Special Session in Low Dimensional Manifolds
AMS Meeting
Florida International University,
Miami - April 2, 2006

TWISTED ALEXANDER POLYNOMIALS AND SYMPLECTIC $\mathbf{S^1 \times N}$

Stefano Vidussi, University of California, Riverside
(joint work with Stefan Friedl, Rice University)

1. **General framework.** (M^4, ω) *symplectic* when $\omega \in \Omega^2 M$ satisfies

$$d\omega = 0, \quad \omega \wedge \omega > 0.$$

Canonical examples: Kähler surfaces. May play role in description of *all* smooth 4-manifolds.

Several characterizing results, but difficult to decide whether M admits symplectic structures (few non obvious obstructions).

Example (Thurston): if $N \xrightarrow{\phi} S^1$, $\phi \in H^1(N)$, $\phi \mapsto$ nondeg 1-form $\alpha \mapsto \omega = dt \wedge \alpha + *\alpha$ on $S^1 \times N$.

Conjecture 1: $S^1 \times N$ symplectic $\Longleftrightarrow N \xrightarrow{\phi} S^1$.

Goal: Find *evidence/obstructions*

Prototype (McCarthy): assuming GC, if $S^1 \times N$ symplectic, N irreducible or $S^1 \times S^2$.

Now on: N irreducible, $H := H_1(N)/\mathrm{Tor}H_1(N)$.

2. Properties of Δ_N for fibered N.

Def: $\|\phi\|_T := \min\{\sum_i -\chi(S_i) \,|\, \sum_i [S_i] = PD(\phi)\}$; extend to (semi)norm on $H^1(N, \mathbb{R})$. Unit ball B_T finite convex polyhedron.

Fibrations: $\mathbb{R}_+ F_T$, where F_T *fibered face* of B_T.

Def: $\|\phi\|_A := 2\max\{\phi(g) | g \in \mathrm{supp}(\Delta_N)\}$ where $\Delta_N \in \mathbb{Z}[H]$; extend to (semi)norm on $H^1(N, \mathbb{R})$. Unit ball B_A finite convex polyhedron.

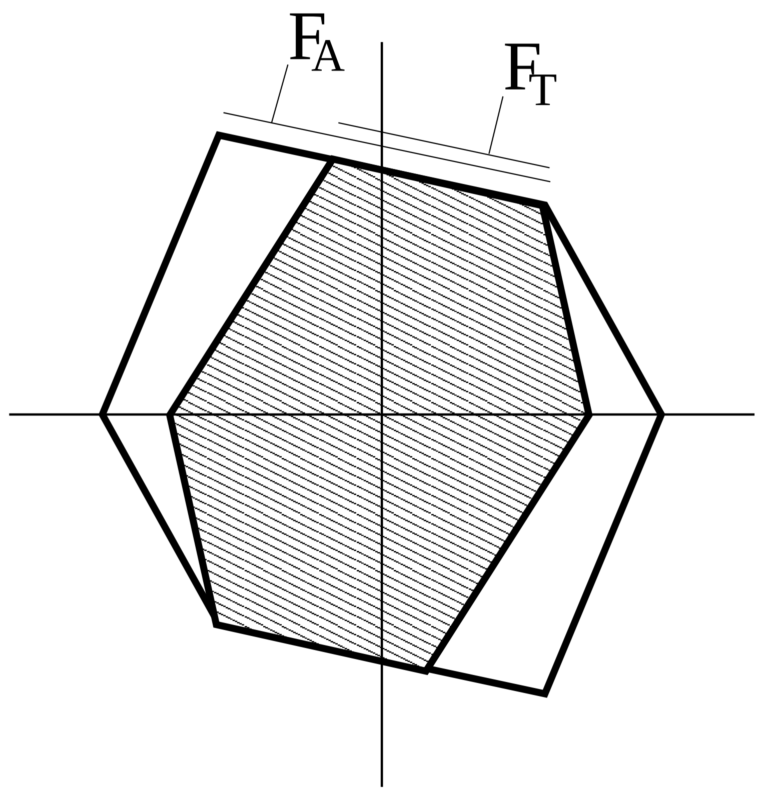
McMullen's inequality:

$$\|\phi\|_A \leq \|\phi\|_T + \begin{cases} 0 & \text{if } b_1(N) > 1, \\ 2\,\mathrm{div}\,\phi & \text{if } b_1(N) = 1. \end{cases}$$

Theorem: N fibered, fibered face F_T, then

1. *Newton polyhedron* $N(\Delta_N)$ has vertex $\kappa \in H$ with coefficient ± 1;

2. $(b_1 \geq 2)$ $F_T \subset F_A$ (face dual to κ) $\square$



Example: $N = N_K$, 0–surgery of S^3 along a knot K; $\Delta_K = \Delta_{N_K}$. If $H_1(N_K) = \mathbb{Z}\langle\phi\rangle$, then

$$\|\phi\|_T = 2g(K) - 2; \quad \|\phi\|_A = \deg\Delta_{N_K}.$$

Neuwirth: K fibered ($\Longleftrightarrow N_K$ fibered):

1. Δ_{N_K} is monic; 2. $\deg \Delta_{N_K} = 2g(K)$.

Theorem (Kronheimer; V.): $S^1 \times N$ symplectic,

1. $N(\Delta_N)$ has vertex $\kappa \in H$ with coefficient ± 1;

2. ($b_1 \geq 2$) exists *symplectic face* $F_T \subset F_A$ (face dual to κ) $\square$

(Uses Meng-Taubes; Donaldson; adjunction.)

Corollary: If $N = N_K$, $S^1 \times N_K$ symplectic, K *"looks fibered"*: Δ_K monic, $\deg \Delta_{N_K} = 2g(K)$.

Problem (Kronheimer): P pretzel knot $(5, -3, 5)$, $\Delta_P = t - 3 + t^{-1}$, $g(P) = 1$, nonfibered;

is $S^1 \times N_P$ symplectic?

3. **Twisted Alexander polynomials.**

G finite group. Consider $\pi_1(N) \xrightarrow{\alpha} G$: $\pi_1(N)$ acts on $G \times H$.

Define the *twisted Alexander module* (X-S. Lin)

$H_*(N, \mathbb{Z}[G][H]) := H(C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} \mathbb{Z}[G][H])$.

We have

$$\mathbb{Z}[H]^r \to \mathbb{Z}[H]^s \to H_1(N, \mathbb{Z}[G][H]) \to 0.$$

Def: $\Delta_N^\alpha :=$ ord Tors $H_1(N, \mathbb{Z}[G][H]) \in \mathbb{Z}[H]$.

- Computable;

- Related with coverings: take $b_1 \geq 2$.

Lemma: Let $\pi : N_G \to N$ associated G–cover:

$$\Delta_N^\alpha = \pi_* \Delta_{N_G}.$$

$\square$

Def: $\|\phi\|_A^\alpha := 2\max\{\phi(g)|g \in \text{supp}(\Delta_N^\alpha)\}$; extend to (semi)norm on $H^1(N,\mathbb{R})$.

McMullen's inequality generalizes, and we have

Theorem (also Friedl-Kim): N fibered, fibered face F_T, then

1. $N(\Delta_N^\alpha)$ has vertex $\kappa_G \in H$ with coefficient ± 1;
2. $(b_1 \geq 2)$ $F_T \subset |G|F_A^\alpha$ (face dual to κ_G) $\square$

(Uses Lemma, and fibration of N_G.)

Can use to verify explicit examples - very effective

Conjecture 2: if 1. + 2. true for all α, N fibered.

Next: Apply twisted Alexander polynomials to study symplectic $S^1 \times N$:

Theorem: $S^1 \times N$ symplectic, symplectic F_T

1. $N(\Delta_N^\alpha)$ has vertex $\kappa_G \in H$ with coefficient ± 1;
2. $(b_1 \geq 2)$ $F_T \subset |G| F_A^\alpha$ (face dual to κ_G) $\square$

(Uses Lemma, symplectic structure on $S^1 \times N_G$.)

Consequences:

1. Conjecture 2 implies Conjecture 1.
2. Effective obstructions: change coefficients to $\mathbb{F}_p$, $(|G|, p) = 1$; change G-module from $\mathbb{Z}[G]$ to smaller rank - Theorem still holds true. Can prove:

- Infinite family K_i with abelian invariants of fibered knot, but $S^1 \times N_{K_i}$ not symplectic;
- Conjecture 1 holds true for K up to 12 crossings;
- $S^1 \times N_P$ not symplectic (Kronheimer's question).

4. **Totally degenerate Thurston norm.**

Now on $\|\phi\|_T = 0$. Example: $N = N_K$, $g(K) = 1$.

If $S^1 \times N$ symplectic, Conjecture 1 implies N is a T^2–bundle over S^1.

Lemma (Kojima): If N has totally degenerate Thurston norm, N a torus bundle or v$b_1 = \infty$. $\square$

Theorem: if N has totally degenerate Thurston norm, Conjecture 2 holds true.

Proof: By contradiction, assume N not torus bundle; by Lemma, $\exists \pi : N_G \to N$, $b_1(N_G) > 3$; corresponding twisted Alexander polynomial

$$\Delta_N^\alpha = \pi_* \Delta_{N_G} = \pi_*(\sum c_g g).$$

For all $\phi \in H^1(N)$, $g \in \text{supp}\Delta_{N_G}$

$$|\phi(\pi_* g)| \leq \|\pi^* \phi\|_A \leq \|\pi^* \phi\|_T = |G| \|\phi\|_T = 0$$

hence $\pi_* g = 0 \in H \implies \Delta_N^\alpha = (\sum c_g) \in \mathbb{Z}[H]$.

As $b_1(N_G) > 3$, $(\sum c_g) = 0$. $\square$

Consequences:

- Twisted Alexander polynomials decide if a genus 1 knot is fibered;
- If N has totally degenerate Thurston norm, Conjecture 1 holds true;
- (Again) $S^1 \times N_P$ not symplectic.

Can prove also:

- if $\pi_1(N)$ satisfies *surface subgroup separability*, then N satisfies Conjecture 2.