the $(p, 1)$-cable of K!)

After this operation, we get a transverse curve of this form:



The resulting symplectic tori of X_K are homologous to F (in particular, primitive) and differ, in some sense, by the sum with different nullhomologous lagrangian tori. If the latter are nonisotopic, the former might be too. $\square$

Statement of the results: Detecting nonisotopy

Different, complementary methods can be employed to detect nonisotopy of two tori $T_p \subset X$

- The SW invariants of the branched covers of X along T_p;
- The lagrangian framing defect (see R.Fintushel's talk);
- The relative SW invariants of (X, T_p), i.e.

$$SW_{(X,T_p)} = SW_{X\#_{T_p=F}E(1)};$$

- The fundamental group of the exterior $X \setminus \nu T_p$.

I will focus on the third and fourth method to outline the proof of the following results.

Theorem (Fintushel-Stern, V, Etgü-Park): Let $E(n)$ be the elliptic surface of fiber F. Then the homology class $m[F]$, for $m \geq 2$, can be represented by infinitely many symplectic nonisotopic tori. In $E(n)$, for $n \geq 3$, also the primitive class $[F]$ can be represented by infinitely many symplectic nonisotopic tori.

Theorem (V, Fintushel-Stern): Let K be a nontrivial fibered knot. Then there exist a homology class $[R]$ in $E(n)_K$, $n \geq 2$, such that $m[R]$ can be represented by infinitely many nonisotopic lagrangian tori.

Idea of the proof: consider a family $S^1 \times \gamma_p \subset E(n)_K$; it is sufficient to prove that the number of SW basic classes of the manifold

$$E(n)_K \#_{S^1 \times \gamma_p = F} E(1)$$

diverges with p. But we have

$$SW_{E(n)_K \#_{S^1 \times \gamma_p = F} E(1)} \text{``} = \text{''} \Delta_{L_p}$$

where L_p is a link built out of K and γ_p. Using Torres' formula, we simplify the computation of Δ_{L_p} and bound the number of basic classes by below in terms of p. □

Theorem (V, Etgü-Park): Let K be a nontrivial fibered knot. Then there exist a primitive homology class $[F]$ in $E(1)_K$ that can be represented by infinitely many nonisotopic symplectic tori.

The original proof(s) goes much as above, but for amusement I will describe the proof of a finite version which has an interesting feature. Consider in $E(1)_K$ the homologous symplectic tori $S^1 \times \gamma_p$ obtained from the circle sum of the meridian with the nullhomotopic curve $\gamma_0 = \partial D^2$ and $p > 0$ copies of the longitude $\lambda(K)$. These are the $(0, 1)$- and the $(p, 1)$-cables of K.

We can rather easily compute the fundamental group of the exteriors of $S^1 \times \gamma_p$, and verify that

$$\pi_1(E(1)_K \setminus \nu(S^1 \times \gamma_p)) = \frac{\pi_1(S^3 \setminus \nu K)}{\mu(K)\lambda(K)^p}.$$

For $p = 0$, this group is trivial, while for $p > 0$ it is the fundamental group of the $1/p$-surgery of S^3 along K. The nonisotopy of $S^1 \times \gamma_p$, for $p = 0$ and $p \neq 0$ follows then by property P for fibered knots! $\square$

Higher genus examples

(work in progress with T.Etgü and D.Park)

Higher genus examples are somehow elusive (in our setup; otherwise, compare Smith and Auroux-Donaldson-Katzarkov).

If we use nullhomologus lagrangian tori in the construction of higher genus examples, we can hope that the fundamental group can detect nonisotopy. In fact, we can obtain the following:

Theorem (Etgü-Park-V): There exist simply connected manifolds containing infinitely many symplectic homologous nonisotopic genus 2 surfaces.

Idea of the proof: For a fibered knot K, consider $E(2)_K$. This contains a genus $g(K)+1$ symplectic surface $\hat{\Sigma}$ of selfintersection 0 (cap off the spanning surface of K, sum F, resolve). The 4-manifold

$$D_K = E(2)_K \#_{\hat{\Sigma}} E(2)_K$$

is symplectic and simply connected (Park). The symplectic tori $S^1 \times \gamma_p \subset E(2)_K$ intersect $\hat{\Sigma}$ once. Therefore in D_K we obtain genus 2 symplectic surfaces of the form

$$\Xi_p := S^1 \times \gamma_p \# S^1 \times \gamma_p.$$

These surfaces remember the fundamental group of the exterior of $S^1 \times \gamma_p$ in $E(2)_K$; in particular, we have

$$\pi_1(D_K \setminus \nu\Xi_p) = \frac{\pi_1(S^3 \setminus \nu K)}{\mu(K)\lambda(K)^p}.$$

Working with a specific class of knots (e.g., K torus knot, or hyperbolic) it is possible to show that these groups distinguish the isotopy class of Ξ_p (at least for infinitely many p's). $\square$

The remaining $g = 3, 4, 5, ...$ cases will be the subject of $n = 1, 2, 3, ...$ other articles.