Let $N$ be a closed 3–manifold. Thurston [Th76] showed that if $N$ admits a fibration over $S^1$, then $S^1 \times N$ is symplectic, i.e. it can be endowed with a closed, non–
degenerate 2–form $\omega$.

It is natural to ask whether the converse of this statement holds true. We can state this problem in the following form:

**Conjecture 1.** Let $N$ be a closed 3–manifold. If $S^1 \times N$ is symplectic, then there exists $\phi \in H^1(N; \mathbb{Z})$ such that $(N, \phi)$ fibers over $S^1$.

Here we say that $(N, \phi)$ fibers over $S^1$ if the homotopy class of maps $N \to S^1$ determined by $\phi \in H^1(N; \mathbb{Z}) = [N, S^1]$ contains a representative that is a fiber bundle over $S^1$.

Assuming the Geometrization Conjecture, it is possible to prove that the problem is reduced to the study of irreducible 3–manifolds, and we will henceforth make that assumption for $N$.

In [FV06a] we related this problem to the study of twisted Alexander polynomials of $N$, and in particular we proved the following, that is a weaker version of the main result of [FV06a]:

**Theorem 1.** Let $N$ be an irreducible 3–manifold such that $S^1 \times N$ admits a symplectic structure. Then there exists a primitive $\phi \in H^1(N; \mathbb{Z})$ such that for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group $G$ the associated 1–variable twisted Alexander polynomial $\Delta_{N, \phi}^\alpha \in \mathbb{Z}[t^{\pm 1}]$ is non–zero.

Recall that the 1–variable twisted Alexander polynomial $\Delta_{N, \phi}^\alpha$ associated to the pair $(N, \phi)$ is defined as the $\mathbb{Z}[t^{\pm 1}]$–order of the twisted Alexander module $H_1(N; \mathbb{Z}[G][t^{\pm 1}])$. Note that $\Delta_{N, \phi}^\alpha \neq 0$ if and only if $H_1(N; \mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$–torsion.

Given $\alpha : \pi_1(N) \to G$, denote the corresponding regular cover of $N$ cover by $N_G$. Note that, if $S^1 \times N$ is symplectic, so is $S^1 \times N_G$. The ingredients of the proof of Theorem 1 are now the following: Taubes’ results on the Seiberg-Witten invariants of symplectic 4–manifolds, the relation, proved by Meng and Taubes, between the Seiberg–Witten invariants of $N_G$ and the ordinary ordinary Alexander polynomial $\Delta_{N_G}$, and finally a relation obtained in [FV06a] between $\Delta_{N_G}$ and $\Delta_{N, \phi}^\alpha$.

Theorem 1 says in particular that the following conjecture implies Conjecture 1 for irreducible manifolds.

**Conjecture 2.** Let $\phi \in H^1(N; \mathbb{Z})$ be a primitive class such that $\Delta_{N, \phi}^\alpha \neq 0$ for all $\alpha : \pi_1(N) \to G$, then $(N, \phi)$ fibers over $S^1$. 

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To state our main theorem we need the following definition.

**Definition.** A subgroup \( A \subset \pi \) is *separable* if for all \( g \in \pi \setminus A \), there exists an epimorphism \( \alpha : \pi \to G \) to a finite group \( G \) such that \( \alpha(g) \notin \alpha(A) \).

We have the following result, proven in [FV06b]:

**Theorem 2.** Let \( N \) be an irreducible 3–manifold. Let \( \phi \in H^1(N; \mathbb{Z}) \) be a primitive class such that \( \Delta_{N,\phi}^0 \neq 0 \) for all epimorphisms \( \alpha : \pi_1(N) \to G \) to a finite group. Let \( \Sigma \subset N \) be an embedded surface dual to \( \phi \) having minimal genus. If \( \pi_1(\Sigma) \subset \pi_1(N) \) is separable, then \((N, \phi)\) fibers.

The question of which subgroups of the fundamental group of a Haken manifold are separable has been studied extensively. In particular, the fact that abelian subgroups are separable (cf. [LN91] and [Ha01]) and that incompressible surfaces in Seifert fibered spaces are classified leads to the following corollary.

**Corollary 1.** Conjecture 1 holds for irreducible manifolds with vanishing Thurston norm and for graph manifolds.

This corollary in particular implies that if \( N_K \) is the 0–surgery on a knot \( K \) of genus \( g(K) = 1 \), and \( S^1 \times N_K \) is symplectic, then \( K \) is a trefoil or the figure–8 knot. This answers a question of Kronheimer [Kr98].

Scott [Sc78] showed that any subgroup of a hyperbolic 2–manifolds is separable. It has been conjectured by Thurston [Th82] that all (surface) subgroups of hyperbolic 3–manifolds are separable. Clearly a positive solution to Thurston’s conjecture would imply Conjecture 1 for hyperbolic manifolds. Furthermore suitable subgroup separability properties of the hyperbolic pieces in the geometric decomposition can be shown to imply Conjecture 1 for all irreducible manifolds.

We conclude with a short outline of the proof of Theorem 2. Let \( M = N \setminus \nu \Sigma \). We have two embeddings \( i_{\pm} : \Sigma \to \partial M \). By Stallings’ theorem, \((N, \phi)\) fibers if the inclusion induced maps \( i_{\pm} : \pi_1(\Sigma) \to \pi_1(M) \) are isomorphisms. Since \( \Sigma \) has minimal genus we know that the maps \( i_{\pm} : \pi_1(\Sigma) \to \pi_1(M) \) are injective and that \( \pi_1(M) \to \pi_1(N) \) is injective.

Assume, by contradiction, that one of the \( i_{\pm} \) is not an isomorphism. We use the corresponding inclusion to view \( \pi_1(\Sigma) \) and \( \pi_1(M) \) as subgroups of \( \pi_1(N) \).

Given an epimorphism \( \alpha : \pi_1(N) \to G \) to any finite group \( G \) we have a long exact Mayer–Vietoris sequence

\[
H_1(N; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(\Sigma; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] \to H_0(M; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] \to H_0(N; \mathbb{Z}[G][t^{\pm 1}]).
\]
Now consider the ranks of the modules over $\mathbb{Z}[t^{\pm 1}]$: we have
\[
\begin{align*}
\text{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_1(N; \mathbb{Z}[G][t^{\pm 1}])) &= 0 \text{ since } \Delta_{N,\phi}^S \neq 0, \\
\text{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N; \mathbb{Z}[G][t^{\pm 1}])) &= 0 \text{ since } \phi \neq 0, \\
\text{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}]) &= \text{rank}_{\mathbb{Z}}(H_0(\Sigma; \mathbb{Z}[G])), \\
\text{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(M; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}]) &= \text{rank}_{\mathbb{Z}}(H_0(M; \mathbb{Z}[G])).
\end{align*}
\]
Therefore
\[
\frac{|G|}{|\alpha(\pi_1(\Sigma))|} = \text{rank}_{\mathbb{Z}}(H_0(\Sigma; \mathbb{Z}[G])) = \text{rank}_{\mathbb{Z}}(H_0(M; \mathbb{Z}[G])) = \frac{|G|}{|\alpha(\pi_1(M))|}.
\]
In particular we get that $\alpha(\pi_1(\Sigma)) = \alpha(\pi_1(M)) \subset G$. On the other hand it follows immediately from the assumption that $\pi_1(\Sigma) \subset \pi_1(N)$ is separable, and from the assumption that $\pi_1(\Sigma) \to \pi_1(M)$ is not an epimorphism, that there exists an epimorphism $\alpha : \pi_1(N) \to G$ to a finite group with $\alpha(\pi_1(\Sigma)) \neq \alpha(\pi_1(M))$. This contradiction concludes the proof of Theorem 2.

References

[Sc78] P. Scott, Subgroups of surface groups are almost geometric, J. London Math. Soc. (2) 17, no. 3: 555–565 (1978)