On the topology of SCY 4–manifolds

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General framework

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Problem: Decide if a 4–manifold admits a symplectic structure.

Few classes of 4–manifolds where we have a complete answer.

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The *Kodaira dimension* κ is a measure of the positivity of K in $H^2(M)$, and it is invariant under diffeomorphism and (symplectic) blow–up.

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Now on we will focus of 4–manifolds with $K = 0$ and we will call them Symplectic Calabi–Yau 4–manifolds (because of the high brand recognizability).

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Observation: For these manifolds,

$$2 \leq b_1(G) \leq vb_1(G) \leq 4, \ \ G \text{ is virtually poly–}\mathbb{Z}, \ \ \chi = \sigma = 0.$$

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Partial result (Baldridge '04, Friedl-V. '10, Chen '11): If M admits a circle action, yes.

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Corollary (see also Baykur, Li-Ni,'12): If G as above is an SCY group, either it is the fundamental group of a T^2–bundle over T^2, or G is the fundamental group of an exotic $S^2 \times T^2$.

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Theorem: If G is a r.f. SCY group with $b_1(G) > 0$ and $H^2(G, \mathbb{Z}G) = 0$ then the corresponding SCY manifold is a $K(G,1)$, hence G determines the manifold up to homotopy. □

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Corollary: let G be the SCY group of an infrasolvmanifold; then G determines the manifold up to homeomorphism.

Proof: Poly–$\mathbb{Z}$ groups are r.f., and for the groups in question $H^2(G, \mathbb{Z}G) = 0$. Also, Borel Conjecture holds in dimension 4 for (virtually) poly–$\mathbb{Z}$ groups. □

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If $g \geq 2$ the statement holds. We consider then $g = 1$, and want to show that if $h \geq 2$ then G is large. Write $\Sigma_1 = S^1_s \times S^1_t$ and look at

$$\begin{array}{ccc} F^3_t & \hookrightarrow & M \\ & & \downarrow \\ & & S^1_s \end{array}$$

where $\Sigma_h \hookrightarrow F_t \to \Sigma^1_t$ is the mapping torus of some $\varphi : \Sigma_h \circlearrowleft$.

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3. φ is reducible; then $\exists$ a (nonseparating) JSJ torus $T \subset F_t$ such that $\pi_1(T)$ is separable in a finite quotient $\alpha : \pi_1(F_t) \twoheadrightarrow Q$.

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Up to taking a suitable cyclic cover, the torus $T \subset F_t$ spans a 3–manifold $\Lambda \subset M$ and α extends to $G = \pi_1(F_t) \rtimes \mathbb{Z}$ so that
$$\alpha(\pi_1(\Lambda)) \lneq \alpha(\pi_1(M \setminus \nu\Lambda)) \leq Q,$$
so G surjects to the HNN extension $\alpha(\pi_1(M \setminus \nu\Lambda))_{*\alpha(\pi_1(\Lambda))}$.

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Finally, $b_2^{(2)}(M) = 0$ as

$$\chi(M) = 2b_0^{(2)}(M) - 2b_1^{(2)}(M) + b_2^{(2)}(M) = 0.$$

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We can now complete the proof by using the following **Theorem** (Eckmann '97): Let M be a 4–manifold with $b_0^{(2)}(M) = b_1^{(2)}(M) = b_2^{(2)}(M) = 0$ whose fundamental group satisfies $H^2(G, \mathbb{Z}G) = 0$; then either G is virtually $\mathbb{Z}$ or $M = K(G, 1)$.