

Problems for Putnam Training.

Problem 1. On a large, flat field, n people ($n > 1$) are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd, show that there is at least one person left dry.

Problem 2. Show that for any positive integer n , there exists a positive multiple of n that contains only the digits 7 and 0.

Problem 3. Find the remainder from division by 7 of the number

$$2^{2005} + 3^{2005}.$$

Problem 4. Let p be a prime number and $p > 3$. Let $k = [2p/3]$, where $[\cdot]$ denotes the integer part of \cdot . Prove that the sum of binomial coefficients

$$C_p^1 + C_p^2 + \dots + C_p^k$$

is divisible by p^2 . Recall that

$$C_n^k = \frac{n!}{(n-k)!k!}.$$

Problem 5. Let p be a prime number and n be a natural number which is not divisible by p . Prove that there exists a natural number m such that

$$n \cdot m = 1 \pmod{p}.$$

If you do not know what is “mod” then the latter is equivalent to “the remainder of the division $n \cdot m$ by p is 1”.

Problem 6. Prove that

$$11^{10} - 1$$

is divisible by 100.

Problem 7. Five points are placed on a sphere. Prove that 4 of them lie on a closed hemisphere.

Problem 8. Prove that one cannot draw a continuous line which intersects each of the 16 segments on Figure 1 exactly once. How to draw Figure 1 on torus so that one can draw a continuous line with the above desired property? (Torus is the surface of bagel)

Problem 9. Find the least number A such that for any two squares of combined area 1, there exists a rectangle of area A such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle.

Hint for problem 12. A function $f(x)$ has a Taylor series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with the infinite radius of convergence. Express $f(A)$ for

$$A = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$$

in terms of $f(x)$ and $f'(x)$.

Problem 13. Prove that if the integers a_1, a_2, \dots, a_n are all distinct then the polynomial

$$(x - a_1)^2(x - a_2)^2 \dots (x - a_n)^2 + 1$$

cannot be written as a product of two other polynomials with integral coefficients.

Problem 14. We have two absolutely identical chicken eggs and a building with 100 floors. We start counting the floors from the ground floor. If we drop any of these eggs from the n -th floor of this building then the egg will break but if we drop it from the previous $(n - 1)$ -th floor then the egg will not break. The number n may equal 1. Describe an algorithm which uses these two eggs to find the number n and which uses the smallest number of steps in its worst case scenario.

PROBLEM 15. COME TO ANY PUTNAM EXAM AND SOLVE ALL THE PROBLEMS! :-)))