Center of Twisted Graded Hecke Algebras for Homocyclic Groups

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Abstract. We determine explicitly the center of the twisted graded Hecke algebras associated to homocyclic groups. Our results are a generalization of formulas by M. Douglas and B. Fiol in [J. High Energy Phys. 2005 (2005), no. 9, 053, 22 pages].

Key words: twisted graded Hecke algebra; homocyclic group

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1 Main results

The notion of twisted graded Hecke algebras was introduced by S. Witherspoon in [10]; they are variants of the graded Hecke algebras of V. Drinfel’d [4] and G. Lusztig [6] (see also [7]) and twisted symplectic reflection algebras of T. Chmutova [2]. To a finite dimensional complex vector space \( V \), a finite subgroup \( G \) of \( \text{GL}(V) \), and a 2-cocycle \( \alpha \) of \( G \), the associated twisted graded Hecke algebra \( H \) is, by definition, a Poincaré–Birkhoff–Witt deformation of the crossed-product algebra \( SV\#_{\alpha} G \), where \( SV \) denotes the symmetric algebra of \( V \). The center of \( SV\#_{\alpha} G \) is \( (SV)^G \), and it is a natural question to determine the center of \( H \). In the non-twisted case, the center of the graded Hecke algebra associated to a finite real reflection group was determined by G. Lusztig in [5, Theorem 6.5]. In this paper, we determine the center of \( H \) for the twisted graded Hecke algebra in [10, Example 2.16], where \( V = \mathbb{C}^n \) and \( G \) is isomorphic to a homocyclic group \( (\mathbb{Z}/\ell\mathbb{Z})^{n-1} \). (By a homocyclic group, we mean a direct product of cyclic groups of the same order.) In this example, the algebra \( H \) is finitely generated as a module over its center; the center of \( H \) therefore plays an important role in the representation theory of \( H \). We show that the center of \( H \) is generated by \( n + 1 \) elements subject to one relation, which we determine explicitly. Our results are a generalization of formulas by M. Douglas and B. Fiol who considered the special case when \( n = 3 \) in their paper [3] on \( \mathbb{C}^3/(\mathbb{Z}/\ell\mathbb{Z})^2 \) orbifolds with discrete torsion.

We state our main results in this section and give the proofs in Section 2. We shall work over \( \mathbb{C} \). Let \( n \) be an integer \( \geq 3 \), and \( \ell \) an integer \( \geq 2 \). Let \( V = \mathbb{C}^n \) and let \( x_1, \ldots, x_n \) be the standard basis of \( V \). Let \( G \) be the subgroup of \( SL_n(\mathbb{C}) \) consisting of all diagonal matrices \( g \) satisfying \( g^\ell = 1 \). Let \( \zeta \) be a primitive \( \ell \)-th root of unity.

Notation 1.1. All subscripts are taken modulo \( n \). For example, \( x_{n+1} = x_1 \).
For $i = 1, \ldots, n$, let $g_i$ be the element of $G$ such that

$$g_i(x_j) = \begin{cases} \zeta x_j, & \text{if } j = i, \\ \zeta^{-1} x_j, & \text{if } j = i + 1, \\ x_j, & \text{else}. \end{cases}$$

Observe that $g_n = g_1^{-1} \cdots g_{n-1}^{-1}$. We have an isomorphism $\mathbb{Z}/\ell\mathbb{Z}^{n-1} \overset{\sim}{\longrightarrow} G$ defined by sending $(1,0,\ldots,0), \ldots, (0,0,\ldots,0,1)$ to $g_1, \ldots, g_{n-1}$, respectively.

Define the 2-cocycle $\alpha : G \times G \to \mathbb{C}^\times$ of $G$ by

$$\alpha(g_1^{i_1} \cdots g_{n-1}^{i_{n-1}}, g_1^{j_1} \cdots g_{n-1}^{j_{n-1}}) = \zeta^{-i_1 j_2 - i_2 j_3 - \cdots - i_{n-2} j_{n-1}}.$$

If $E$ is an algebra, an action of $G$ on $E$ is a homomorphism $G \to \text{Aut}(E)$. Recall that for any algebra $E$ and an action of $G$ on $E$, one has the crossed product algebra $E \#_\alpha G$. As a vector space, $E \#_\alpha G$ is $E \otimes \mathbb{C}G$; the product is defined by

$$(r \otimes g)(s \otimes h) = \alpha(g, h) r(g \cdot s) \otimes gh$$

for all $r, s \in E$ and $g, h \in G$. If $g, h \in G$, then we shall denote their product in $E \#_\alpha G$ by $g \ast h$; thus,

$$g \ast h = \alpha(g, h)gh.$$

One has, for any $i, j \in \{1, \ldots, n\}$ with $|i - j| \notin \{1, n - 1\}$,

$$g_{i+1} \ast g_i = \zeta g_i \ast g_{i+1}, \quad g_i \ast g_j = g_j \ast g_i.$$

Let $t = (t_1, \ldots, t_n) \in \mathbb{C}^n$, and write $TV$ for the tensor algebra of $V$. Following [10, Example 2.16], we make the following definition.

**Definition 1.2.** Let $H$ be the associative algebra defined as the quotient of $TV \#_\alpha G$ by the relations:

$$x_i x_{i+1} - x_{i+1} x_i = t_i g_i, \quad x_i x_j - x_j x_i = 0$$

for all $i, j \in \{1, \ldots, n\}$ with $|i - j| \notin \{1, n - 1\}$.

**Remark 1.3.** By [10, Theorem 2.10] and [10, Example 2.16], the algebra $H$ in Definition 1.2 is a twisted graded Hecke algebra for $G$. (However, when $n > 3$ and $\ell = 2$, this is not the most general twisted graded Hecke algebra for $G$; see [10, Example 2.16] and [9, Example 5.1].)

Let $\mathbb{C}[y_1^\pm, \ldots, y_n^\pm]$ be the algebra of Laurent polynomials in the variables $y_1, \ldots, y_n$. The group $G$ acts on $\mathbb{C}[y_1^\pm, \ldots, y_n^\pm]$ by

$$g_i y_1^{p_1} \cdots y_n^{p_n} = \zeta^{p_i - p_{i+1}} y_1^{p_1} \cdots y_n^{p_n}$$

for all $i \in \{1, \ldots, n-1\}$ and $p_1, \ldots, p_n \in \mathbb{Z}$.

**Proposition 1.4.** There is an injective homomorphism

$$\Theta : H \rightarrow \mathbb{C}[y_1^\pm, \ldots, y_n^\pm] \#_\alpha G$$

such that

$$\Theta(x_i) = y_i - \left(\frac{\zeta t_i}{\zeta - 1}\right) y_{i+1}^{-1} g_i, \quad (1.1)$$

$$\Theta(g_i) = g_i \quad (1.2)$$

for all $i \in \{1, \ldots, n\}$. 
Let
\[
I = \{\{i_1 < \cdots < i_k\} \mid k \geq 0; \ i_1, \ldots, i_k \in \{1, \ldots, n\}\},
\]
\[
J = \{\{i_1 < \cdots < i_k\} \in I \mid |i_r - i_s| \notin \{1, n - 1\} \text{ for all } r, s\}.
\]

Define the elements \(\delta, \varepsilon_1, \ldots, \varepsilon_n\) of \(\mathbb{Z}^n\) by
\[
\delta = (1, 1, \ldots, 1), \quad \varepsilon_1 = (1, 1, 0, \ldots, 0), \quad \varepsilon_2 = (0, 1, 1, 0, \ldots), \quad \ldots, \quad \varepsilon_n = (1, 0, \ldots, 0, 1).
\]

**Notation 1.5.** For any variables \(\omega_1, \ldots, \omega_n\) and \(p = (p_1, \ldots, p_n) \in \mathbb{Z}^n\), we denote by \(\omega^p\) the expression \(\omega_1^{p_1} \cdots \omega_n^{p_n}\).

We shall set
\[
\tau_i = \frac{t_i}{\zeta - 1} \quad \text{for } \ i = 1, \ldots, n - 1, \quad \tau_n = \frac{\zeta t_n}{\zeta - 1}.
\]

Define the element \(w \in H\) by
\[
w = \sum_{\{i_1 < \cdots < i_k\} \in J} \tau_{i_1} \cdots \tau_{i_k} x_1^{\delta - \varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} g_{i_1} \cdots g_{i_k}.
\]

**Example 1.6.** If \(n = 3\), then
\[
w = x_1 x_2 x_3 + \tau_1 x_3 g_1 + \tau_2 x_1 g_2 + \tau_3 x_2 g_3 = x_1 x_2 x_3 + \frac{1}{\zeta - 1} (t_1 x_3 g_1 + t_2 x_1 g_2 + \zeta t_3 x_2 g_3).
\]

In particular, if \(n = 3\) and \(\ell = 2\), the formula for \(w\) is in [1, Lemma 7.1].

**Theorem 1.7.** The center of \(H\) is generated as an algebra by \(x_1^{\ell}, \ldots, x_n^{\ell}\), and \(w\).

Let \(Z\) be the center of \(H\). For \(r = 0, \ldots, \lfloor \ell/2 \rfloor\), set
\[
\nu_r = (-1)^r \frac{\ell}{\ell - r} \left(\begin{array}{c} \ell - r \\ r \end{array}\right),
\]
and set
\[
\tilde{\tau}_i = \tau_i^\ell \quad \text{for } \ i = 1, \ldots, n - 1, \quad \tilde{\tau}_n = (-1)^{n(\ell - 1)} \tau_n^\ell.
\]

We define a polynomial \(F\) in the \(n + 1\) variables \(a_1, \ldots, a_n\) and \(b\) by
\[
F = \sum_{\{i_1 < \cdots < i_k\} \in J} \tilde{\tau}_{i_1} \cdots \tilde{\tau}_{i_k} a_1^{\delta - \varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} - \sum_{r=0}^{[\ell/2]} (-1)^n \zeta^{(n-2)r} \nu_r (\tau_1 \cdots \tau_n)^r b^{\ell - 2r}. \tag{1.3}
\]

**Corollary 1.8.** The assignment
\[
a_i \mapsto x_i^{\ell} \quad \text{for } \ i = 1, \ldots, n, \quad b \mapsto w \tag{1.4}
\]
defines an isomorphism
\[
\mathbb{C}[a_1, \ldots, a_n, b]/(F) \overset{\sim}{\longrightarrow} Z. \tag{1.5}
\]

In the undeformed case, when \(t_1 = \cdots = t_n = 0\), the polynomial \(F\) is equal to \(a_1 \cdots a_n - b^{\ell}\).
2 Proof of main results

Proof of Proposition 1.4. For \( i = 1, \ldots, n - 1 \), we define \( \Theta(x_i), \Theta(x_n), \) and \( \Theta(g) \) by (1.1) and (1.2). It follows from a straightforward verification that \( \Theta \) is a well-defined homomorphism.

It remains to see that \( \Theta \) is injective. Observe that \( H \) is spanned by the monomials \( x^p g \) for \( p = (p_1, \ldots, p_n) \in \mathbb{Z}^n \) and \( g \in G \), where \( p_1, \ldots, p_n \geq 0 \). We call \( p_1 + \cdots + p_n \) the total degree of the monomial \( x^p g \). The image of \( x^p g \) under \( \Theta \) is the sum of \( y^p g \) with terms of strictly smaller total degrees. Therefore, if \( \alpha \in H \) is nonzero, we can write it as a sum \( \alpha_0 + \alpha_1 + \cdots \), where \( \alpha_k \) is a linear combination of monomials \( x^p g \) with total degree \( k \). If \( k \) is the maximal integer with \( \alpha_k \) nonzero, then \( \Theta(\alpha_k) \) is nonzero, and hence \( \Theta(\alpha) \) is also nonzero.

Remark 2.1. It follows from Proposition 1.4 that the monomials \( x_1^{p_1} \cdots x_n^{p_n} g \) for non-negative integers \( p_1, \ldots, p_n \) and \( g \in G \) form a basis for \( H \) (called the PBW basis of \( H \)). This was first proved in [10, Example 2.16] using [10, Theorem 2.10].

We have an increasing filtration on \( H \) defined by setting \( \deg(x_i) = 1 \) and \( \deg(g) = 0 \) for all \( i \in \{1, \ldots, n\} \), \( g \in G \). It is immediate from Remark 2.1 that the natural homomorphism \( SV \#_\alpha G \to \text{gr} H \) is an isomorphism, where \( \text{gr} H \) denotes the associated graded algebra of \( H \).

The proof of (2.3) in the following lemma is the key calculation in this paper.

Lemma 2.2.

(i) One has:

\[
\Theta(x_i^\ell) = y_i^\ell - \tau_i y_{i+1}^{-\ell},
\]

(2.1)

\[
\Theta(x_n^\ell) = y_n^\ell - (-1)^{(\ell-1)n} \tau_n y_1^{-\ell},
\]

(2.2)

for all \( i \in \{1, \ldots, n-1\} \).

(ii) One has:

\[
\Theta(w) = y_1 \cdots y_n + (-1)^n \zeta^{n-2} \tau_1 \cdots \tau_n y_1^{-1} \cdots y_n^{-1}.
\]

(2.3)

Proof. (i) To prove (2.1), we need to show that

\[
\left( y_i - \zeta \tau_i y_{i+1}^{-1} g_i \right) \cdots \left( y_i - \zeta \tau_i y_{i+1}^{-1} g_i \right) = y_i^\ell - \tau_i^\ell y_{i+1}^{-\ell}.
\]

(2.4)

Since \( g_i y_i = \zeta y_i g_i \) and \( g_i y_i^{-1} = \zeta y_i^{-1} g_i \), the product on the left hand side of (2.4) is a linear combination of \( y_i^k y_{i+1}^{-\ell-k} g_i \) for \( k = 0, 1, \ldots, \ell \). Moreover, the coefficient of \( y_i^k y_{i+1}^{-\ell-k} g_i \) in this linear combination is the same as the coefficient of \( u^k \) when we expand the product

\[
(u - \zeta^\ell \tau_i) (u - \zeta^{\ell-1} \tau_i) \cdots (u - \zeta \tau_i)
\]

(2.5)

in the polynomial ring \( \mathbb{C}[u] \). Since the polynomial in (2.5) is equal to \( u^\ell - \tau_i^\ell \), the identity (2.1) follows. The proof of (2.2) is similar except that

\[
\underbrace{g_1 \cdots g_n}_{\ell} = (-1)^{(\ell-1)n}.
\]

(ii) For any \( h_* = \{h_1 < \cdots < h_j\} \in I \), we let

\[
h'_* = \{h_r \in h_* \mid h_s - h_r \in \{1, 1 - n\} \text{ for some } s\},
\]

\[
\chi(h_*) = |\{h_r \in h'_* \mid h_r \neq n\}| - |\{h_r \in h'_* \mid h_r = n\}|.
\]
Now suppose \( i_* = \{ i_1 < \cdots < i_k \} \in J \). Let \( D \) be the subset of \( \{ 1, \ldots, n \} \) consisting of all \( d \) such that \( d \not\equiv i_r, i_r + 1 \) (mod \( n \)) for all \( r \). We denote by \( d_1 < \cdots < d_p \) the elements of \( D \). Then

\[
\begin{align*}
\Theta(\tau_{i_1} \cdots \tau_{i_k} x^{\delta - \varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} g_{i_1} \cdots g_{i_k}) &= \tau_{i_1} \cdots \tau_{i_k} \left( y_{d_1} - \frac{\xi t_{d_1} - 1}{1 - y_{d_1}} \right) \cdots \left( y_{d_p} - \frac{\xi t_{d_p} - 1}{1 - y_{d_p}} \right) g_{i_1} \cdots g_{i_k} \\
&= \tau_{i_1} \cdots \tau_{i_k} \sum_{S \subseteq D} Y_{d_1}(S) \cdots Y_{d_p}(S) g_{i_1} \cdots g_{i_k},
\end{align*}
\]

where, for \( r = 1, \ldots, p \),

\[
Y_{d_r}(S) = \begin{cases} 
  y_{d_r}, & \text{if } d_r \not\in S, \\
  -\zeta \left( \xi - 1 \right)^{-1} t_{d_r} y_{d_r}^{-1} g_{d_r}, & \text{if } d_r \in S.
\end{cases}
\]

Setting \( h_* = i_* \cup S \), we obtain

\[
\Theta(\tau_{i_1} \cdots \tau_{i_k} x^{\delta - \varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} g_{i_1} \cdots g_{i_k}) = \sum_{h_* \in I \mid h_* \in h_*'} (-1)^{|h_*| - |i_*|} E(h_*).
\]

Hence,

\[
\Theta(w) = \sum_{\{i_1 < \cdots < i_k\} \in J} \Theta(\tau_{i_1} \cdots \tau_{i_k} x^{\delta - \varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} g_{i_1} \cdots g_{i_k})
\]

\[
= \sum_{i_* \in J} \left( \sum_{h_* \in I \mid h_* \in h_*'} (-1)^{|h_*| - |i_*|} E(h_*) \right) = \sum_{h_* \in I} \left( E(h_*) \sum_{i_* \in h_* - h_*'} (-1)^{|h_*| - |i_*|} \right).
\]

If \(|h_*| = n\), then \( h_*' = h_* \). If \(|h_*| \not\in \{0, n\}\), then \( h_*' \not= h_* \). Therefore,

\[
E(h_*) \sum_{i_* \in h_* - h_*'} (-1)^{|h_*| - |i_*|} = \begin{cases} 
  y_1 \cdots y_n, & \text{if } |h_*| = 0, \\
  (-1)^n \zeta^{-2} \tau_1 \cdots \tau_n y_1^{-1} \cdots y_n^{-1}, & \text{if } |h_*| = n, \\
  0, & \text{else}.
\end{cases}
\]

**Proof of Theorem 1.7.** It is easy to see that the center of \( SV\#_\alpha G \) is the algebra of \( G \)-invariant elements \( (SV)^G \) of \( SV \), and moreover, the algebra \( (SV)^G \) is generated by \( x_i^\ell \) \((i = 1, \ldots, n)\) and \( x_1 \cdots x_n \).

Using Lemma 2.2, we see that

\[
\Theta(x_i^\ell) \text{ for } i = 1, \ldots, n, \quad \text{and} \quad \Theta(w)
\]

are in the center of \( C[y_1^\pm, \ldots, y_n^\pm]\#_\alpha G \). Since the homomorphism \( \Theta \) is injective, the elements \( x_i^\ell \) \((i = 1, \ldots, n)\) and \( w \) are in the center of \( H \). Since the principal symbols of \( x_1^\ell, \ldots, x_n^\ell \) and \( w \) in \( SV\#_\alpha G \) are, respectively, \( x_1^\ell, \ldots, x_n^\ell \) and \( x_1 \cdots x_n \), the theorem follows from a standard argument.

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\(^1\)Note that if \( d_* \in S \) but \( d_* + 1 \in D - S \), then the term \( g_{d_*} \) in \( Y_{d_*}(S) \) appears on the left of the term \( y_{d_* + 1} \) of \( Y_{d_* + 1}(S) \) and one has \( g_{d_*} y_{d_* + 1} = \xi^{-1} y_{d_* + 1} g_{d_*} \). However, if \( n \in S \) but \( 1 \in D - S \), then the term \( g_n \) in \( Y_n(S) \) already appears to the right of the term \( y_1 \) of \( Y_1(S) \). This is the reason why the definition of \( \tau_n \) differs from the corresponding definitions of \( \tau_1, \ldots, \tau_{n-1} \) by a factor of \( \zeta \).
Proof of Corollary 1.8. Let \( \bar{a}_1 = \Theta(x_1^\ell), \ldots, \bar{a}_n = \Theta(x_n^\ell), \) and \( \bar{b} = \Theta(w) \). By Lemma 2.2,
\[
\bar{a}_i = y_i - \tau_i y_{i+1} \quad \text{for} \quad i = 1, \ldots, n,
\]
\[
\bar{b} = y_1 \cdots y_n + (-1)^n \zeta^{-2} \tau_1 \cdots \tau_n y_1^{-1} \cdots y_n^{-1}.
\]
By a calculation completely similar to the proof of (2.3), one has
\[
\sum_{\{i_1 < \cdots < i_k\} \in J} \bar{\tau}_1 \cdots \bar{\tau}_k \bar{d}^{-\varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} = (y_1 \cdots y_n)^\ell + (-1)^n (\tau_1 \cdots \tau_n)^\ell (y_1 \cdots y_n)^{-\ell}. \quad (2.6)
\]
We claim that we also have
\[
\sum_{r=0}^{[\ell/2]} (-1)^n r^{n-2} \nu_r (\tau_1 \cdots \tau_n)^\ell \bar{b}^{\ell-2r} = (y_1 \cdots y_n)^\ell + (-1)^n (\tau_1 \cdots \tau_n)^\ell (y_1 \cdots y_n)^{-\ell}. \quad (2.7)
\]
To see this, recall that the Chebyshev polynomials of the first kind are defined recursively by
\[
T_0(\xi) = 1, \quad T_1(\xi) = \xi, \quad \text{and} \quad T_m(\xi) = 2\xi T_{m-1}(\xi) - T_{m-2}(\xi) \quad \text{for} \quad m = 2, 3, \ldots.
\]
It is well known (and can be easily proved by induction) that
\[
2T_\ell \left( \frac{\xi}{2} \right) = \sum_{r=0}^{[\ell/2]} \nu_r \xi^{\ell-2r}, \quad (2.8)
\]
\[
2T_\ell \left( \frac{\xi + \xi^{-1}}{2} \right) = \xi^\ell + \xi^{-\ell}. \quad (2.9)
\]
By (2.8) and (2.9), one has the identity
\[
\xi^\ell + \xi^{-\ell} = \sum_{r=0}^{[\ell/2]} \nu_r (\xi + \xi^{-1})^{\ell-2r},
\]
and hence the identity
\[
\xi^\ell + q^{2\ell} \xi^{-\ell} = \sum_{r=0}^{[\ell/2]} \nu_r q^{2r} (\xi + q^2 \xi^{-1})^{\ell-2r}
\]
where \( \xi \) and \( q \) are formal variables. By setting \( \xi = y_1 \cdots y_n \) and choosing \( q \) to be a square-root of \( (-1)^n \zeta^{-2} \tau_1 \cdots \tau_n \), we obtain (2.7).

By Proposition 1.4, Theorem 1.7, and the equations (2.6) and (2.7), the assignment (1.4) defines a surjective homomorphism
\[
\Phi : \mathbb{C}[a_1, \ldots, a_n, b] \to \mathbb{Z}
\]
such that \( \Phi(F) = 0 \). Suppose \( D \in \mathbb{C}[a_1, \ldots, a_n, b] \) and \( \Phi(D) = 0 \). We can write
\[
D = \sum_{r=0}^{\ell-1} D_r(a_1, \ldots, a_n) b^r + R,
\]
where \( D_r(a_1, \ldots, a_n) \in \mathbb{C}[a_1, \ldots, a_n] \) for \( r = 0, \ldots, \ell - 1 \), and \( R \in (F) \). Thus,
\[
\sum_{r=0}^{\ell-1} D_r(\ell_1^\ell, \ldots, \ell_n^\ell) w^r = 0. \quad (2.10)
\]
We claim that $D_r(a_1, \ldots, a_n) = 0$ for all $r$. Suppose not; then let $m$ be the maximal integer such that $D_m(a_1, \ldots, a_n) \neq 0$. Let $x_1^{\ell_1} \cdots x_n^{\ell_n}$ be a monomial in $D_m(x_1^\ell, \ldots, x_n^\ell)$ with nonzero coefficient. Since $0 \leq m < \ell$, when we write the left hand side of (2.10) in terms of the PBW basis, the coefficient of $x_1^{\ell_1+m} \cdots x_n^{\ell_n+m}$ is nonzero, a contradiction. Hence, the kernel of $\Phi$ is $(F)$. This proves (1.5).

\textbf{Remark 2.3.} When $n = 3$, the algebra $H$ is Morita equivalent to a deformed Sklyanin algebra $S_{\text{def}}$ defined by C. Walton in [8, Definition IV.2]. More precisely, if $n = 3$ and

$$e = \frac{1}{\ell} \sum_{r=0}^{\ell-1} g_1^r,$$

one has $HeH = H$ and $eHe \cong S_{\text{def}}$ where the parameters for $S_{\text{def}}$ (following the notations in [8, Definition IV.2]) are $a = 1$, $b = \zeta$, $c = d_i = 0$, and $e_i = -\zeta t_i$ for $i = 1, 2, 3$. This follows from the observation that, for $n = 3$, setting $\phi_i = x_ig_i$, one has $\phi_i\phi_{i+1} - \zeta \phi_{i+1}\phi_i = \zeta t_i$ for all $i$. The algebra $S_{\text{def}}$ (with above parameters) was first studied by M. Douglas and B. Fiol, see [3, (3.10)]. Our formulas (1.1)–(1.2) are a generalization of [3, (4.6)], and our equation (1.3) is a generalization of [3, (4.7)]. The formulas in (2.1)–(2.3) are generalizations of [3, (4.8)].

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\textbf{References}


