Twisted Alexander Polynomials and Fibered 3–manifolds

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Introduction - 1

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Our approach: Use constraints on twisted Alexander polynomials of N arising from symplectic condition to show N fibers.

Introduction - 2

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$\forall\ \hat{\pi} \leq_{f.i.} \pi := \pi_1(N)$ the TAP $\Delta_{N,\phi}^{\pi/\hat{\pi}}$ satisfies

1. $\Delta_{N,\phi}^{\pi/\hat{\pi}} \in \mathbb{Z}[t^{\pm 1}]$ is monic;
2. $\deg\ (\Delta_{N,\phi}^{\pi/\hat{\pi}}) = [\pi : \hat{\pi}]\|\phi\|_T + 2\mathrm{div}\ \phi|_{\hat{\pi}}$.

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(ϕ is the Künneth component of the class of an integral symplectic form on $S^1 \times N$.)

Introduction - 3

Tools for the proof:

1. Taubes' constraints on SW invariants of $S^1 \times N$;
2. Meng–Taubes relation "$SW = \Delta$";
3. Donaldson's Theorem on $PD[\omega]$;
4. Kronheimer's refined adjunction inequality;
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Theorem (Friedl-V. '06): Conjecture 2 holds true when either

1. N has vanishing Thurston norm;
2. N is a graph manifold;
3. $\pi = \pi_1(N)$ is LERF.

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Step 3. Use results above on suitable cover of N.

Remark

Remark: Combining work of Kutluhan–Taubes and Ni, it is possible to prove "directly" Conjecture 1 (under some restrictions) using SWF theory.

Preliminaries: Twisted Alexander Polynomials

Given a $\mathbb{Z}$–module V that carries a representation α of π, we can associate to (N, ϕ) the twisted Alexander module

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when the module is $\mathbb{Z}[t^{\pm 1}]$–torsion, we define the TAP

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Given $\hat{\pi} \leq_{f.i.} \pi$ we consider the permutation module $V = \mathbb{Z}[\pi/\hat{\pi}]$ and the corresponding TAP $\Delta^{\pi/\hat{\pi}}_{N,\phi}$, related to the AP of the $\hat{\pi}$–cover of N.

Preliminaries: Prosolvable Completions - 1

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for all maps $A \longrightarrow G \in \mathcal{C}$.
It satisfies all appropriate functorial properties, in particular given a homomorphism $\varphi : A \longrightarrow B$, $\exists$ a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ \hat{A}_{\mathcal{C}} & \xrightarrow{\hat{\varphi}} & \hat{B}_{\mathcal{C}}. \end{array}$$

Preliminaries: Prosolvable Completions - 2

We will use the following characterizing property:

Claim: A homomorphism $\varphi : A \to B$ induces an isomorphism of pro–$\mathcal{C}$ completions iff $\forall G \in \mathcal{C}$, $\varphi^* : \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ is a bijection.

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We will be mostly interested in the case where $\mathcal{C}$ is the variety of finite solvable groups. In that case, pro–$\mathcal{C}$ is referred to as *prosolvable*.

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Lemma: Let (N, ϕ) satisfy Condition $(*)$; then $\forall \alpha : \pi \to G$ finite,

$$i_\pm : H_1(A; \mathbb{Z}[G]) \xrightarrow{\cong} H_1(B; \mathbb{Z}[G]).$$

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$$i_\pm : H_1(A; \mathbb{Z}[G]) \xrightarrow{\cong} H_1(B; \mathbb{Z}[G]).$$

Proof (sketch): We have a free square resolution

$$H_1(A; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] \xrightarrow{ti_+ - i_-} H_1(B; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] \twoheadrightarrow H_1(\pi; \mathbb{Z}[G][t^{\pm 1}])$$

hence $\Delta^\alpha_{N,\phi} = \det(ti_+ - i_-)$; Condition $(*)$ implies $\det(i_\pm) = \pm 1$. $\square$

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Let **H(n)** be the statement that, for any epimorphism $\beta : B \to S$ solvable group of $l(S) \leq n$,

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S(0) holds by *fiat*, H(0) by previous Lemma.

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We have

$$\begin{array}{ccccccccc} 1 & \to & G^{(n)} & \to & G & \to & S & \to & 1 \\ & & & & \uparrow & \nearrow & \uparrow & & \\ & & & & A & \to & B & & \end{array}$$

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As $G^{(n)}$ is abelian the map $A \to G$ factorizes through a quotient of A (determined by the map $A \to S$) that, by S(n) & H(n), is a quotient of B as well.

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Condition $(*)$ (applied to $\pi \to \mathbb{Z}_k \ltimes B/B(S)$) implies now

$$i_{\pm}^{*} : H_1(A; \mathbb{Z}[B/B(S)]) \xrightarrow{\cong} H_1(B; \mathbb{Z}[B/B(S)]).$$

$H(n)$ follows from there. $\square$

Step 2 - 1

Definition: A group π is called RFRS if $\exists$ a filtration $\pi = \pi_0 \geq \ldots \geq \pi_i \geq \ldots$ s.t.

1. $\cap_i \pi_i = \{1\}$;
2. $\pi_i \trianglelefteq_{f.i.} \pi_0$;
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Let $M = N \setminus \nu\Sigma$; define $DM := M \cup M$ with folding map $r : DM \to M$.

Theorem (Agol '08): With the notation above, if $\pi_1(M)$ is RFRS, $\exists$ an epimorphism $\alpha : \pi_1(M) \to S$ solvable s.t. in the cover $p : \widehat{DM} \to DM$ determined by $\alpha \circ r_* : \pi_1(DM) \to S$ the class of $p^{-1}\Sigma^{\pm}$ lies in the closure of a fibered cone.

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Theorem: With the notation above, assume that
1. $i_\pm : \pi_1(\Sigma) \to \pi_1(M)$ induces an isomorphism of prosolvable completions;
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We know that $[\hat{\Sigma}^{\pm}] \in H_2(\widehat{DM})$ is in the closure of a fibered cone; but 1. implies $\exists$ an isomorphism $f : H_2(\hat{\Sigma} \times S^1) \to H_2(\widehat{DM})$ with $f([\hat{\Sigma}]) = \pm[\hat{\Sigma}^{\pm}]$ preserving the Alexander norm, hence $[\hat{\Sigma}^{\pm}]$ sits on the cone over a top–dimensional face of the Alexander unit ball.

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Agol's theorem and the existence of an involution on $\widehat{DM}$ imply that $[\hat{\Sigma}^\pm]$ sits on the cone over a top–dimensional, fibered face of the Thurston unit ball as well; in particular $\hat{\Sigma}^\pm$ is a fiber. The statement is then straightforward. $\square$

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Claim: Let N be an irreducible 3–manifold; $\exists$ a cover $p : \hat{N} \longrightarrow N$ s.t. the fundamental group of each JSJ component $\hat{N}_i$ of $\hat{N}$ is residually finite solvable.
(Well–known if N is hyperbolic, as in that case π is linear hence virtually residually finite solvable)

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Claim: Let N_i be a JSJ component of N. Then if $\pi_1(\Sigma) \to \pi_1(M)$ induces an isomorphism of prosolvable completions, $\pi_1(\Sigma \cap N_i) \to \pi_1(M \cap N_i)$ induces an isomorphism of prosolvable completions.