An Introduction to 3-Dimensional Contact Topology

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Preface

In the summer of 2001, I taught a short course on contact geometry as a part of the Fourth Special Mathematics Lectures in Peking University. This short course was meant to be an introduction to the subject, covering mainly the study of 3-dimensional contact geometry before earlier 1990's. In the past decade, the study of contact geometry entered into a new stage with the advent of Gromov-Witten theory. The so-called Symplectic Field Theory of Eliashberg, Givental, and Hofer relates the studies of contact geometry and symplectic geometry in a much deeper and boarder setting. In the meantime, the study of 3-dimensional contact geometry using geometric and topological methods is also flourishing. It was with these new developments of contact geometry in mind that we hope to offer this book, which is based on my lecture notes of the aforementioned short course, to a wider readership as an introduction to this fascinating subject.

Originated in Hamiltonian mechanics and geometric optics, contact geometry caught geometers' attention very earlier. In 1953, S.-S. Chern showed that the structure group of a contact manifold M^{2n+1} can be reduced to the unitary group $U(n)$ and therefore all of its odd characteristic classes vanish[1]. Since all the characteristic classes of a closed, orientable 3-manifold vanish, Chern posed the question of whether such a manifold always admits a contact structure and whehter there are non-isomorphic contact structures on one manifold in 1966.

One of the milestones in the study of contact geometry is Bennequin's proof of the existence of exotic contact structures (i. e., contact structures not isomorphic to the standard one) on $\mathbb{R}^3$. Bennequin recognized that the induced singular foliation on a surface embedded in a contact 3-manifold plays a crucial role for the classification of contact structures. This role was further explored in the work of Eliashberg, who distinguished two classes of contact structures in dimension 3, overtwisted and tight, and gave a homotopy classification for overtwisted contact structures on 3-manifolds. Eliashberg also proved that on $\mathbb{R}^3$ and S^3, the standard contact structure is the only tight contact structure. For the majority of 3-manifolds, the classification of tight contact structures is unknown. These works of Bennequin and Eliashberg are the main topics of this book.

Studying transverse and Legendrian knots in a contact 3-manifold is the approach that Bennequin used to distinguish non-standard contact structures on $\mathbb{R}^3$ and S^3. A deep result in this direction is the so-called Bennequin's inequality, which implies that

[1]Thus, all characteristic numbers of M vanish and M is cobordant to zero.

for the standard contact structure on $\mathbb{R}^3$ and S^3, the self-linking number of a Legendrian knot has a topological upper bound. Thus the study of transverse and Legendrian knots is an interesting and rich subject in its own. We shall give an introduction to this subject.

Chapter 1

Basics

1.1 Contact structures

A contact structure may be regarded as an atlas for a smooth manifold M whose coordinate changes belong to the pseudogroup of local "contact transformations". But the modern point of view is to regard a contact structure as a nowhere integrable hyperplane distribution.

Definition 1.1. *Let M be a smooth manifold of dimension $2n-1$. A smooth 1-form α on M is called a* contact form *if $\alpha \wedge (d\alpha)^{n-1} \neq 0$ everywhere on M.*

Definition 1.2. *A smooth tangent hyperplane field (a smooth distribution of hyperplanes) $\mathcal{C}$ on M is called a* contact structure *if, for every point of M, there is a neighborhood U and a contact form α on U such that $C|_U = ker(\alpha)$.*

In the above definition, at a point $x \in U$, we consider the 1-form α as a linear functional on T_xM. Thus $\ker(\alpha)$ is a hyperplane in T_xM.

Recall that a tangent hyperplane field $\mathcal{D}$ is called *involutive* if for any two vector fields X, Y on M, $X, Y \in \mathcal{D}$ implies $[X, Y] \in \mathcal{D}$.

Lemma 1.3. *Let α be a 1-form and $\mathcal{D} = ker(\alpha)$. Then $\mathcal{D}$ is involutive iff $\alpha \wedge d\alpha \equiv 0$. Thus, following Frobenius Theorem, a contact structure is nowhere integrable.*

Proof. By definition, $\mathcal{D}$ is involutive iff $\alpha(X) = 0$ and $\alpha(Y) = 0$ imply $\alpha([X, Y]) = 0$. Since

$$\alpha([X, Y]) = -d\alpha(X, Y) + X(\alpha(Y)) - Y(\alpha(X)),$$

we see that $\mathcal{D}$ is involutive iff $\alpha(X) = 0$ and $\alpha(Y) = 0$ imply $d\alpha(X, Y) = 0$.

Now pick three vector fields X, Y, Z on M so that $X, Y \in \mathcal{D}$ and Z is not necessarily in $\mathcal{D}$. Then

$$\alpha \wedge d\alpha(X, Y, Z) = \alpha(X)\, d\alpha(Y, Z) - \alpha(Y)\, d\alpha(X, Z) + \alpha(Z)\, d\alpha(X, Y) \equiv 0$$

iff $\alpha(X) = 0$ and $\alpha(Y) = 0$ imply $d\alpha(X, Y) = 0$. □

By the proof of Lemma 1.3, we see that if α is a contact form, then $d\alpha$ is non-degenerate on the tangent hyperplane $\ker(\alpha)$.

Definition 1.4. *A contact structure $\mathcal{C}$ is called* co-oriented *if there is an oriented line field $\mathcal{L}$ such that $\mathcal{C} \oplus \mathcal{L} = TM$.*

Lemma 1.5. *A contact structure $\mathcal{C}$ is defined by a (global) contact form α by $\mathcal{C} = ker(\alpha)$ iff it is co-oriented.*

Proof. Let $\mathcal{C} = \ker(\alpha)$. There is a line field $\mathcal{L}$ such that $\mathcal{C} \oplus \mathcal{L} = TM$. The restriction of α on $\mathcal{L}$ is nowhere zero. So we may give $\mathcal{L}$ a direction so that $\alpha > 0$ in this direction. So $\mathcal{L}$ is an oriented line bundle.

No assume $\mathcal{C} \oplus \mathcal{L} = TM$ for an oriented line bundle $\mathcal{L}$. This is equivalent to have a vector field X on M so that nowhere X lies in the hyperplane in $\mathcal{C}$. Since the cotangent bundle T^*M is the dual of the tangent bundle TM, we may find a 1-form α such that $\mathcal{C} = \ker(\alpha)$ and α is a contact form. □

It is easy to verify that if α is a contact form and f is a nowhere zero smooth function on M, then $f\alpha$ is also a contact form. These two contact forms, α and $f\alpha$, determine the same contact structure on M. Furthermore, if two contact forms α and β determine the same contact structure, then $\beta = f\alpha$ for a nowhere zero smooth function f.

Definition 1.6. *Integral submanifolds of dimension $n-1$ of a contact structure on M are called* Legendrian submanifolds*. A smooth fibration of M, all of whose fibres are Legendrian, is called a* Legendrian fibration.

Example 1.7. Let $M = \mathbb{R}^3$ with Cartesian coordinates x, y, z. Then $\alpha = dz - ydx$ is a contact form. Indeed,

$$\alpha \wedge d\alpha = dx \wedge dy \wedge dz$$

is equal to the standard volume form of $\mathbb{R}^3$. We will call the contact structure $\mathcal{C}_0$ determined by this contact form the *standard contact structure* on $\mathbb{R}^3$. It is a Legendrian fibration with fibres $x, z =$ constants.

Example 1.8. Let $M = \mathbb{R}^2 \times S^1$ with coordinates $x, y, \theta \pmod{2\pi}$. Then

$$\alpha = (\cos\theta)\, dx + (\sin\theta)\, dy$$

is a contact form ($\alpha \wedge d\alpha = -dx \wedge dy \wedge d\theta$). This is the case when we have M as the space of "contact elements" of $\mathbb{R}^2$. Like the previous example, it is also a Legendrian fibration with fibres $x, y =$ constants. We will discuss such spaces in details later.

A contact structure determines an orientation of the $(2n-1)$-manifold M via the nowhere zero $(2n-1)$-form $\alpha \wedge (d\alpha)^{n-1}$. If we change the contact form to $f\alpha$, since

$$(f\alpha) \wedge (d(f\alpha))^{n-1} = f^n \alpha \wedge (d\alpha)^{n-1},$$

the derived orientation does not change when n is even. In such a case, if M is oriented, we say that a contact structure is positive if the orientation derived from the contact structure agrees with the given orientation of M. Otherwise, the contact structure is negative.

Definition 1.9. *Let α be a contact form. The* Reeb vector field *of α is a vector field R_α determined uniquely the following conditions:*

$$\alpha(R_\alpha) = 1 \qquad \textit{and} \qquad \iota_{R_\alpha}(d\alpha) = 0.$$

Recall ι_X is the contraction with respect to the vector field X. I.e., $\iota_X(d\alpha)(Y) = d\alpha(X, Y)$ for all Y. Since $d\alpha$ is an odd-dimensional anti-symmetric bilinear form, its determinant under a basis is zero. Thus, we may find a non-zero vector R_α so that $\iota_{R_\alpha}(d\alpha) = 0$. This vector can not lie in $\ker(\alpha)$ since $d\alpha$ is non-degenerate on this tangent hyperplane. Therefore $\alpha(R_\alpha) \neq 0$ and we may normalize R_α so that $\alpha(R_\alpha) = 1$.

The Lie derivative L_X with respect to X on forms satisfies the following Cantan formula:

$$L_X = \iota_X \cdot d + d \cdot \iota_X.$$

Thus the conditions characterizing the Reeb vector field can be given as

$$\alpha(X_\alpha) = 1 \qquad \text{and} \qquad L_{X_\alpha}\alpha = 0.$$

Let Φ_t be the flow of the Reeb vector field X_α. Since

$$\frac{d}{dt}(\Phi_t^*(\alpha)) = \Phi_t^*(L_{X_\alpha}(\alpha)) = 0,$$

we see that α is preserved under the flow of X_α, i.e., $\Phi_t^*(\alpha) = \alpha$. This is Liouville's theorem. We also have $L_{X_\alpha} d\alpha = 0$ and $\Phi_t^*(d\alpha) = d\alpha$. Thus, the flow of the Reeb vector field preserves the volume form $\alpha \wedge d\alpha$.

1.2 Symplectic structures

Symplectic structures and contact structures are closely related. They share the same property of having no local invariants (Darboux Theorem). In some special cases, symplectic structures and contact structures can be translated from one to another by the processes of symplectification and contactification. In recent years, exploration of this transition is proved to be one of the most fruitful approaches in the study of contact geometry.

Definition 1.10. *A* symplectic form ω *on a smooth manifold W of dimension $2n$ is a closed 2-form which is nondegenerate everywhere.*

Thus a symplectic form ω satisfies two conditions: (1) $d\omega = 0$; and (2) $\omega(X, Y) = 0$ for all Y implies $X = 0$ at every point of W.

Definition 1.11. *Let H be a smooth function of W equipped with a symplectic form ω. The* Hamiltonian vector field *X_H associated with H is determined uniquely by the condition*

$$\iota_{X_H}(\omega) = dH.$$

The condition defining a Hamiltonian vector field X_H implies

$$L_{X_H}(\omega) = \iota_{X_H}(d\omega) + d(\iota_{X_H}(\omega)) = 0.$$

So, if we let Φ_t be the flow of the Hamiltonian vector field X_H, then $\Phi_t^*(\omega) = \omega$. This is again Liouville's theorem.

Theorem 1.12. *Suppose we have two symplectic forms ω_0 and ω_1 on W. Let $x_0 \in W$ such that $\omega_0|_{x_0} = \omega_1|_{x_0}$. Then there is a neighborhood U of x_0 and a diffeomorphism $\Phi : U \to \Phi(U)$ such that $\Phi(x_0) = x_0$ and $\Phi^*(\omega_1) = \omega_0$.*

Proof. Consider the closed form $\omega_t = (1-t)\omega_0 + t\omega_1$. This is a closed 2-form and nondegenerate at x_0. Therefore, there is a neighborhood U_1 of x_0 such that ω_t is nondegenerate on U_1. Since $\omega_0 - \omega_1$ is a closed 2-form on U_1, by Poincaré's lemma, there is a 1-form α on U_1 such that

$$d\alpha = \omega_0 - \omega_1 \qquad \text{and} \qquad \alpha|_{x_0} = 0.$$

Now we define a time-dependent vector field X_t satisfying

$$\iota_{X_t}(\omega_t) = \alpha.$$

Since ω_t is nondegenerate on U_1, this condition determines X_t uniquely. We have $X_t(x_0) = 0$ for all t.

Let Φ_t be the diffeomorphism given by the flow of X_t at time t, i.e., it is determined by

$$\frac{d}{dt}\Phi_t(x) = X_t(\Phi_t(x)) \quad \text{and} \quad \Phi_0(x) = x \quad \text{for all } x \in U_1.$$

Then, we have

$$\begin{aligned}\frac{d}{dt}\Phi_t^*(\omega_t) &= \Phi_t^*\left(\frac{d}{dt}\omega_t + L_{X_t}(\omega_t)\right)\\ &= \Phi_t^*(\omega_1 - \omega_0 + \iota_{X_t}(d\omega_t) + d(\iota_{X_t}(\omega_t)))\\ &= \Phi_t^*(\omega_1 - \omega_0 + d\alpha)\\ &= 0\end{aligned}$$

Therefore,

$$\Phi_t^*(\omega_t) = \Phi_0^*(\omega_0) = \omega_0.$$

In particular, $\Phi_1(\omega_1) = \omega_0$ and $\Phi_1(x_0) = x_0$. So this is the desired diffeomorphism Φ defined in a neighborhood of x_0 □

Corollary 1.13. (Darboux's Theorem) *Let ω be a symplectic form on W. For every point $x_0 \in W$, there is a local coordinate system $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ in a neighborhood of x_0 such that $x_0 = (0, 0, \ldots, 0)$ and*

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

1.3 Symplectification and contactification

Let α be a contact form on M. Let $W = M \times \mathbb{R}$ and $t \in \mathbb{R}$ be the coordinate. Define a closed 2-form on W by

$$\omega = d(e^t \alpha) = e^t dt \wedge \alpha + e^t d\alpha.$$

Since

$$(\omega)^n = e^{nt} dt \wedge \alpha \wedge (d\alpha)^{n-1},$$

we get a symplectic form on W. This is called the *symplectification* of the contact form α.

Consider a function $H : W \to \mathbb{R}$ such that $dH = -e^t dt$. Let X_α be the Reeb vector field associated with the contact form α. The Hamiltonian vector field of H is denoted by X_H.

Lemma 1.14. *We have*

$$X_H = X_\alpha.$$

Proof. We have

$$(e^t dt \wedge \alpha + e^t d\alpha)(X_\alpha, \partial/\partial t) = -e^t dt(\partial/\partial t)\, \alpha(X_\alpha) = -e^t,$$

and

$$(e^t dt \wedge \alpha + e^t d\alpha)(X_\alpha, Y) = e^t d\alpha(X_\alpha, Y) = 0$$

for every vector field Y of M. Thus

$$d(e^t \alpha)(X_\alpha, \cdot) = -e^t dt(\cdot) = dH(\cdot)$$

and $X_H = X_\alpha$. □

For a symplectic form ω on a manifold W of dimension $2n$, a vector field X is called *Liouville vector field* if $L_X(\omega) = \omega$, or equivalently $d(\iota_X(\omega)) = \omega$. Thus, if we let

$$\alpha = \iota_X(\omega),$$

this gives rise to a contact form on every submanifold $M \subset W$ of dimension $2n - 1$ transverse to X. Indeed,

$$\alpha \wedge (d\alpha)^{n-1} = (\iota_X(\omega)) \wedge (d(\iota_X(\omega)))^{n-1} = \iota_X(\omega) \wedge \omega^{n-1} = \frac{1}{n}\, \iota_X(\omega^n),$$

which is nondegenerate on M. This is called the *contactification* of the symplectic form ω.

Consider the case of $W = M \times \mathbb{R}$ and $\omega = d(e^t\alpha)$ being the symplectification of the contact form α on M.

Lemma 1.15. *$X = \frac{\partial}{\partial t}$ is a Liouville vector field of the symplectic form $\omega = d(e^t\alpha)$. Moreover the contactification of this symplectic form on $M \times \{t\}$ is $e^t\alpha$.*

Proof. We check the condition of X being a Liouville vector field:

$$L_X(\omega) = d(\iota_X(\omega)) = d(\iota_X(e^t dt \wedge \alpha + e^t d\alpha)) = d(e^t\alpha) = \omega.$$

This also shows that $\iota_X(\omega) = e^t\alpha$. ☐

Definition 1.16. *Vector fields whose local flows preserve a contact structure are called* contact vector fields. *Let α be a contact form on M, a smooth function $K : M \to \mathbb{R}$ is called a* contact Hamiltonian *if there is a contact vector field V such that $K = \iota_V(\alpha) = \alpha(V)$.*

Lemma 1.17. *The correspondence $K = \iota_V(\alpha)$ maps the space of contact vector fields bijectively onto the space of smooth functions.*

Proof. Given a smooth function K on M with a contact form α, we want to find a contact vector field V such that $K = \iota_V(\alpha)$. Let R_α be the Reeb vector field of α. The following two equations

$$\iota_X(\alpha) = 0 \quad \text{and} \quad \iota_X(d\alpha) = \iota_{R_\alpha}(dK) \cdot \alpha - dK$$

determines a vector field X uniquely. Notice that the right hand side of the second equation above vanishes on R_α. So the second equation is an equation in $\ker(\alpha)$.

Now let $V = KR_\alpha + X$. We have

$$L_V(\alpha) = K \cdot L_{R_\alpha}(\alpha) + dK \cdot \iota_{R_\alpha}(\alpha) + L_X(\alpha) = dK + \iota_X(d\alpha) = \iota_{R_\alpha}(dK) \cdot \alpha$$

So V is a contact vector field. We have $\alpha(V) = K$ and K is a contact Hamiltonian, ☐

Denote by $S = \{x \in M\,;\, K(x) = 0\}$ and assume that it is a smooth submanifold of M. The tangent space of S is $\ker(dK)$. If $x \in S$ such that $\ker(\alpha) = \ker(dK)$ at x, then by the construction of Lemma 1.17, $V(x) = 0$ and x is a fixed point of the contact flow Φ_t.

1.4 Local and homotopy rigidity of contact structures

Just like symplectic structures, contact structures are all locally isomorphic.

Theorem 1.18. (Darboux's Theorem) *Let* α *be a contact form on* M. *For every point* $x_0 \in M$, *there is a local coordinate system* $(z, p_1, \ldots, p_n, q_1, \ldots, q_n)$ *in a neighborhood of* x_0 *such that* $x_0 = (0, 0, \ldots, 0)$ *and*

$$\alpha = dz - \sum_{i=1}^{n} q_i dp_i.$$

Proof. This is a consequence of Darboux Theorem for symplectic forms (Corollary 1.13).

Suppose α is a contact form defined in a neighborhood U of $x_0 \in M$, with R_α its Reeb vector field. We can form the quotient smooth manifold U/R_α such that $\pi : U \to U/R_\alpha$ is a trivial principal $\mathbb{R}$-bundle with the connection 1-form α (so that its curvature is $d\alpha$). The push-forward of $d\alpha$, $\omega = \pi_*(d\alpha)$, is a symplectic form on U/R_α. Thus, there is a local coordinate system $(p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1})$ on U/R_α, with $\pi(x_0) = (0, 0, \ldots, 0)$ in this coordinate system, such that

$$\omega = \sum_{i=1}^{n-1} dp_i \wedge dq_i.$$

We use z to denote the coordinate of the fibre through x_0, $z(x_0) = 0$. Starting at x_0 , we lift each coordinate axis of U/R_α to U as a geodesic with respect to the connection form α. This gives us a coordinate system (with notation abused) $(z, p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1})$ in a neighborhood of x_0, which has coordinates all equal to 0. In this coordinate system

$$\alpha = dz - \sum_{i=1}^{n-1} q_i \, dp_i.$$

□

One of the earlier significant results in contact geometry is the following theorem of Gray. It was proved first along the line of Kodaira-Spencer deformation theory. The modern proof of this theorem follows the so called *Moser trick* which is much simpler than the original proof.

Theorem 1.19. (Gray) *If* α_t, $t \in [0, 1]$, *is a smooth 1-parameter family of contact forms on a closed smooth manifold* M, *then there is a diffeomorphism* $\Phi : M \to M$ *(isotopic to identity) such that* $\Phi^*(\alpha_1) = f\alpha_0$ *for some function* $f \neq 0$.

Proof. We want to construct a time-dependent vector field X_t such that its induced diffeomorphism Φ_t satisfies $\Phi_t^*(\alpha_t) = \lambda_t \alpha_0$ for some positive function λ_t on M. Differentiate both sides of the required equality, we get

$$\frac{d}{dt}\Phi_t^*(\alpha_t) = \Phi_t^*(\dot{\alpha}_t + L_{X_t}(\alpha_t)) = \dot{\lambda}_t \alpha_0 = \frac{\dot{\lambda}_t}{\lambda_t}\Phi_t^*(\alpha_t).$$

Let

$$\mu_t = \frac{d}{dt} \log \lambda_t \circ \Phi_t^{-1},$$

then X_t has to satisfy

$$\Phi_t^*(\dot{\alpha}_t + d(\alpha_t(X_t)) + \iota_{X_t}(d\alpha_t)) = \Phi_t^*(\mu_t\alpha_t).$$

Assume further $X_t \in \ker(\alpha_t)$, we get

$$\dot{\alpha}_t + \iota_{X_t}(d\alpha_t) = \mu_t\alpha_t.$$

Let R_{α_t} be the Reeb vector field of α_t, we get

$$\dot{\alpha}_t(R_{\alpha_t}) = \mu_t.$$

Thus μ_t is completely determined and so is X_t. Since M is closed, we can get the diffeomorphism Φ_t by solving

$$\frac{d}{dt}\Phi_t(x) = X_t(\Phi_t(x)) \quad \text{for all } x \in M, \quad \Phi_0 = \mathrm{Id}$$

for all t. Eventually, we have

$$\lambda_t(x) = \exp\left(\int_0^t \mu_t(\Phi_t(x))\, dt\right).$$

□

Corollary 1.20. *Two (co-orientable) contact structures on a closed smooth manifold M that are homotopic in the space of contact structures on M are isotopic.*

1.5 Almost contact structures and Gromov's theorem

Denote by TM the tangent bundle of M. Let ξ be a co-orientable, co-dimension 1 subbundle of TM with a complex structure $J : \xi \to \xi$ ($J^2 = -\mathrm{Id}$). The pair $\{\xi, J\}$ is called an *almost contact structure* of M.

If M admits an almost contact structure, then the structure group of M can be reduced to $U(n-1)$.

If α be a contact form on M, then $\xi = \ker(\alpha)$ is a co-orientable, co-dimension 1 subbundle of TM. Moreover, $d\alpha$ is a symplectic form on ξ. Then there is a complex structure J on ξ, unique up to homotopy, that is compatible with the symplectic structure $d\alpha$, i.e.

$$d\alpha(JX, JY) = d\alpha(X, Y) \quad \text{for all } X, Y \in \xi$$

and

$$d\alpha(X, JX) > 0 \quad \text{if } X \neq 0.$$

This pair $\{\xi, J\}$ is the almost contact structure induced by the contact form α.

Theorem 1.21. (Gromov) *If M is open, then two co-oriented contact structures on M are homotopic iff their induced almost contact structures are homotopic.*

This theorem is a consequence of Gromov's *h-principle*.

1.6 More examples

Example 1.22. Consider the 3-dimensional Lie group $G = SO(3)$, the rotation group of the Euclidean space $\mathbb{R}^3$. Its Lie algebra $\mathfrak{g} = T_1G$ has a basis:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We have

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y.$$

Thus, translating the subspace of $\mathfrak{g}$ spanned by X, Y, say, over G by left multiplication gives rise to a contact structure on G.

Let $\gamma(s)$ be an arclength parameterized curve in $\mathbb{R}^3$. Let $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$ be the Frenet framing of $\gamma(s)$. Consider $\hat{\gamma}(s) = (\mathbf{T}(s), \mathbf{B}(s), \mathbf{N}(s))$ as a curve in $SO(3)$. Since

$$(\mathbf{T}'(s), \mathbf{N}'(s), \mathbf{B}'(s)) = (\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)) \begin{pmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix},$$

where $\kappa(s)$ and $\tau(s)$ are curvature and torsion of $\gamma(s)$, respectively, we conclude that $\hat{\gamma}(s)$ is a Legendrian curve in $SO(3)$.

Example 1.23. (Lutz) On $S^1 \times \mathbb{R}^2$, we have the coordinate system (θ, r, ϕ), where θ is the coordinate of S^1 and (r, ϕ) is the polar coordinates of $\mathbb{R}^2$. Let

$$\alpha = d\theta + r^2 d\phi.$$

This is a contact form on $S^1 \times \mathbb{R}^2$. Indeed, with Cartesian coordinates

$$x^2 + y^2 = r^2, \quad \arctan \frac{y}{x} = \phi,$$

we have

$$\alpha = d\theta + xdy - ydx$$

and

$$\alpha \wedge d\alpha = 2d\theta \wedge dx \wedge dy.$$

The roles played by θ and ϕ can in fact be switched. This is an important fact for the study of contact structures in dimension 3. Let consider 1-forms of the form

$$\alpha = h_1(r)d\theta + h_2(r)d\phi.$$

We have

$$\alpha \wedge d\alpha = \det \begin{pmatrix} h_1(r) & h_2(r) \\ h_1'(r) & h_2'(r) \end{pmatrix} d\theta \wedge dr \wedge d\phi.$$

So, as far as the curve $r \mapsto (h_1(r), h_2(r)) \in \mathbb{R}^2$ has the property that its position and velocity vectors never parallel, α is a contact form.

We may now change the contact form $\alpha = d\theta + r^2\phi$ by changing the curve $(1, r^2)$ to $(h_1(r), h_2(r))$ so that it spirals around $(0,0)$ several times and still ends at $(1,0)$. This new contact form is so-called "overtwisted". These contact forms are in fact not homotopic if the number of twists are different.

On the other hand, we can have the new curve $(h_1(r), h_2(r))$ equal to $(1, r^2)$ for r large but equal to $(-r^2, 1)$ for r small. This is the case when the roles played by θ and ϕ are getting switched and "contact surgery" becomes possible.

1.7 Existence of contact structures in dimension 3

The existence of a contact structure on every closed, orientable 3-manifold was asked by S.-S. Chern in 1966 [3]. The question was raised because the obvious obstruction for the existence of a contact strucuture vanishes for a closed, orientable 3-manifold.

Theorem 1.24. (Lutz-Martinet) *There is a contact structure on every closed orientable 3-manifold.*

This theorem was first proved by Lutz and Martinet in 1971. The proof given here is due to Thurston and Winkelnkemper. It is based on the following theorem of Alexander on a construction which produces all closed orientable 3-manifold.

Theorem 1.25. *Every closed, smooth, orientable 3-manifold M is diffeomorphic to*

$$F(h) \cup_f \partial F \times D^2,$$

where F is an orientable, compact 2-manifold with one boundary component ∂F, $h : F \to F$ is a diffeomorphism which restricts to the identity on ∂F; $F(h)$ denotes the mapping torus of h, i.e. the 3-manifold with boundary obtained from $F \times [0,1]$ by identifying $(x, 0)$ with $h(x), 1)$, and $f : \partial F(h) \to (S^1 \times D^2)$ is a diffeomorphism.

Proof of Theorem 1.24. We may assume that h fixes a collar neighborhood of C of ∂F, and C has coordinates (t, θ) with $t = 0$ for points on ∂F. We have $h(t, \theta) = (t, \theta)$.

Let α_1 be a 1-form on F equal to $(1 + t)d\theta$ on C. By Stokes' theorem, $\int_F d\alpha_1 = 1$. If Ω is any volume form of F which equals to $dt \wedge d\theta$ in C, then, by de Rham's theorem, $\Omega - d\alpha_1 = d\beta$, where β is a 1-form which is zero in C. Let $\alpha_2 = \alpha_1 + \beta$. Then α_2 satisfies (1) $d\alpha_2$ is a volume form, and (2) $\alpha_2 = (1 + t)d\theta$ in C.

The set of 1-forms on F satisfying (1) and (2) above is convex. So we may joint α_2 and $h^*(\alpha_2)$ by a 1-parameter family of 1-forms on F satisfying (1) and (2), and think of this 1-parameter family as a 1-form α on $F(h)$. Since $C \times S^1$ is a collar neighborhood of $\partial F(h)$, we may use (t, θ, ϕ) as the coordinates of this collar neighbor hood of $\partial F(h)$.

Notice that $F(h)$ is the total space of a F-bundle over S^1. We denote by $d\phi$ the 1-form on $F(h)$ which is the pull-back of $d\phi$ on S^1. Then $d\phi \wedge d\alpha$ is a volume form of $F(h)$.

Let $\gamma = \alpha + K d\phi$. Since

$$\gamma \wedge d\gamma = K(d\phi \wedge d\alpha) + \alpha \wedge d\alpha,$$

when K is sufficiently large, γ is a contact form on $F(h)$. In the collar neighborhood $C \times S^1$ of $\partial F(h)$, $\gamma = (1+t)d\theta + K d\phi$.

Let θ_1, r, ϕ_1 be the coordinates of $S^1 \times D^2$. We have a contact form $\beta = d\theta_1 + r^2\, d\phi_1$ on $S^1 \times D^2$. Let $f : \partial F(h) \to S^1 \times S^1$ be given by

$$f^{-1} : (\theta_1, \phi_1) \longrightarrow (\theta, \phi) \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = \pm 1.$$

Then $f^*(\beta) = (a+b)\, d\theta + (c+d)\, d\phi$.

Pick a small $\epsilon > 0$ and let $t \in [-\epsilon, \epsilon]$, we may now choose a curve $(h_1(t), h_2(t))$ such that

(1) $(h_1(t), h_2(t)) = (1+t, K)$ when $t \geq 0$, and $(h_1(-\epsilon), h_2(-\epsilon)) = (a+b, c+d)$;

(2) $(h_1(t), h_2(t))$ and $(h_1'(t), h_2'(t))$ are linearly independent for all t.

Then the contact form $h_1(t)\, d\theta + h_2(t)\, d\phi$ is defined in a thin neighborhood of $\partial F(h)$ in $M = F(h) \cup_f S^1 \times D^2$ which glues together contact forms α on $F(h)$ and β on $S^1 \times D^2$ to give us a contact form on M. □

Chapter 2

Bennequin

2.1 Introduction

Lutz showed in his thesis of 1971 that in every homotopy class of co-oriented tangent hyperplane fields on S^3, there is a contact structure. Lutz's examples are basically these described in Example 1.23. Since the homotopy classes of co-oriented tangent hyperplane fields on S^3 can be identified with the homotopy group $\pi_3(S^2) \cong \mathbb{Z}$, we see that there are infinitely many non-isomorphic contact structures on S^3.

When M is of dimension 3, homotopy classes of co-oriented tangent hyperplane fields on M can be identified with homotopy classes of almost contact structures. Indeed, assuming that M is equipped with a Riemannian metric, the complex structure on a co-oriented tangent hyperplane field can be given as a count-clockwise rotation of 90° in each hyperplane. And two almost contact structures n M are homotopic iff their underlying co-oriented tangent hyperplane fields are homotopic. Thus the isotopic classification problem of contact structures in dimension 3 could be addressed as follows: Is it true that two co-oriented contact structures within the same homotopy classes of co-oriented tangent hyperplane fields necessarily isotopic?

For closed 3-dimensional manifolds, due to Gray's theorem (Theorem 1.19), the question above can be reduced to the question of whether two contact structures are homotopic if they are in the same homotopy class of co-oriented tangent hyperplane fields. For open manifolds, due to Gromov's theorem (Theorem 1.21), the question above becomes whether there are homotopic but non-isotopic co-oriented contact structures. These questions were answered by Bennequin in 1982 for S^3 and $\mathbb{R}^3$. Thus, according to Bennequin, there are non-homotopic contact structures on S^3 whose underlying hyperplane fields are homotopic. And there are homotopic but non-isotopic contact structures on $\mathbb{R}^3$.

Let $\mathcal{C}$ be a contact structure of a 3-dimensional smooth manifold M. Let S be a smooth 2-dimensional surface and $f : S \to M$ be an immersion. For every point $z \in S$, the tangent map $Tf : T_zS \to T_{f(z)}M$ is injective. This immersion is in general position with respect to the contact structure $\mathcal{C}$ if the points $z \in S$ where $Tf(T_zS) = \mathcal{C}_{f(z)}$ are all isolated.

If the immersion f is in general position, we may get a line field on S away from some isolated points as follows. For a point z where $\dim(Tf(T_zS) \cap \mathcal{C}_{f(z)}) = 1$, we simply bring back this tangent line to M at $f(z)$ to a tangent line to S at z via Tf. We may think of this line field as a *singular line field* on S.

Integral curves of this singular line field on S gives rise to a *singular foliation* on S, where the leave at an isolated singular point is degenerated to a point.

Theorem 2.1. (Bennequin) *Let $\mathcal{C}_0$ be the standard contact structure on $\mathbb{R}^3$. For every imbedded disk V in $\mathbb{R}^3$ in general position with respect to $\mathcal{C}_0$, the induced singular foliation on D has no closed leaves except for its singular points.*

Example 2.2. Let us now consider the contact form

$$\alpha = r \sin r \, d\phi + \cos r \, dz$$

on $\mathbb{R}^3$ with cylindrical coordinates (z, r, ϕ). On the paraboloid $z = r^2$, the normal vector of the paraboloid is

$$N = \frac{\partial}{\partial z} - 2r \frac{\partial}{\partial r}.$$

So we have $\alpha(N) = 2r \cos r$. Thus, in a small neighbor hood of the the curve $r = \pi$ on the paraboloid $z = r^2$, $\alpha(N) \neq 0$. Therefore, the curve $r = \pi$, $z = r^2$ is a closed leave of the singular foliation on a disk imbedded in $\mathbb{R}^3$. By Theorem 2.1, the contact structure on $\mathbb{R}^3$ determined by this contact form α is not isomorphic to the standard contact structure on $\mathbb{R}^3$.

Theorem 2.1 is closely related with the subject of knot theory. If V is a disk imbedded in $\mathbb{R}^3$, the boundary of V, ∂V, is an imbedding of S^1 in $\mathbb{R}^3$ called an *unknotted curve*, or simply an *unknot*. The reason is that there are many other imbeddings of S^1 in $\mathbb{R}^3$, called *knotted curves*, or simply *knots*, which can not be realized as the boundary of imbedded disks. The curve γ given by $r = \pi$, $z = r^2$ in Example 2.2 is unknotted as well as Legendrian in the contact structure of Example 2.2. Let R_α be the Reeb vector field of the contact form α in Example 2.2, let $\gamma' = \gamma + \epsilon R_\alpha$ be a curve obtained from γ by a small (ϵ) shift in the direction of R_α. Give the curve γ an orientation. The γ' inherits an orientation from γ. It is easy to check that the *Gauss linking number* of γ and γ', $\mathrm{lk}(\gamma, \gamma')$, is zero. The following theorem says that such a situation never happens in $\mathbb{R}^3$ with the standard contact structure.

Theorem 2.3. (Bennequin) *Let γ be a unknotted Legendrian curve in $\mathbb{R}^3$ with respect to the standard contact structure $\mathcal{C}_0$. Let γ' be obtained from γ by a small shift in the direction of the Reeb vector of $\mathcal{C}_0$. Then* $\mathrm{lk}(\gamma, \gamma') < 0$.

We can also use Theorem 2.3 to conclude that the contact structure in Example 2.2 is not isomorphic to the standard contact structure. In fact, Theorem 2.1 is a consequence of Theorem 2.3. This claim is not hard to verify.

2.2 The Hopf contact structure

We shall use the cylindrical coordinates (r, ϕ, z) in $\mathbb{R}^3$ with the transformation from and to the Cartesian coordinates (x, y, z) given by

$$\phi = \arctan y/x \quad \text{and} \quad r = \sqrt{x^2 + y^2}.$$

The standard contact structure $\mathcal{C}_0$ on $\mathbb{R}^3$, under the Cartesian coordinates (x, y, z), is determined by the equation

$$dz - y\,dx = 0.$$

Lemma 2.4. *Under the cylindrical coordinates (r, ϕ, z), the standard contact structure $\mathcal{C}_0$ on $\mathbb{R}^3$ is isomorphic to the contact structure $\hat{\mathcal{C}}_0$ determined by the contact form $dz + r^2\,d\phi$.*

Proof. We have

$$d(z + xy) - (2y)\,dx = dz + x\,dy - y\,dx = dz + r^2\,d\phi.$$

Thus $(x, y, z) \mapsto (x, 2y, z + xy)$ is the contactomorphism between $\mathcal{C}_0$ and $\hat{\mathcal{C}}_0$. $\square$

On each cylinder $r =$ constant, we may solve the equation $dz + r^2\,d\phi = 0$ to get its integral curves

$$z = -r^2\phi + \text{constant}.$$

Exercise 2.5. Show that the integral curve of the Reeb vector field of $\hat{\mathcal{C}}_0$ passing through (a, b, c) is given by $x = ae^t, y = be^t, z = t + c$.

The standard contact structure $\mathcal{C}_0$ or $\hat{\mathcal{C}}_0$ has a more intrinsic definition coming from complex geometry. Let $\mathbb{C}^2$ be equipped with the Hermitian inner product

$$<(z_1, z_2)\,,\,(w_1, w_2)> = z_1\bar{w}_1 + z_2\bar{w}_2.$$

Then S^3 is given as the subset of $\mathbb{C}^2$ specified by the equation $|(z_1, z_2)| = 1$. For each point $(z_1, z_2) \in S^3$, the complex line

$$L_{(z_1,z_2)} = \{(w_1, w_2)\,;\, z_1\bar{w}_1 + z_2\bar{w}_2 = 0\}$$

perpendicular to (z_1, z_2) can be thought of as a tangent hyperplane of S^3. Thus we get a tangent hyperplane field on S^3.

If we use the polar coordinates, $z_1 = r_1e^{i\phi_1}$ and $z_2 = r_2e^{i\phi_2}$, S^3 is given by the equation $r_1^2 + r_2^2 = 1$ and the tangent hyperplane $L_{(z_1,z_2)}$ is given by the equation

$$0 = z_1\,d\bar{z}_1 + z_2\,d\bar{z}_2 = \frac{1}{2}\,d(r_1^2 + r_2^2) - i(r_1^2\,d\phi_1 + r_2^2\,d\phi_2) = -i(r_1^2\,d\phi_1 + r_2^2\,d\phi_2).$$

Therefore the tangent hyperplane field $L_{(z_1,z_2)}$ is a contact structure on S^3 with the contact form $r_1^2\,d\phi_1 + r_2^2\,d\phi_2$. We will call this contact structure on S^3 the *Hopf contact structure*.

The Hopf fibration of S^3 is given by the circle action

$$e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2).$$

So the tangent direction of the circle action at the point (z_1, z_2) is (iz_1, iz_2), which is perpendicular to the tangent hyperplane $L_{(z_1,z_2)}$ at that point.

Consider the 2-sphere $S^2 = \{(z_1, z_2)\,;\, -1 \le z_2 \le 1\}$ in S^3. Each component of $S^3 \setminus S^2$ is an open 3-dimensional ball with center $z_1 = \pm 1$, respectively. The contact structure in each of these components given by $L_{(z_1,z_2)}$ can be described by the equation

$$\frac{r_1^2}{r_2^2}\, d\phi_1 + d\phi_2 = 0.$$

So in each these components, the contact structure is isomorphic to the standard contact structure on $\mathbb{R}^3$.

Theorem 2.6. *There is a contact flow Φ_t on S^3 of the Hopf contact structure with following properties:*

(1) It has exactly two fixed points $z_1 = \pm 1$;

(2) For every epsilon > 0, there is a $T > 0$, such that the contactomorphism Φ_T sends the complement of the ball neighborhood of $z_1 = -1$ of radius ϵ into the ball neighborhood of $z_1 = 1$ of radius ϵ.

Proof. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Consider the smooth function $K(z_1, z_2) = y_1$. We define the contact flow Φ_t to be the flow of the contact vector filed V with K as its contact Hamiltonian. The subset $K = 0$ of S^3 is the 2-sphere

$$\{(z_1, z_2) \in S^3\,;\, -1 \le z_1 \le 1\}.$$

Since

$$\alpha := r_1^2\, d\phi_1 + r_2^2\, d\phi_2 = x_1\, dy_1 - y_1\, dx_1 + x_2\, dy_2 - y_2\, dx_2,$$

we see that $z_1 = \pm 1$ are the only fixed points of the flow Φ_t.

We may use the construction of Lemma 1.17 to get an explicit expression for the contact vector field with K as its contact Hamiltonian. First of all, the Reeb vector field of the Hopf contact structure is

$$R_\alpha = -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_2}.$$

Next, since $d\alpha = 2\, dx_1 \wedge dy_1 + 2\, dx_2 \wedge dy_2$, the solution of

$$\iota_X(\alpha) = 0 \quad \text{and} \quad \iota_X(d\alpha) = \iota_{R_\alpha}(dK) \cdot \alpha - dK = x_1 \alpha - dy_1$$

is

$$X = (x_1^2 + y_1^2 - 1) \frac{\partial}{\partial x_1} + (x_1 x_2 + y_1 y_2) \frac{\partial}{\partial x_2} + (x_1 y_2 - x_2 y_1) \frac{\partial}{\partial y_2}.$$

Thus, the contact vector field $V = KR_\alpha + X$ is

$$V = (x_1^2 - 1)\frac{\partial}{\partial x_1} + x_1 y_1 \frac{\partial}{\partial y_1} + x_1 x_2 \frac{\partial}{\partial x_2} + x_1 y_2 \frac{\partial}{\partial y_2}.$$

One may check the conclusion of the theorem by using this explicit form of the contact vector field. □

2.3 Links, braids, and closed braids

A *link* consists of finitely many disjointly imbedded oriented circles in $\mathbb{R}^3$. A *closed braid* around the z-axis is a link disjoint from the z-axis so that when one moves on each circle in the direction of the orientation, the angle ϕ increases monotonically. Thus a closed braid will intersect any half plane $\phi =$ constant with the same number of points. We call this number n the *braid index*.

Closed braids are *closures* of *braids*, and the latter form a group B_n called the *braid group* under the equivalence relation of isotopy. The braid group B_n admits the following standard presentation with generators and relations, all of them having clear geometric meaning:

generators: $\sigma_2, \sigma_2, \ldots, \sigma_{n-1}$

relations: $\sigma_i\sigma_j = \sigma_j\sigma_i,\ |i = j| \leq 2; \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

The following are two basic theorems about the relationship between links and braids.

Theorem 2.7. (Alexander) *Every link in $\mathbb{R}^3$ is isotopic to a closed braid.*

Theorem 2.8. (Markov) *Let $b \in B_n$ and $b' \in B_m$ be two braids. Their closures are isotopic as links in $\mathbb{R}^3$ iff b can be changed to b' by a finite sequence of the following two elementary operations and their inverses (called* Markov moves*):*

(1) Conjugation in a braid group;

(2) Transform $w \in B_k$ to $w\sigma_k^{\pm 1} \in B_{k+1}$.

For a braid $b \in B_n$ written as a words in the generators $\sigma_2, \sigma_2, \ldots, \sigma_{n-1}$:

$$b = \sigma_{i_2}^{\epsilon_2}{\sigma_2}^{\epsilon_2} \cdots \sigma_{i_k}^{\epsilon_k},$$

the exponent sum, called the *algebraic length* of the braid b,

$$e = \epsilon_2 + \epsilon_2 + \cdots + \epsilon_k \in \mathbb{Z},$$

is well defined. It is gives rise to a homomorphism $B_n \to \mathbb{Z}$, which can be considered as the abelianization of B_n.

For every link L in $\mathbb{R}^3$, there is a connected, compact, orientable surface S imbedded in $\mathbb{R}^3$ such that $\partial S = L$. This is called a *Seifert surface* of L. Of course L may have many different Seifert surface, we define $\chi(L)$ be the largest Euler characteristic of Seifert surfaces of L. We called $\chi(L)$ the *characteristic* of L. In the case when L is a knot, i.e. it has only one component, $\chi(L) = 1 - 2g$ where g is called the *genus* of the knot.

The diffeomorphism $\mathbb{R}^3 \to \mathbb{R}^3 : (r, \phi, z) \mapsto (r, \phi, -z)$ sends a closed braid to a closed braid. The image of a closed braid under this diffeomorphism is called the mirror image of that closed braid. Let $\{\chi, n, e\}$ be the characteristic, braid index, and algebraic length of a closed braid, respectively, then for its mirror image, the corresponding set of numbers is $\{\chi, n, -e\}$.

Theorem 2.9. (Bennequin inequality) *Let L be a closed braid with χ its characteristic, n its braid index, and e its algebraic length, then $|e| - n \leq -\chi$.*

Corollary 2.10. *If L is an unknotted closed braid, then $|e| < n$.*

2.4 Transversal links and transversal closed braids

Definition 2.11. *Let $\mathcal{C}$ be a contact structure on M. A smooth curve $\gamma = \gamma(t)$ in M is* transverse *to the contact structure if the tangent vector $\dot{\gamma}$ is transverse to the hyperplane of $\mathcal{C}$ at every point $\gamma(t)$ on the curve.*

If each component of a link L is transverse to a contact structure, L is called a *transverse link*.

Assume that the contact structure $\mathcal{C}$ is co-oriented, or it is determined by a contact form α. For a transverse link L, if the direction of a component agrees with the co-orientation of the underlying tangent hyperplane field of $\mathcal{C}$, this component is called *increasing*. In the case when $\mathcal{C}$ is determined by a contact form α, a component of a transverse link L is increasing iff $\alpha(\dot{\gamma}(t)) > 0$ for every t, where $\gamma(t)$ is a parameterization of the (directed) component of L in question.

A contact structure is called *parallelizable* if its underlying tangent hyperplane field is trivial. If a contact structure is given as $\ker(\alpha)$ for a contact form α, then it is parallelizable iff there is a vector field X such that $0 \neq X \in \ker(\alpha)$ everywhere.

Lemma 2.12. *Let M be a closed orientable 3-manifold. A contact structure $\ker(\alpha)$ in M is parallelizable if $H_1(M; \mathbb{Z}) = 0$.*

Proof. Suppose that M is triangulated. We may construct a vector field X on the 1-skeleton first such that $0 \neq X \in \ker(\alpha)$ everywhere. This gives rise a 2-cocycle whose value on each 2-simplex is the rotation number of X around the boundary of this 2-simplex. One can show that if this 2-cocycle is homologous to zero, after some modification, X can be extended to the 2-skeleton with the same property. There is no obstruction to extend it further to the whole M. By Poincaré duality $H_1(M; \mathbb{Z}) = H^2(M; \mathbb{Z})$, we can find a vector field X on M such that $0 \neq X \in \ker(\alpha)$ everywhere. ∎

Suppose M is oriented and is equipped with a parallelizable contact structure. Let K be a transverse knot that is homologically trivial in M. We can define a self-linking number $l(K)$ as follows:

Since the underlying tangent hyperplane bundle of the contact structure is trivial, we can have a non-zero vector field X on M such that $X \in \ker(\alpha)$ everywhere. We may get another knot K' by push K in the direction of the vector field X slightly so that K and K' are disjoint. Now, by assumption, K is homologically trivial in M, so the linking number $\mathrm{lk}(K, K')$ is defined. By definition, $l(K) = \mathrm{lk}(K, K')$.

Lemma 2.13. *$l(K)$ is independent of the choice of the vector field X.*

Proof. Let Y be another non-zero vector field in $\ker(\alpha)$. Let J be the almost complex structure on $\ker(\alpha)$ compatible with the symplectic form $d\alpha$. Without loss of generality, we may assume $d\alpha(X, JX) = d\alpha(Y, JY) = 1$. Then $Y = e^{i\theta}X$, where $e^{i\theta}$ defines a map $M \to S^1$. Since K is homologically trivial in M, this map restricted on K is homotopically trivial. Thus we may get a family of push-offs K_t of K so that $K_1 = K'$ is the push-off of K in the direction of X, and K_2 is the push-off of K in the direction of Y. Therefore $\mathrm{lk}(K, K') = \mathrm{lk}(K, K_2)$. □

Example 2.14. Consider the contact form $dz + r^2\, d\phi$ on $\mathbb{R}^3$ with the cylindrical coordinates (r, ϕ, z). The corresponding contact structure is denoted by $\hat{\mathcal{C}}_0$.

A curve $\gamma = \gamma(t) = (r(t), \phi(t), z(t))$ is increasing in $\hat{\mathcal{C}}_0$ if

$$\dot{z} + r^2\dot{\phi} > 0$$

for all t. Notice that for a closed braid L, we have $\dot{\phi} > 0$. So, we can change the closed braid L through isotopy so that its r-coordinate becomes very large and its z- and ϕ-coordinate remain the same. The resulting closed braid is an increasing transverse link in the contact structure $\hat{\mathcal{C}}_0$. Combined with the theorem of Alexander that every link in $\mathbb{R}^3$ is isotopic to a closed braid, we get the following theorem.

Theorem 2.15. *Every link in $\mathbb{R}^3$ is isotopic to an increasing transverse link in the standard contact structure of $\mathbb{R}^3$.*

Example 2.16. Consider the Hopf contact structure given by

$$\alpha = x_1\, dy_1 - y_1\, dx_1 + x_2\, dy_2 - y_2\, dx_2 = 0$$

on $S^3 = \{(z_1, z_2) \in \mathbb{C}^2\, ;\, |z_1|^2 + |z_2|^2 = 1\}$, with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. The non-zero vector field

$$X = -x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial x_2} - y_1 \frac{\partial}{\partial y_2}$$

on S^3 is in $\ker(\alpha)$ everywhere. So we may use this vector field X to calculate $l(K)$ for a transverse knot in the Hopf contact structure.

Fix a point $(z_1, z_2) \in S^3$. Let

$$K_1 = \{(e^{i\theta} z_1, e^{i\theta} z_2) \in S^3\,;\, e^{i\theta} \in S^1\}.$$

This is an increasing transverse knot in the Hopf contact structure.

It will be convenient to use quaternion numbers to describe the geometry of the Hopf contact structure. With the standard notation, i, j, k are unit quaternions satisfying

$$i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j.$$

We write $(z_1, z_2) \in \mathcal{C}^2$ as quaternion $z = z_1 + z_2 j$. Assume that $z \in S^3$ is a unit quaternion. Then iz, jz, kz are all perpendicular to z and therefore we can think of them as vectors in the tangent space of S^3 at z, $T_z S^3$. In fact, they form a basis of $T_z S^3$. The Hopf contact structure is the plane field spanned by jz and kz at every point $z \in S^3$. The vector iz is normal to this plane field everywhere, so it is the Reeb vector field of the Hopf contact structure.

Starting at a point $1 \in S^3$, the curve K_1 written as $e^{i\theta}$ is the integral curve of the Reeb vector field passing through 1. To calculate $l(K_1)$, we consider a disk D in S^3 whose boundary is K_1. It can be given as follows:

$$D = \{e^{i\theta} \cos r + j \sin r\,;\, 0 \le r \le \pi/2,\ 0 \le \theta \le 2\pi\}.$$

The tangent vector of this disk in the radius direction when $r = 0$ (on the boundary $\partial D = K_1$) is j. On the other hand, the direction toward which K_1 should be pushed in order to calculate $l(K_1)$ is $je^{i\theta} = j \cos\theta - k \sin\theta$. Therefore, $l(K_1) = -1$.

In the Hopf contact structure on S^3, consider two disjoint positive transverse curves $e^{i\theta}$ and $-e^{i\theta}$. Let K be a knot in S^3 disjoint with the curve $e^{i\theta}$. If K is also increasing transverse, it can be considered as a closed braid with the curve $e^{i\theta}$ as its axis.

Theorem 2.17. *Let K be an increasing transverse knots in the complement of $e^{i\theta}$. Consider K as a closed braid with the braid index n and the algebraic length e. Then $l(K) = e - n$.*

Proof. We may use the contactomorphism Φ_T in Theorem 2.6 to bring K to a closed braid in $\mathbb{R}^3$ with the z-axis as its axis and is increasing transverse in the contact structure $\hat{\mathcal{C}}_0$. We may assume that a plane projection of K formed by n circles $z = 0$, $r =$ constant jointing together by some elementary braids. To calculate $l(K)$, we may assume that the vector field X in the definition of $l(K)$ is vertical $(= \frac{\partial}{\partial z})$ at each elementary braid. Since the vector field X rotating around each circle $z = 0$, $r =$ constant once in counter-clockwise direction, we get $l(K) = e - n$. □

The following theorem about transverse links in $\hat{\mathcal{C}}_0$ is a generalization of Theorem 2.7. Its corollary will be useful in the proof of Theorem 2.3.

Theorem 2.18. *Every transverse link in $\hat{\mathcal{C}}_0$ is isotopic in the class of transverse links to a transverse closed braid.*

Proof. **To be written.** □

Corollary 2.19. *Let L be a transverse link in $\hat{\mathcal{C}}_0$ and $\chi(L)$ be its characteristic. Then $l(L) \leq -\chi(L)$.*

Proof. This is a corollary of Theorem 2.9 and Theorem 2.18. □

2.5 Reeb foliation and Markov surfaces

We give a proof of Theorem 2.9 in this section.

Let us consider a foliation of co-dimension 1 on the annulus $z = re^{i\phi}$, $1/\sqrt{2} \leq r \leq 1$, determined by the equation

$$\sin(r^2\pi)\,dr + \cos(r^2\pi)\,d\phi = 0.$$

This foliation has only one close leaf $r = 1/\sqrt{2}$.

We may rotate the annulus with $r = 1$ as the axis to get a solid torus. The above foliation may also be rotated to get a foliation of co-dimension 1 on the solid torus. This is called the *Reeb foliation* on the solid torus. Notice that the boundary torus is the only closed leaf of the Reeb foliation. For each ϕ, there is only one non-compact leaf $\mathcal{R}_\phi$ passing through the curve $r = 1$ at the angle ϕ.

The 3-sphere $S^3 = \{(r_1e^{i\phi_1}, r_2e^{i\phi_2})\,;\, r_1^2 + r_2^2 = 1\}$ is the union of two solid tori $N_1 = \{1/2 \leq r_1^2 \leq 1\}$ and $N_2 = \{1/2 \leq r_2^2 \leq 1\}$. The core circle of $N - 1$ is $r_1 = 1$ of $(e^{i\phi_1}$ and the core circle of N_2 is $r_2 = 1$ or $(0, e^{i\phi_2}$. With respect to the standard orientation of S^3, these two core circles, when oriented in the increasing direction of the angles ϕ_1 and ϕ_2, respectively, have their linking number equal to 1.

We shall think of $N_{1,2}$ as obtained from rotating annuli respectively so that both of them carry the Reeb foliation with the only closed leaf $T = \{r_1^2 = r_2^2 = 1/2\}$. Put these two solid tori with the Reeb foliation together gives us a foliation of co-dimension 1 on the 3-sphere. It is called the *Reeb foliation* on S^3. We orient the fibers of this Reeb foliation continuously so that the oriented core circle $(e^{i\phi_1}, 0)$ gives rise to the positive normal direction for each $\mathcal{R}_{\phi_1}$ with respect to the standard orientation of S^3. On the torus T, the positive normal direction points toward N_1.

On the torus T, we will call curves given by $\phi_2 =$ constant *longitudes* and $\phi_1 =$ constant *meridians*, respectively. A longitude on T bounds a disk in N_2. We may assume that the Reeb foliation in N_2 induces a (singular) foliation on this disk whose leaves consisting of a singular point and concentric circles round it.

Definition 2.20. *Let L be a closed braid in N_1. A* Markov surface *of L is a compact orientable surface V in S^3 with the following properties:*

(1) $\partial V = L$ and V has no closed component;

(2) V intersects T along a finite collection of longitudes and $V \cap N_2$ is the collection of disks these longitudes bound;

(3) every point where V is tangent to a leaf of the Reeb foliation is a saddle point.

Thus, the singular foliation on a Markov surface induced by the Reeb foliation has only two types of singularities: hyperbolic (saddle points) and elliptic (local max/min points).

Theorem 2.21. *Every closed braid L in N_1 bounds a Markov surface with maximal characteristic $\chi(L)$. In fact, a Seifert surface of L with maximal characteristic $\chi(L)$ is isotopic to a Markov surface.*

Proof. Let F be an oriented Seifert surface of the closed braid L whose Euler characteristic is $\chi(L)$. We assume F is in general position so that it intersects transversally the core circle of N_2 at finitely many points. Thus, we may assume that $F \cap N_2$ is a finite collection of disjoint disks $\{D_i\}$ such that each ∂D_i is a longitude of T. Let us assume further that F is in general position with respect with the Reeb foliation so that the intersections of F with leaves of the Reeb foliation gives rise to a singular foliation $\mathcal{F}$ on F. The leaves of this singular foliation of F are oriented according to the standard orientation of S^3, the orientations of leaves of the Reeb foliation, and the orientation of F. The singular foliation on each disk D_i consists of concentric circles of constant radii and the center point.

The oriented singular foliation $\mathcal{F}$ has only two types of singularities: The hyperbolic points and the elliptic points. Consider now an elliptic point a. There is a maximal open disk D on F containing a so that $\mathcal{F}|_D$ foliates D by concentric circles. Consider $C = \partial \bar{D}$. Take into consideration of the orientation of the leaves of $\mathcal{F}$, we can conclude that either C is circle which is a leaf of $\mathcal{F}$ and thus $\bar{D}$ is one of D_i, or there is another open disk B on F such that (1) $\mathcal{F}$ foliates B by concentric circles; and (2) $\partial \bar{B}$ is a circle which is a leaf of $\mathcal{F}$ with a single hyperbolic point on it.

Then we can perform a surgery on F to eliminate an elliptic point. Since we have assumed that F is of maximal characteristic $\chi(L)$, such a surgery must be trivial and therefore can be archived by an isotopy of F. This finishes the proof. □

Let F be a Markov surface of the closed braid L as obtained in the proof of Theorem 2.21. We see from the construction that L is homologous to $\cup \partial D_i$. Let S_+ (S_-, respectively) the number of positive (negative, respectively) longitudes ∂D_i. Let n be the braid index of L. The following lemma should be clear.

Lemma 2.22. $n = S_+ - S_-$.

We may assign a sign to each hyperbolic point on the Markov surface F. The sign is positive (negative, respectively) if the orientations of F and the leaf $\mathcal{R}_{\phi_1}$ of the Reeb fibration agree (do not agree, respectively) at the hyperbolic point. Let A_+ (A_-, respectively) be the number of positive (negative, respectively) hyperbolic points. Recall that we denote by e the algebraic length of L.

Lemma 2.23. $e = A_+ - A_-$.

Proof. **To be written.** □

Theorem 2.24. $S_- \leq A_-$.

Proof. **To be written.** □

Proof of Theorem 2.9. We have

$$e - n = A_+ - A_- - S_+ + S_-.$$

On the other hand, by Hopf-Poincaré theorem, we have

$$-\chi(L) = A_+ + A_- - S_+ - S_-.$$

Thus we get

$$e - n \leq -\chi(L)$$

and this proves Theorem 2.9. □

2.6 Invariants of Legendrian knots

Let $\mathcal{C} = \ker(\alpha)$ be a parallelizable contact structure on a 3-manifold M. Recall that a curve γ in M is called Legendrian if $\alpha(\dot\gamma) = 0$ everywhere.

Now let γ be a Legendrian knot, which is homologous to zero in M. Let a knot γ' be obtained by slightly moving γ in the direction of the Reeb vector field of α so that γ' and γ are disjoint.

We assume that M is oriented by the volume form $\alpha \wedge d\alpha$. Each hyperplane in $\mathcal{C} = \ker(\alpha)$ is oriented by $d\alpha$.

Definition 2.25. *The* Bennequin-Thurston number *of γ, denoted by $\tau(\gamma)$, is the linking number of γ and γ'.*

Since $\mathcal{C} = \ker(\alpha)$ is parallelizable, by definition, we may have two linearly independent vector fields X, Y form a correctly oriented frame for each hyperplane in $\mathcal{C}$.

Definition 2.26. *The* Maslov index *of γ, denoted by $\mu(\gamma)$, is the rotation number of $\dot\gamma$ with respect to the oriented frame $\{X, Y\}$.*

Let ν be the normal vector field of γ in the hyperplane field $\ker(\alpha)$ so that $\{\dot{\gamma}, \nu\}$ is an oriented frame of $\ker(\alpha)$ everywhere along γ. Shift γ slightly in the direction of ν, we get a knot γ_+. And shift γ slightly in the direction of $-\nu$, we also get a knot γ_-.

Lemma 2.27. *γ_+ is a decreasing transverse knot, and γ_- is an increasing transverse knot.*

Proof. Infinitesimally, $\gamma_+ = \gamma + \epsilon\nu$ for a small $\epsilon > 0$. We have

$$\alpha(\dot{\gamma}_+) = d\alpha(\epsilon\nu, \dot{\gamma}) < 0.$$

This can be obtained by applying Stocks' theorem to the infinitesimal rectangle with one side on γ, one side on γ_+, and other two sides point in the direction of ν. So γ_+ is a decreasing transverse knot. Similarly, we can show that γ_- is an increasing transverse knot. ☐

Theorem 2.28. *With the notations and assumptions as in the above discussion, we have*

$$l(\gamma_+) = \tau(\gamma) + \mu(\gamma)$$

and

$$l(\gamma_-) = \tau(\gamma) - \mu(\gamma).$$

Proof. Let X be a nowhere zero vector field in the parallelizable hyperplane field $\ker(\alpha)$. The number $l(\gamma_+)$ is the linking number of $\gamma_+ = \gamma + \epsilon\nu$ and $\gamma_+ + \epsilon' X = \gamma + \epsilon\nu + \epsilon' X$, for a small $\epsilon' > 0$. Notice that since ν is perpendicular to γ, the number $\tau(\gamma)$ is the linking number of γ and $\gamma + \epsilon\nu$, and therefore also the linking number of $\gamma + \epsilon' X$ and $\gamma + \epsilon\nu + \epsilon' X$. Thus, $l(\gamma_+) - \tau(\gamma)$ is the rotation number of ν with respect to X, which is equal to $\mu(\gamma)$. This proves the first equality $l(\gamma_+) = \tau(\gamma) + \mu(\gamma)$. The second equality is proved similarly. ☐

Corollary 2.29. *If γ is a Legendrian knot in the standard contact structure $\hat{\mathcal{C}}_0$ on $\mathbb{R}^3$. Then*

$$\tau(\gamma) \leq -\chi(\gamma) - |\mu(\gamma)|.$$

Proof. Since γ_+ and γ_- are transverse knots in $\hat{\mathcal{C}}_0$, and they are all isotopic to γ, by Corollary 2.19, we have

$$\tau(\gamma) \leq -\chi(\gamma) - \mu(\gamma) \quad \text{and} \quad \tau(\gamma) \leq -\chi(\gamma) + \mu(\gamma).$$

Therefore, we have the desired inequality. ☐

Proof of Theorem 2.3. We may use Corollary 2.29 to prove Theorem 2.3. First of all, the contact structures $\mathcal{C}_0$ and $\hat{\mathcal{C}}_0$ on $\mathbb{R}^3$ are isomorphic. So we may not distinguish them. Since γ is the unknot, we have $\chi(\gamma) = 1$. Thus

$$\tau(\gamma) = \mathrm{lk}(\gamma, \gamma') \leq -1 - |\mu(\gamma)| < 0.$$

This completes the proof of Theorem 2.3. ☐

2.7 Non-standard contact structures on S^3

In this section, we shall construct a contact structure on S^3 which is not isomorphic to the Hopf contact structure, yet its underlying hyperplane field is homotopic to that of the Hopf contact structure.

Consider the standard contact structure on the solid torus $S^1 \times D^2$ with coordinates θ, r, ϕ given by the contact form

$$\alpha_0 = d\theta + r^2\, d\phi.$$

As in Example 1.23, we consider the form

$$h_1(r)\, d\theta + h_2(r)\, d\phi$$

such that $(h_1(1), h_2(1) = (1,1)$ and $(h_1(0), h_2(0)) = (1,0)$ with $h_2(r) = O(r^2)$ at $r = 0$. Such a form is a contact form on $S^1 \times D^2$ if $h_1 \cdot h_2' - h_2 \cdot h_1' \neq 0$.

For a positive integer n, let $\alpha_n = h_1(r)\, d\theta + h_2(r)\, d\phi$ be a contact form as described above such that the curve $(h_1(r), h_2(r))$, $0 \leq r \leq 1$, circles around $(0,0)$ monotonically and counterclockwisely for n times, going from $(1,0)$ to $(1,1)$. This contact form α_n agrees with α_0 on the boundary $S^1 \times S^1$, where they are both equal to $d\theta + d\phi$.

Lemma 2.30. *The form α_n is homotopic to α_0 in the class of nowhere zero forms equal to $d\theta + d\phi$ on the boundary of $S^1 \times D^2$.*

Proof. Let $(f_t(r), g_t(r))$ is a homotopy from $(1, r^2)$ to $(h_1(r), h_2(r)$. Let

$$\alpha(t) = r(1-r)t(1-t)\, dr + f_t(r)\, d\theta + g_t(r)\, d\phi.$$

Then $\alpha(t)$ is a homotopy from α_0 to α_n within the class of nowhere zero 1-forms equal to $d\theta + d\phi$ on the boundary of $S^1 \times D^2$. □

We may extend the form α_n to $S^3 = S^1 \times D^2 \cup D^2 \times S^1$ by $\alpha_n|_{D^2 \times S^1} = r^2\, d\theta + d\phi$. The contact structure $\ker(\alpha_n)$ and the Hopf contact structure $\ker(\alpha_0)$ have homotopic underlying hyperplane fields, but they are not isotopic since $\ker(\alpha_n)$ has a Legendrian unknot. Indeed, for $\alpha_n = h_1(r)\, d\theta + h_2(r)\, d\theta$, when $h_2(r_0) = 0$ for some r_0, the unknotted curve given by $r = r_0$ and $\theta =$ constant is Legendrian. Using the relationship between the Hopf contact structure and the standard contact structure on $\mathbb{R}^3$ and Theorem 2.1, we know this is impossible for the Hope contact structure to have a Legendrian unknot.

This method of modifying an existing contact structure can be carried out in general cases due to the following result of Martinet.

Theorem 2.31. *Let γ be an increasing transverse knot in a contact structure ker(α) on a 3-manifold M. Then there is a tubular neighborhood U of γ with coordinates (θ, r, ϕ) such that γ is identified with the curve $r = 0$ and $\alpha|U = \alpha_0 = d\theta + r^2\, d\phi$.*

Proof. **To be written.** □

Thus we may modify the contact structure by change α_0 to α_n inside the tubular neighborhood U. The new contact structure on M and the original one have homotopic underlying hyperplane fields. Such a modification and its inverse will be called a *Lutz modification.*

The following two questions were asked by Bennequin.

Question 2.32. *Let M be a closed orientable 3-manifold. Is it true that every two contact structures on M with homotopic underlying hyperplane fields can be changed from one to the other by a finite sequence of Lutz modifications?*

Question 2.33. *When two successive Lutz modifications leave a contact structure unchanged?*

Chapter 3

Legendrian knots

3.1 Space of contact elements

Definition 3.1. *A* contact element *of a point* $(x, y) \in \mathbb{R}^2$ *is a line in* $T_{(x,y)}\mathbb{R}^2$*. The* co-orientation *of a contact element is a choice of one of the two half planes of* $T_{(x,y)}\mathbb{R}^2$ *divided by this contact element.*

Thus the space of all co-oriented contact element can be identified with $\mathbb{R}^2 \times S^1$, with coordinates $(x, y, \phi \,(\mathrm{mod}\, 2\pi))$, so that the contact element is given by the equation

$$\cos\phi\, dx + \sin\phi\, dy = 0$$

with the co-orienting vector $(\cos\phi, \sin\phi)$ at every point.

Notice that the 1-form $\alpha = \cos\phi\, dx + \sin\phi\, dy$ on $\mathbb{R}^2 \times S^1$ is a contact form. The hyperplane field $\ker(\alpha)$ at a point $(x, y, \phi) \in \mathbb{R}^2 \times S^1$ is spanned by the contact element at (x, y) orthogonal to $(\cos\phi, \sin\phi)$ and $\partial/\partial\phi$. Thus, the circle $(x_0, y_0) \times S^1$ is Legendrian. In order words, the projection $\mathbb{R}^2 \times S^1 \to \mathbb{R}^2$ is a Legendrian fibration.

More generally, let F be an oriented closed surface. Let $PT(F)$ be the projectivized tangent space of F. The projection $PT(F) \to F$ is a fibration such that the fibre over a point $z \in F$ is the projectivization of $T_z(F)$, i.e. it is an S^1-fibration. The Euler characteristic of this S^1-fibration is $2\chi(F)$.

Given a Riemannian metric on F, we may also consider the unit tangent bundle $ST(F)$ over F. This is again an S^1-fibration over F. The natural map $ST(F) \to PT(F)$ is a fibre-wise 2-fold covering map. The Euler characteristic of the S^1-fibration is $\chi(F)$.

The 3-manifold $ST(F)$ admits a tautological contact structure. First, a tangent hyperplane field is given as follows. To each point $(z, \nu) \in ST(F)$, we assign a tangent hyperplane which is the preimage of the tangent line spanned by the unit tangent vector $\nu \in T_z(F)$ under the tangent map of the projection $PT(F) \to F$. This hyperplane field is a contact structure because it can be identified with the contact structure on $\mathbb{R}^2 \times S^1$ in each neighborhood of F diffeomorphic to $\mathbb{R}^2$. This contact structure is co-oriented.

A tautological contact structure can be defined similarly on $PT(F)$. This contact structure is not co-orientable.

Notice in both cases of $ST(F)$ and $PT(F)$, the S^1-fibrations are Legendrian fibrations with respect to their tautological contact structure.

Lemma 3.2. *Let $\eta : M \to F$ be an S^1-fibration over F. Denote by $e(\eta)$ be its Euler characteristic. The total space M admits a contact structure so that η is a Legendrian fibration iff $ke(\eta) = 2\chi(F)$ for some integer $k \neq 0$.*

Proof. If η is a Legendrian fibration, the image of a contact plane on F will be a tangent line. Thus, we may construct a fibre-wise covering map $M \to PT(F)$. This implies that for an integer $k \neq 0$, $ke(\eta)$ is the Euler characteristic of $PT(F)$. Therefore $ke(\eta) = 2\chi(F)$.

On the other hand, if $e(PT(F))$ is divisible by $e(\eta)$, we can construct a fibre-wise covering map $M \to PT(F)$. Then the pull-back of the tautological contact structure on $PT(F)$ gives rise to a contact structure on M which makes η a Legendrian fibration. □

Let $\gamma : S^1 \to \mathbb{R}^2$ be a generic immersion, i.e. an immersion with only transverse double points. We can define a unique Legendrian lifting $\Gamma : S^1 \to \mathbb{R}^2 \times S^1$ of γ as follows:

Let $\gamma(t) = (x(t), y(t))$ and $\Gamma(t) = (x(t), y(t), \phi(t))$. Then the angle $\phi(t)$ is uniquely determined by the equation

$$\cos(\phi(t))\,\dot{x}(t) + \sin(\phi(t))\,\dot{y}(t) = 0$$

together with the condition that $\dot{\gamma} = (\dot{x}, \dot{y})$ and $(\cos\phi, \sin\phi)$ give the standard orientation for R^2. Since we have assumed that γ is a generic immersion, its lifting Γ just defined must be an imbedding. I.e. Γ is a Legendrian knot in $\mathbb{R}^2 \times S^1$.

In general, if Γ is a Legendrian knot in $\mathbb{R}^2 \times S^1$, its projection in $\mathbb{R}^2$ may not be an immersion.

Lemma 3.3. *The projection of a generic Legendrian knot Γ has only transverse double points and cusp points as its singularities.*

Proof. Let $\Gamma(t) = (x(t), y(t), \phi(t))$ be a Legendrian knot in $\mathbb{R}^2 \times S^1$. First we may assume that there are only finitely many t's such that $\dot{x}(t) = \dot{y}(t) = 0$. Away from those t's, the projection $\gamma(t) = (x(t), y(t))$ is an immersion. So we may assume that γ has only transverse double points.

Near a point on γ where $\dot{x} = \dot{y} = 0$, notice that the normal vector $(\cos\phi, \sin\phi)$ passes this point smoothly due to the condition of Γ being Legendrian. If the orientations of $\mathbb{R}^2$ given by $\dot{\gamma}$ and $(\cos\phi, \sin\phi)$ before and after this singular point are the same, we may reparametrize γ to eliminate this singular point.

Assume that the orientations of $\mathbb{R}^2$ given by $\dot{\gamma}$ and $(\cos\phi, \sin\phi)$ before and after this singular point are different. A typical model of this situation is the following cusp singularity:

$$(x(t), y(t)) = (t^3, -t^2) \quad t \in (-\epsilon, \epsilon)$$

with a small $\epsilon > 0$. The singularity is at $t = 0$. Solve the equation

$$\dot{x}\cos\phi + \dot{y}\sin\phi = 0$$

gives us $\phi = \arctan(3t/2)$. This is a smooth function. It is not hard to see that we can always modify γ near each singular point by a small isotopy and reparametrization so that it has this standard form. □

Lemma 3.4. *Every knot in $\mathbb{R}^2 \times S^1$ is isotopic to a Legendrian knot.*

Proof. If the vector $(\cos\phi, \sin\phi)$ turns from $\phi = \pi/2$ to $\phi = 3\pi/2$ counterclockwise along the line segment $I = \{(t, 0)\,;\, t \in -1, 1\}$ in $\mathbb{R}^2$, we may replace this line segment by the cusp curve $(t^3, \epsilon(t^2 - 1))$ so that its normal vector makes the same turn as $(\cos\phi, \sin\phi)$. We can choose ϵ small so that this modification can be achieved by an isotopy in $\mathbb{R}^2 \times S^1$. This is one of the basic constructions.

Another basic construction deals with the case when ϕ is near its local extremal points. In the case of a Legendrian knot, such points correspond to inflection points on its projection. Consider the case when the vector $(\cos\phi, \sin\phi)$ turns from $\phi = \pi/2$ slightly to $\phi = \pi/2 + \epsilon$ and then turns back to $\phi = \pi/2$ along the line segment I. We may replace I by a flat S-shaped curve with its normal vector having three local extremal points. This can be achieved by an isotopy in $\mathbb{R}^2 \times S^1$.

With these two basic modifications, it should not be hard to see that we can isotopy an arbitrary knot in $\mathbb{R}^2 \times S^1$ to a Legendrian knot. □

Question 3.5. (Arnold) *What topological knot types in $\mathbb{R}^2 \times S^1$ have Legendrian representatives with smooth projections onto $\mathbb{R}^2$?*

3.2 Maslov index

For an oriented Legendrian knot Γ in $\mathbb{R}^2 \times S^1$, its projection γ is both oriented and co-oriented. By the latter, we mean that the normal vector field of γ is oriented. Define the index of Γ, $\mathrm{ind}(\Gamma)$, to be the total angle variation of the normal vector field of γ divided by 2π. We may consider the map $S^1 \to S^1$ obtained by the composition of $\Gamma : S^1 \to \mathbb{R}^2 \times S^1$ and the projection $\mathbb{R}^2 \times S^1 \to S^1$. The $\mathrm{ind}(\Gamma)$ is simply the degree of this map.

Let us now consider the space of unit tangent vectors of $\mathbb{R}^2 \times S^1$ that lie in the contact plane. This space has coordinates (x, y, ϕ, θ) which corresponds to the unit tangent vector

$$\cos\theta \frac{\partial}{\partial\phi} + \sin\theta \left(\sin\phi \frac{\partial}{\partial x} - \cos\phi \frac{\partial}{\partial y}\right).$$

Definition 3.6. *The* Maslov index *of the Legendrian knot Γ, $\mu(\Gamma)$, defined to be the rotation number of $\dot{\Gamma}$ in the contact plane.*

Notice that if we projectivize each contact plane, we get a (trivial) S^1-bundle on $\mathbb{R}^2 \times S^1$. The Maslov index in this case is twice of the Maslov index in the previous case.

Consider a generic Legendrian knot Γ so that its projection γ in $\mathbb{R}^2$ has only transverse double and cusp singularities. At a cusp point on γ,

$$\dot{\Gamma} = \pm \frac{\partial}{\partial \phi}$$

where the sign is positive (negative) if the normal angle ϕ is increasing (decreasing) when one moves along γ in its given direction. These two cases correspond to the points $\theta = 0$ and $\theta = \pi$, respectively. If when γ passes through a cusp point, the normal angle is increasing and the sign of $\dot{x} \sin \phi - \dot{y} \cos \phi$ changes from $-$ to $+$, then $\dot{\Gamma}$ moves through $\theta = 0$ counterclockwisely. Thus this cusp point contributes 1 to the Maslov index.

There are other three cases for each cusp point. A cusp point is defined to be positive if when one passes through this point along γ in the given direction, either

(1) ϕ is increasing and $\dot{x} \sin \phi - \dot{y} \cos \phi$ changes sign from $-$ to $+$; or

(2) ϕ is decreasing and $\dot{x} \sin \phi - \dot{y} \cos \phi$ changes sign from $+$ to $-$.

A negative cusp point is the opposite of a positive cusp point.

Theorem 3.7. *Let μ_+ and μ_- be the numbers of positive and negative cusp points on γ, respectively. Then $\mu(\Gamma) = (\mu_+ - \mu_-)/2$.*

Notice that the index and the Maslov index can be defined for (not necessarily imbedded) Legendrian curves. Two Legendrian curves are called Legendrian homotopic if they are homotopic through Legendrian curves.

Theorem 3.8. (Gromov) *Two Legendrian curves are Legendrian homotopic iff they have the same index and Maslov index.*

Proof. This follows from Gromov's h-principle. But we will provide an elementary proof here.

To be written. □

3.3 Arnold's invariant J^+

Since $\mathbb{R}^2 \times S^1$ is not simply connected, we can not defined directly the Bennequin-Thurston number for Legendrian knots in this space. Arnold's invariant J^+ offers an alternative in this case.

Let us consider the universal covering $\mathbb{R}^3 \to \mathbb{R}^2 \times S^1$. The pull-back of the contact form $\cos \phi \, dx + \sin \phi \, dy$ is standard on $\mathbb{R}^3$. Indeed, if we use the *Legendrian transform*:

$$z = x \cos \phi + y \sin \phi, \quad p = x \sin \phi - y \cos \phi,$$

we have

$$\cos\phi\, dx + \sin\phi\, dy = dz - p\, d\phi.$$

Let Γ be a Legendrian knot in $\mathbb{R}^2 \times S^1$. We consider first the case when $\mathrm{ind}(\Gamma) = 0$. In this case, we may lift Γ to $\mathbb{R}^3$ to infinitely many Legendrian knots Γ_n for $n \in \mathbb{Z}$. Let τ_0 be the Bennequin-Thurston number of Γ_0, and τ_n the linking number of Γ_0 and Γ_n for $n \neq 0$.

When $\mathrm{ind}(\Gamma) = k > 0$, consider the k-fold covering $R^2 \times S^1 \to \mathbb{R}^2 \times S^1$. Γ has k liftings $\Gamma_0, \Gamma_1, \ldots, \Gamma_{k-1}$. Let Γ_0^* be the Legendrian knot upstairs specified by the equations $x = N, y = 0$ for sufficiently large N. Then Γ_0 and Γ_0^* cobound a 2-chain, i.e. there is an oriented compact surface F such that $\Gamma_0 - \Gamma_0^* = \partial F$. Now let τ_n be the intersection number of Γ_n with F, for $n \neq 0$. Finally, we push Γ_0 slightly in the direction of the Reeb vector field to get a curve Γ_0' and τ_0 is the intersection number of Γ_0' with F.

Definition 3.9. *The* linking polynomial *of* Γ *is*

$$L(t) = \sum_n \tau_n t^n.$$

The J^+ *invariant of* Γ *is defined by*

$$J^+(\Gamma) = 1 - L(1).$$

The J^+ invariant of a Legendrian knot Γ can be calculated from its projection γ by the following theorem.

Consider a regular homotopy of the projection γ in $\mathbb{R}^2$. The following list of local regular homotopy can be realized as the projection of an isotopy in $\mathbb{R}^2 \times S^1$:

(1) passage of a cusp point through a smooth branch of γ;

(2) passage of a double point through a smooth branch of γ;

(3) creation and elimination of a tangent point between two smooth branches of γ oriented in opposite orientations.

On the other hand, the following type of local regular homotopy will change the Legendrian knot type:

(4) creation and elimination of a tangent point between two smooth branches of γ oriented in the same direction.

Theorem 3.10. *If* Γ *and* Γ' *are two Legendrian knots in* $\mathbb{R}^2 \times S^1$*, whose projection* γ *and* γ' *differ by a local regular homotopy of type (4). Assume that* γ *has more double points than* γ*. Then*

$$J^+(\gamma') - J^+(\gamma) = -2.$$

Proof. **To be written.** □

Since J^+ will not change its value when γ is changed by local regular homotopy of types (1), (2), and (3), and we can always simplify the projection γ by local regular homotopy of types (1)–(4) until it becomes one of the standard forms. The evaluation of J^+ is then reduced to the evaluation of J^+ on these standard projections.

The standard projections are $K_{i+1,k}$, for $i, k \leq 0$, and $K_{0,k}$, for $k \leq 0$. We have

$$J^+(K_{i+1,k}) = -2i - k, \quad J^+(K_{0,k}) = -k.$$

3.4 Legendrian knots in $\mathbb{R}^3$

We now consider the standard contact structure $\alpha = dz - y\,dx = 0$ in $\mathbb{R}^3$. We want to study the projection of a Legendrian knot to the (x, z)-plane and to the (x, y)-plane. And we shall be able to calculate the Bennequin-Thurston number and the Maslov index from their projection.

Let us consider the (x, z)-plane projection first. If $\Gamma(t) = (x(t), y(t), z(t))$ is a Legendrian curve, we have

$$y(t) = \frac{\dot{z}(t)}{\dot{x}(t)}.$$

Thus, if $\dot{x}(t) \neq 0$, the Legendrian curve $\Gamma(t)$ is determined by its projection $\gamma(t) = (x(t), z(t))$ in the (x, z)-plane.

For a generic Legendrian knot Γ in $\mathbb{R}^3$, its projection γ in the (x, z)-plane is a piecewise immersion with finitely many transverse double points and cusp points as singularities. At a point on Γ where $\dot{x} = 0$, then we also have $\dot{z} = 0$. The projection of such a point in the (x, z)-plane is exactly a cusp point. Thus the tangent line of a cusp point is horizontal in the (x, z)-plane. We may assume that $\dot{x} = 0$ only at cusp points.

It is easy to see that the Reeb vector field is in this case is $\partial/\partial z$. So the Bennequin-Thurston number of Γ can be calculated as the linking number of Γ with Γ', which is a Legendrian knot with the projection γ' obtained by shifting γ up in the z-direction in the (x, z)-plane.

To calculate the Maslov index, we use the global vector field $X = \partial/\partial y$. We have

$$d\alpha(\dot{\Gamma}, X) = \dot{x}.$$

Thus the Maslov index of Γ is equal to the algebraic number of cusp points of γ with $\dot{y} > 0$ (the slope of the tangent line is increasing), where such a cups point of γ is counted as $+1$ (or -1, respectively) if $\dot{x}$ changes from $+$ to $-$ (or $-$ to $+1$, respectively).

Let us quote the following two theorems, which were first formulated by Eliashberg and proved by Swiatkowski.

Theorem 3.11. *The space of Legendrian knots in $\mathbb{R}^3$ with generic projections into the (x, z)-plane having only transverse double points and cusp points is open and dense in the space of all Legendrian knots with topology induced from that of $C^\infty(S^1, \mathbb{R}^3)$.*

Theorem 3.12. *Two Legendrian knots with generic projections in the (x, z)-plane are Legendrian isotopic iff we can pass from one projection to another through a sequence of finitely many elementary moves of the following four types:*

(1) composition with an orientation preserving diffeomorphism of the (x, z)-plane or reparametrization;

(2) creation or elimination of a "swallow tail";

(3) passage of a cups point through a smooth piece of the projection;

(4) passage of a cups point through a smooth piece of the projection.

We next consider the projectin of a Legendrian knot onto the (x, y)-palne. The basic relation among the coordinates of a Legendrian knot is

$$z = \int y\, dx$$

in this case. Thus, generically, the projection of Γ in the (x, y)-plane is a smooth curve γ with only transverse double points as singularities. We may divide this curve γ into pieces such that each piece is the graph of a function $y = y(x)$. The z-coordinate of Γ is then determined by the integration of this function.

Lemma 3.13. *The Maslov index of Γ is equal to the winding number of its projection in the (x, y)-plane.*

Proof. By the previous calculation of the Maslov index using the (x, z)-palne projection, we see that the Maslov index is equal to the algebraic number of points of the (x, y)-plane projection γ where the tangent vector is in the direction of $\partial/\partial y$. Such a point is positive if the tangent vector of *gamma* turns counterclockwise through the $\partial/\partial y$ direction near this point, and negative otherwise. So the Maslov index Γ is the same of the winding number of γ. □

It seems that no specific rule for the calculation the Bennequin-Thurston number of Γ from its projection in the (x, y)-plane is known.

A fundamental question in the study of Legendrian knots is whether the oriented topological knot type, the Bennequin-Thurston number and the Maslov index are sufficient to determine an oriented Legendrian knot type. More specifically, we have a following definition.

Definition 3.14. *An oriented topological knot type is called* Legendrian simple *if any two oriented Legendrian knots in this knot type are Legendrian isotopic when they have the same Bennequin-Thurston number and Maslov index.*

It is known that the unknot, the figure 8 knot, and all torus knots are Legendrian simple.

Theorem 3.15. (Chekanov, Eliashberg-Givental-Hofer) *There exist infinitely many non-Legendrian simple knots.*

The simplest example of a non-Legendrian simple knot type given by Chekanov is 5_2. We shall return to this topic later.

Chapter 4

Eliashberg

4.1 Overtwisted contact structures

4.1.1 Singular foliations on a 2-sphere

Consider the following two contact forms on $\mathbb{R}^3$:

$$\alpha_0 = dz + r^2\, d\phi \quad \text{and} \quad \alpha_1 = \cos r\, dz + r \sin r\, d\phi$$

The contact structures $\mathcal{C}_0 = \ker(\alpha_0)$ and $\mathcal{C}_1 = \ker(\alpha_1)$ are called the *standard contact structure* and the *standard overtwisted contact structure*, respectively.

Let S^2 be the 2-sphere with radius $5\pi/4$ centered at the origin. The intersection of contact planes in the contact structure $\mathcal{C}_0$ with the tangent planes on S^2 gives rise to a vector field on S^2. The integral curves of this vector field form a singular foliation S^2, which is depicted in the following picture. The singular foliation has two singular points, the north and south poles of S^2.

Choose a longitude from the south pole to the north pole. Then this longitude is transverse to the singular foliation everywhere except the south and north poles. Identify this longitude with the unit interval $I = [0, 1]$ with 0 being the south pole and 1 the north pole. The Poincaré map of the flow of the vector field $f_0 : I \to I$ satisfies (1) $x < f_0(x)$ for all $x \in (0, 1)$ and (2) $f_0(0) = 0$, $f_0(1) = 1$. The singular foliation on S^2 can be thought of the suspension of the map f_0. So we denote it by $\mathcal{F}(f_0)$.

The singular foliation on S^2 obtained by considering the intersection of the contact planes in $\mathcal{C}_1$ and the tangent planes of S^2 has the same nature. It is now the suspension of a diffeomorphism $f_1 : I \to I$ having 4 fixed points including $x = 0$ and $x = 1$. The singular foliation in this case is then denoted by $\mathcal{F}(f_1)$.

Notice that in both cases, the point $x = 0$ is an unstable fixed point and the point $x = 1$ is a stable fixed point.

If we consider the 2-sphere S^2 as being obtained by rotating the a vertical half circle in the plane $\phi = 0$, we may perturb this circle so that it intersects with the line $r = \pi$

transversely finitely many times and then rotate this arc to obtain another 2-sphere. The singular foliation induced by $\mathcal{C}_1$ on this distorted 2-sphere will have several limit circles corresponding to the intersection points of the rotating arc with the line $r = \pi$, $\phi = 0$. The Poincaré map in this case will have several fixed points including the points $x = 0$ and $x = 1$. See Figure.

In general, let $f : I \to I$ be a diffeomorphism having only isolated fixed points including $x = 0$ and $x = 1$. Let $\mathcal{F}(f)$ be the corresponding singular foliation on S^2.

Let $g : I \to I$ be another diffeomorphism with isolated fixed points including $x = 0$ and $x = 1$. Assume the point $x = 1$ is an unstable fixed point of f and $x = 0$ is a stable fixed point of g. Then we can form the union of these two diffeomorphisms $f \cup g$ after an appropriate reparametrization, such that $f \cup g : I \to I$ is again a diffeomorphism with isolated fixed points including $x = 0$ and $x = 1$.

Definition 4.1. $\mathcal{F}(f)\#\mathcal{F}(g) = \mathcal{F}(f \cup g)$.

Lemma 4.2. *Let $f : I \to I$ be a diffeomorphism with isolated fixed points including $x = 0$ and $x = 1$. Assume that $f(x) > x$ for x near 0. Then the singular foliation $\mathcal{F}(f)\#\mathcal{F}(f_1)$ on S^2 is induced by a contact structure on a 3-ball B^3 with $\partial B^3 = S^2$.*

Proof. This becomes clear if one draws the graph of the diffeomorphism f and consider the parity of the number of fixed points. Since $x = 0$ is an unstable fixed point of f, if there are even number of fixed points, $x = 1$ is a stable fixed point. Otherwise, $x = 1$ is an unstable fix point. In either cases, by adding $\mathcal{F}(f_1)$ on the top of $\mathcal{F}(f)$, the resulting foliation on S^2 is diffeomorphic to the induced foliation of the standard overtwisted contact structure $\mathcal{C}_1$ on a distort S^2 as shown in Figure. $\square$

The following parametrized version of the previous lemma is a key ingredient in Eliashberg's classification of "overtwisted contact structure". Let P be a compact parameter space, and Q a closed subspace of P.

Lemma 4.3. *Suppose $f_t : I \to I$, $t \in P$ be a family of diffeomorphisms with isolated fixed points. For $t \in Q$, the singular foliation $\mathcal{F}(f_t)$ is induced by the standard contact structure $\mathcal{C}_0$ on a 2-sphere centered at the origin of $\mathbb{R}^3$. Then Then the singular foliation $\mathcal{F}(f_t)\#\mathcal{F}(f_1)$ on S^2 is induced by a contact structure $\mathcal{C}_t$ on a 3-ball B^3 with $\partial B^3 = S^2$ such that for $t \in Q$, $\mathcal{C}_t$ is isomorphic to $\mathcal{C}_1$ in the 3-ball of radius $5\pi/4$ centered at the origin of $\mathbb{R}^3$.*

4.1.2 Norm of tangent hyperplane fields

Let $A \subset \mathbb{R}^3$ be a compact subset. Let $\mathcal{D}$ be a co-oriented tangent hyperplane field defined in a neighborhood of A. The normal vector to $\mathcal{D}$ defines a Gauss map $g : A \to S^2$. For $x \in A$, denote $\|dg(x)\|$ the norm of the tangent map $dg(x) : T_x\mathbb{R}^3 \to T_{g(x)}S^2$, i.e.

$$\|dg(x)\| = \max\{\|dg(x)(v)\|\,;\, v \in T_x\mathbb{R}^3,\ \|v\| = 1\}.$$

Then we define

$$\|\mathcal{D}\| = \max\{\|dg(x)\|\,;\, x \in A\}.$$

Suppose we have 2-sphere S^2 in $\mathbb{R}^3$ whose Gaussian curvature lies between $K+\epsilon$ and K everywhere for $K > 0$ and small $\epsilon > 0$. Let $\mathcal{D}$ be a tangent hyperplane field defined in a neighborhood of the S^2 such that $\|\mathcal{D}\| \leq K$. Assume that this S^2 is in general position with respect to $\mathcal{D}$ so that D induces a singular foliation of S^2. Choose a simple geodesic c on S^2 connecting two singularities of this foliation on S^2.

Lemma 4.4. *Possibly after a small perturbation, c is transverse to the singular foliation on S^2 induced by $\mathcal{D}$ everywhere.*

4.1.3 Classification of overtwisted contact structure

Let M be an oriented connected 3-manifold. Let is fix a point $x_0 \in M$ and an imbedded 2-disk $\Delta \subset M$ centered at the point x_0. Denote by $\mathbf{D}(M)$ the space of all tangent hyperplane fields on M fixed at the point x_0 provided with the C^∞-topology. Denote by $\mathbf{C}(M)$ the subspace of $\mathbf{D}(M)$ consisting of all positive contact structures with their underlying tangent hyperplane fields, and $\mathbf{C}^{\mathrm{ot}}(M)$ the subspace of $\mathbf{C}(M)$ consisting of contact structures whose induced foliation on the 2-disk Δ is the *standard overtwisted foliation*, which is the foliation induced by the standard overtwisted contact structure $\mathcal{C}_1 = \ker(\alpha_1)$ on the upper hemisphere of radius π in $\mathbb{R}^3$.

Theorem 4.5. (Eliashberg) *The inclusion*

$$\mathbf{C}^{\mathrm{ot}}(M) \longrightarrow \mathbf{D}(M)$$

is a weak homotopy equivalence.

Thus, two overtwisted (positive) contact structures iff their underlying tangent hyperplane fields are homotopic. If the 3-manifold is open, Theorem 4.5 is a corollary of Gromov's theorem (Theorem 1.21).

The following theorem is a more technical version of Theorem 4.5.

Theorem 4.6. *Let M be a closed orientable 3-manifold and let $A \subset M$ be a colsed subset of M. Let P be a compact space and $Q \subset P$ a closed subset. Let $\mathcal{D}_t$, $t \in P$ be a family of tangent hyperplane fields which are contact everywhere for $t \in Q$ and contact near A for $t \in P$. Suppose there is an imbedded 2-disk $\Delta \subset M \setminus A$ such that for all $t \in P$, $\mathcal{D}_t$ is contact near Δ and whose induced foliation on Δ is the standard overtwisted foliation. Then there is a family of contact structures $\mathcal{C}_t$, $t \in P$, on M such that its underlying tangent hyperplane field coincides with $\mathcal{D}_t$ near A for all $t \in P$ and everywhere for all $t \in Q$. Moreover the underlying tangent hyperplane field of $\mathcal{C}_t$ can be connected with $\mathcal{D}_t$ by a homotopy of tangent hyperplane fields fixing $A \times P \cup M \times Q$.*

The first step in the proof of Theorem 4.6 is to modify $\mathcal{D}_t$ near the 2-skeleton of a triangulation of M.

Lemma 4.7. *Under the hypotheses of Theorem 4.6, there exist disjoint 3-balls* $B_1, B_2, \ldots, B_N \subset M \setminus (A \cup \Delta)$ *and a family of tangent hyperplane fields* $\mathcal{D}'_t$, $t \in P$, *on* M *such that*

(1) $\mathcal{D}'_t$ *coincides with* $\mathcal{D}_t$ *on* $A \cup \Delta$ *for* $t \in P$ *and everywhere for* $t \in Q$*;*

(2) $\mathcal{D}'_t$ *is contact on* $M \setminus \cup \operatorname{Int}(B_i)$*;*

(3) the singular foliation on ∂B_i *induced by* $\mathcal{D}'_t$ *has a transverse arc connecting its zeros, for each* $i = 1, 2, \ldots, N$*;*

(4) $\mathcal{D}'_t$ *is* C^0*-close to* $\mathcal{D}_t$, $t \in P$.

Item (3) is the delicate part of the lemma. Its proof involves some careful consideration of the norm $\|\mathcal{D}_t\|$.

The second step in the proof of Theorem 4.6 is to connect the 3-balls $B_1, B_2, \ldots, B_N$ in Lemma 4.7 together using Legendrian arcs. This is based on the following lemma.

Lemma 4.8. *Let* P *be a compact space,* M *be a oriented 3-manifold,* $\mathcal{C}_t$, $t \in P$, *be a family of contact structures on* M*, and* $\phi_t : I \to M$, $t \in P$, *be a family of imbeddings. Then* C^0*-close to* ϕ_t*, there exists a family of imbeddings* ϕ'_t *with* $\phi'_t|_{\partial I} = \phi_t|_{\partial I}$ *consisting of any of followings: (a) positive transverse arcs, (b) negative transvers arcs, and (c) Legendrian arcs.*

Finally, to finish the proof of Theorem 4.6, we use transverse arcs to joint $\partial B_1, \partial B_2, \ldots, \partial B_N$ and the boundary of a small neighborhood of Δ at their poles. Use the method discussed in Section 4.1.1, we may extend the contact structures $\mathcal{D}'_t$, $t \in P$, in Lemma 4.8 from $M \setminus \cup \operatorname{Int} B_i$ to M.

4.2 Tight contact structures

Definition 4.9. *A contact structure is called* tight *if for any imbedded 2-disk* Δ*, the induced foliation on* Δ *contains no limit cycles.*

Lemma 4.10. *A contact structure is tight iff it is not overtwisted.*

Proof. It follows from definition that tight implies not overtwisted. Let's prove the other direction. Assume that a contact structure not overtwisted, i.e. it contains no 2-disks with the standard overtwisted foliation as the induced foliation. Let us assume that there is an imbedded disk whose induced foliation has a limit circle C. We may asusme that there is no other limit circle inside of C. Let D be the disk bounded by C. Notice that the sum of indices of singular points inside of D is 1. Therefore, if there is no hyperbolic singular points, there can be only one elliptic point in D and we find that D has the stardard overtwisted foliation as its induced foliation. We now try to eliminate hyperbolic singular points inside of D. This is based on the Elimination Lemma and the Creation Lemma. (To be continued.) □

A oriented 3-manifold M is equipped with a co-oriented contact structure. Let F be an imbedded oriented surface in M which in general position with respect to the contact structure. The induced foliation on F has two types of singularities: elliptic and hyperbolic. Each singularity can have a sign, it is positive when the co-orientation of the contact plane is consistent with the orientation of the surface and it is negative otherwise. The leaves of the induced foliation have induced directions. We assume that positive elliptic singularities are sources of the flow.

Lemma 4.11. (Elimination Lemma) *Suppose the induced foliation on F has a leaf connecting an elliptic singularity p and a hyperbolic singularity q of the same sign. Let U be a neighborhood of this leaf, which contains no other singularities except p and q. Then a small C^0-isotopy of F in M, which is supported in U and fixed at the leaf, such that the new imbedding of F has no singularities in U. If p and q belong to the Legendrian boundary of F, one can eliminate them leaving ∂F fixed.*

Lemma 4.12. (Creation Lemma) *By a small C^0-isotopy of F near a non-singular point of the induced foliation, we can always create a pair singular points for the induced foliation, one elliptic and one hyperbolic, having the same pre-specified sign.*

Proof of Lemma 4.10 (continued). Suppose we find a hyperbolic singular point q in D. Suppose the stable leaf of q comes from an elliptic singular point p. If q is positive, we can use the Elimination Lemma to kill p and q. If q is negative, then in the case that the unstable leaf of q comes from an elliptic point p', p' and q have the same sign. Thus we can kill p' and q.

If these cases do not happen, both unstable leaves of q will be attracted by the limit circle C. So, then will become very close after a certain time. Then we can create a pair of singularities, an elliptic point e and a hyperbolic point h so that the unstable leaves of q now meet at e. Conside the circle C' which is the union of these two unstable leaves of q. It bounds a disk D'. We can use the Elimination Lemma again to change C' a limit circle bounding the disk D'. This reduces the number of hyperbolic singularities. □

Theorem 4.13. (Eliashberg) *A tight contact structure on S^3 is isotopic to the standard contact structure.*

Theorem 4.14. (Eliashberg) *For any closed 3-manifold only finite many homotopy classes of tangent hyperplane fields can contain tight contact structures.*

Conjecture 4.15. (Eliashberg) *Any closed 3-manifold admits, up to isotopy, only finitely many tight contact structures.*

Consider the lens space $L(p,q)$, where $p > q > 0$ and $(p,q) = 1$. Assume we have the continued fraction expansion

$$-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 - \cdots \frac{1}{r_k}}},$$

with all $r_i < -1$.

Theorem 4.16. (Honda) *There exist exactly* $|(r_0 + 1)(r_1 + 1) \cdots (r_k + 1)|$ *tight contact structures on the lens space* $L(p, q)$ *up to isotopy. Moreover, all the tight contact structures on* $L(p, q)$ *can be obtained from Legendrian surgery on links in* S^3*, and are therefore holomorphically fillable.*

Bibliography

[1] V. I. Arnold, Topological Invariants of Plane Curves and Caustics, University Lecture Series, vol. 5, AMS, Providence, 1994.

[2] D. Bennequin, *Entrelacements et équations de Pfaff*, Astérisque **107–108** (1983), pp. 87–161. English transaltion: *Links and Pfaff's equations*, Russian Math. Surveys **44**:3 (1989), pp. 1-65.

[3] S.-S. Chern, *The geometry of G-structure*, Bull. Amer. Math. Soc. **72**(1966), pp. 167–219.

[4] Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. **98** (1989), pp. 623–637.

[5] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier, Grenoble **42** (1992), pp. 165–192.

[6] Y. Eliashberg and W. Thurston, Confoliation, University Lecture Series, vol. 13, AMS, Providence, 1998.

[7] K. Honda, *On the classification of tight contact structures I*, Geometry and Topology **4** (2000), pp. 309–368.