ON THE BRIDGE NUMBER OF KNOT DIAGRAMS WITH MINIMAL CROSSINGS

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ABSTRACT. Given a diagram D of a knot K, we consider the number $c(D)$ of crossings and the number $b(D)$ of overpasses of D. We show that, if D is a diagram of a nontrivial knot K whose number $c(D)$ of crossings is minimal, then $1 + \sqrt{1 + c(D)} \leq b(D) \leq c(D)$. These inequalities are shape in the sense that the upper bound of $b(D)$ is achieved by alternating knots and the lower bound of $b(D)$ is achieved by torus knots. The second inequality becomes an equality only when the knot is an alternating knot. We prove that the first inequality becomes an equality only when the knot is a torus knot.

1. INTRODUCTION

A diagram D of a knot K is a nice representative of the knot type $[K]$ which is the isotopy class of K. It can be obtained from a regular planar projection (or simply, regular projection) P of K as follows. Let us take a sufficiently small neighborhood of each double point of P so that the intersection of the neighborhood and P is of the 'shape X' on the plane. Then modify the interior of each neighborhood so that we get a knot D which is isotopic to K and regularly projected to P. In this sense, a knot diagram can be 'almost planar', i.e., it lies in the plane except for a sufficiently small neighborhood of each double point of the regular planar projection.

Tabulation of knots is usually done by listing knots using their diagrams of minimal numbers of crossings. In general, such a minimal knot diagram is not unique within its isotopy class. Can this nonuniqueness be fixed by considering some other quantities associated with knot diagrams? In this paper, we study the number of overpasses of a minimal knot diagram. An immediate concern in this investigation is the relationship between the number of crossings and the number of overpasses of a knot diagram. As it turns out, the number of crossings can be estimated from below and above by that of overpasses if the number of crossings is minimal among all diagrams of the same knot type (Theorem 2.9).

As an interesting consequence of this estimation of the minimal crossing number by the number of overpasses, we may put two nice and relatively well understood classes of knots, that of alternating knots and that of $(k-1, \pm k)$-torus knots, on the opposite

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extremes of a measurement of nonalternatingness of knot diagrams (Theorem 2.10). Under this measurement, we prove that the 'most nonalternating' knot diagrams are exactly the standard minimal $(k-1, \pm k)$-torus knot diagrams (Theorem 3.1).

Notice that the minimal number of overpasses of a knot was the classical *bridge number* of a knot, first studied by Schubert in [5], where the effect of various operations on knots (satellite, cabling, connected sum) on this number was investigated.

Throughout this paper, all knots are oriented and lie in the 3-dimensional sphere S^3. Also, all knots are tame, i.e., they are isotopic to polygonal or smooth knots. Hence, every knot has a diagram whose number of crossings is finite. For convenience, we distinguish knot diagrams and knots. From now on, a knot means an isotopy knot type $[D]$ for some knot diagram D. We will refer the reader to textbooks of knot theory (such as [1] or [2]) for standard terminologies. Also, we assume the *well ordering property* of the set $N \cup \{0\}$ of nonnegative integers to guarantee existence of the smallest element of any nonempty subset of $N \cup \{0\}$.

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2. Minimal crossings and bridge number of knot diagrams

The number of crossings of a knot diagram D is denoted by $c(D)$. For each knot K, we denote $min\{c(D) \mid D \text{ is a diagram of } K\}$ by $c(K)$. Note that we may assume a knot diagram D lies in the plane by indicating 'overcrossings' and 'undercrossings'. A 'crossing', in normal sense, of a knot diagram D means a 'double point' of the regular projection of D. Hence, $c(D)$ is the number of all double points of the regular projection of D. If x is a crossing of a knot diagram D, we denote the overcrossing and the undercrossing of D projected to x by x_+ and x_-, respectively, where x_+ is above x and x_- is below x. Alternatively, we may regard a crossing of D as a pair of two points, overcrossing and undercrossing, in D which are projected to the same double point. In this case, a crossing is considered as the preimage of a double point under the projection map.

Proposition 2.1. *Let D be a knot diagram with $c(D) \geq 1$. Then there is a unique positive integer k such that there is a finite sequence*

$$s_1, f_1, s_2, f_2, ..., s_k, f_k$$

of $2k$ points of D, each of which is neither an overcrossing nor an undercrossing of D, such that

$$[s_1, f_1], [s_2, f_2], ..., [s_{k-1}, f_{k-1}], [s_k, f_k]$$

and

$$[f_1, s_2], [f_2, s_3], ..., [f_{k-1}, s_k], [f_k, s_1]$$

are the overpasses *and the* underpasses *of D with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$, respectively, where $[s_i, f_i]$, for each $i \in \{1, ..., k\}$, is the closed arc of D from s_i to f_i which contains at least one overcrossing but has no undercrossing; similarly, $[f_i, s_{i+1}]$, for each $i \in \{1, ..., k\}$, with the subscript counted modulo k, is the closed arc of D from f_i to s_{i+1} which contains at least one undercrossing but has no overcrossing.*

Proof. We may assume $c(D) \geq 2$. If $c(D) = 1$, then D is of the 'shape 8', and hence, $k = 1$.

Existence: We will construct a finite sequence as stated above. The following procedure is one way to get it. Let us start at a point $*$ on D slightly before an undercrossing and go along with D until arriving at the first overcrossing a_1 from $*$. Let a_2 be the undercrossing just before a_1. Take a point s_1 of D between a_2 and a_1. Then go along with D from s_1 until arriving at the first undercrossing a_3 from s_1, and let a_4 be the overcrossing just before a_3. Take a point f_1 of D between a_4 and a_3. We may now repeat this procedure to take the other points $s_2, f_2, s_3, f_3, ..., s_k, f_k$ until there is no overcrossing between f_k and $*$. Since the number $c(D)$ of crossings of D is finite, k must be finite. By its construction, the sequence of points $s_1, f_1, s_2, f_2, ..., s_k, f_k$ of D is a one as stated in the proposition.

Uniqueness: Suppose that l is a positive integer and $s'_1, f'_1, s'_2, f'_2, ..., s'_l, f'_l$ is a sequence of D as stated in the proposition. Let

$$a_2, a_1, a_4, a_3, ..., a_{2(2k-1)}, a_{2(2k-1)-1}, a_{2(2k)}, a_{2(2k)-1}$$

be the sequence of points on D as described above. Then s'_1 must be contained in an arc of D which contains s_i for some $i \in \{1, ..., k\}$ but no overcrossing and no undercrossing of D. In other words, s'_1 must be between $a_{2(2i-1)}$ and $a_{2(2i-1)-1}$ for some $i \in \{1, ..., k\}$. Hence, f'_1 must be between $a_{2(2i)}$ and $a_{2(2i)-1}$. Keep going on like this. If $i = 1$, then f'_l must be between $a_{2(2k)}$ and $a_{2(2k)-1}$, and if $1 < i \leq k$, then f'_l must be between $a_{2(2(i-1))}$ and $a_{2(2(i-1))-1}$. Hence, we have $l = k$. This proves the uniqueness of k. □

Such a sequence $s_1, f_1, s_2, f_2, ..., s_k, f_k$ as in Proposition 2.1 is said to be an *over-underpass sequence* of D. Since any over-underpass sequence of D consists of $2k$ points, the number of overpasses (or underpasses) with respect to any over-underpass sequence of D is k. Hence, we define the number of overpasses (or underpasses) of the knot diagram D as k, and denote it by $b(D)$, sometimes called the length of over-underpass sequence. If a knot diagram D has no crossing, we define $b(D)$ as 0. Also, for each knot K, we denote $min\{b(D) \mid D \textit{ is a diagram of } K\}$ by $b(K)$. This number $b(K)$ is called the *bridge number* of K.

Notice that (1) $c(D)$ and $b(D)$ are plane isotopy invariants of knot diagrams, i.e., if D_1 and D_2 are plane isotopic knot diagrams, then $c(D_1) = c(D_2)$ and $b(D_1) = b(D_2)$; (2) $c(K)$ and $b(K)$ are isotopy invariants of knots, i.e., if K_1 and K_2 are isotopic knots, then $c(K_1) = c(K_2)$ and $b(K_1) = b(K_2)$.

Corollary 2.2. *If $b(D) \leq 1$, then D is a diagram of a trivial knot. Therefore, a knot K is trivial if and only if K has a diagram D with $b(D) \leq 1$.*

Obviously, for any positive integer k, there is a diagram of a trivial knot whose number of overpasses is greater than k. On the other hand, given a knot diagram D with at least one crossing, we can add crossings to D as many as we want without changing the knot type and the number of overpasses of D. For example, take a sufficiently small arc of D from s_1 to a point between s_1 and the first overcrossing of D from s_1, and twist it alternatively so that the number of overpasses of D is not changed. Or, we may modify the interior of a sufficiently small neighborhood of a crossing of D to achieve the goal of increasing $c(D)$ arbitrarily while keeping $b(D)$ and the knot type of D fixed. Hence, we have the following corollary.

Corollary 2.3. *If D is a diagram of a knot K such that $c(D) \geq 1$, then for every positive integer n, there is a diagram D' of K such that $b(D') = b(D)$ and $c(D') \geq c(D) + n$.*

Remark that the number of crossings of a knot diagram with a minimal number of overpasses can be arbitrarily large.

Lemma 2.4. *$b(D) \leq c(D)$ for any knot diagram D. The equality holds if and only if D is an alternating knot diagram. Furthermore, $b(K) \leq c(K)$ for any knot K.*

Proof. Suppose that $b(D) = k \geq 1$ and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of D. Let m_i be the number of overcrossings of D on $[s_i, f_i]$ for each $i \in \{1, ..., k\}$, n_i the number of undercrossings of D on $[f_i, s_{i+1}]$ for each $i \in \{1, ..., k-1\}$, and n_k the number of undercrossings of D on $[f_k, s_1]$. Then

$$2c(D) = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} n_i \geq \sum_{i=1}^{k} 1 + \sum_{i=1}^{k} 1 = 2b(D).$$

D is an alternating knot diagram if and only if every overpass and underpass has exactly one overcrossing and undercrossing, respectively, i.e., $m_i = n_i = 1$ for each $i \in \{1, ..., k\}$.

Also, it follows immediately that $b(K) \leq c(K)$ for any knot K. □

The following lemma is the first key to prove the first main theorem of this paper (Theorem 2.9). By this lemma, if an overpass of a knot diagram D with respect to an over-underpass sequence of D crosses an underpass more than once, then the number of crossings of D is not minimal any more.

Lemma 2.5. *If D is a minimal diagram of a knot K with respect to crossings, i.e., $c(D) = c(K)$, $b(D)$ is a nonnegative integer k, and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of D, then every overpass of D with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses each underpass at most once.*

Proof. We may assume K is nontrivial. If K is trivial, then $c(D) = c(K) = 0$, hence, $b(D) = 0$. Let D be a diagram of a knot K such that $c(D) = c(K)$. Suppose that $b(D) = k \geq 1$ and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of D. Then, by Lemma 2.4, $c(D) \geq k \geq 1$. Let $o_i = [s_i, f_i]$ and $u_i = [f_i, s_{i+1}]$ for each $i \in \{1, ..., k-1\}$, and let $o_k = [s_k, f_k]$ and $u_k = [f_k, s_1]$. Suppose that the number of crossings between o_i and u_j is at least 2 for some $i, j \in \{1, ..., k\}$. Let x_1 and x_2 be two crossings of D such that x_{1+} and x_{2+} are the first and the second overcrossings from s_i among all overcrossings between o_i and u_j, respectively, and let y_1 and y_2 be two crossings of D such that y_{1-} and y_{2-} are the first and the second undercrossings from f_j in $\{x_{1-}, x_{2-}\}$, respectively. Then either $y_1 = x_1$, $y_2 = x_2$ or $y_1 = x_2$, $y_2 = x_1$. There are two different diagrams for each case. One of them for the case of $y_1 = x_1$, $y_2 = x_2$ is shown in Figure 1. It should be not hard for the reader to figure out the other diagrams.

Let r be the number of overcrossings on the arc $\overrightarrow{x_{1+}x_{2+}}^{o_i}$ of o_i from x_{1+} to x_{2+}, and let s be the number of undercrossings on the arc $\underrightarrow{y_{1-}y_{2-}}_{u_j}$ of u_j from y_{1-} to y_{2-}. To prove this lemma, for each of the cases $r \leq s$ and $r > s$, we will construct a diagram D' of K by modifying an arc of o_i or u_j such that $c(D') < c(D)$. Temporarily, we regard the knot diagram D as its regular projection. Hence, D lies in the plane, and all overcrossings and all undercrossings of D are the double points of D as the regular projection. Consider the r double points $d_1, d_2, ..., d_r$ on the arc $\overrightarrow{x_1x_2}^{o_i}$ of o_i from x_1 to x_2 such that $d_1 < d_2 < ... < d_r$ with respect to the order from s_i and the s double points $e_1, e_2, ..., e_s$ on the arc $\underrightarrow{y_1y_2}_{u_j}$ of u_j from y_1 to y_2 such that $e_1 < e_2 < ... < e_s$ with respect to the order from f_j. Then $x_1 = d_1$, $x_2 = d_r$ and $y_1 = e_1$, $y_2 = e_s$. Notice that we can take a sufficiently small positive real number ϵ, an ϵ-neighborhood $U_{o_i,\epsilon}$ of $\overrightarrow{x_1x_2}^{o_i}$, and an ϵ-neighborhood $V_{u_j,\epsilon}$ of $\underrightarrow{y_1y_2}_{u_j}$ such that $\overline{U_{o_i,\epsilon}} \cap \{s_1, f_1, s_2, f_2, ..., s_k, f_k\} = \varnothing$, $\overline{V_{u_j,\epsilon}} \cap \{s_1, f_1, s_2, f_2, ..., s_k, f_k\} = \varnothing$, the set of all double points of D contained in $\overline{U_{o_i,\epsilon}}$ is $\{d_1, d_2, ..., d_r\}$, the set of all double points of D contained in $\overline{V_{u_j,\epsilon}}$ is $\{e_1, e_2, ..., e_s\}$, and, for every positive real number $\epsilon' \leq \epsilon$, $|Bd(U_{o_i,\epsilon'}) \cap D| = 2(r+1)$ and $|Bd(V_{u_j,\epsilon'}) \cap D| = 2(s+1)$, where $Bd(U_{o_i,\epsilon'})$ and $Bd(V_{u_j,\epsilon'})$ are the boundaries of $U_{o_i,\epsilon'}$ and $V_{u_j,\epsilon'}$, respectively. Hence, we may assume that $\overline{U_{o_i,\epsilon}} \cap D = \overrightarrow{a_1a_2}^{o_i} \cup (l_1 \cup l_2 \cup ... \cup l_r)$ and $\overline{V_{u_j,\epsilon}} \cap D = \underrightarrow{b_1b_2}_{u_j} \cup (l^1 \cup l^2 \cup ... \cup l^s)$, where a_1 and a_2 are the first and the second points from s_i in $Bd(U_{o_i,\epsilon}) \cap o_i$, respectively, b_1 and b_2 are the first and the second points from f_j in $Bd(V_{u_j,\epsilon}) \cap u_j$, respectively, l_t is an arc of the underpass passing through d_t whose endpoints are on $Bd(U_{o_i,\epsilon})$ for each $t \in \{1, ..., r\}$, and l^t is an arc of the overpass passing through e_t whose endpoints are on $Bd(V_{u_j,\epsilon})$ for each $t \in \{1, ..., s\}$. Remark that $\{l_1, ..., l_r\}$ is pairwise disjoint, $l_t \cap o_i = \{d_t\}$ and $|l_t \cap Bd(U_{o_i,\epsilon})| = 2$ for each $t \in \{1, ..., r\}$, and $|Bd(U_{o_i,\epsilon}) \cap o_i| = 2$; similarly, $\{l^1, ..., l^s\}$ is pairwise disjoint, $l^t \cap u_j = \{e_t\}$ and $|l^t \cap Bd(V_{u_j,\epsilon})| = 2$ for each $t \in \{1, ..., s\}$, and $|Bd(V_{u_j,\epsilon}) \cap u_j| = 2$.

Let p_1, p_2, p_3, p_4 be the first, the second, the third, the fourth points from f_j in $Bd(U_{o_i,\epsilon}) \cap (l_1 \cup l_r)$, respectively, and let α be a point on $\underrightarrow{p_1p_2}_{u_j}$ between p_1 and y_1. Draw an arc A in $U_{o_i,\epsilon}$ starting at α so that A intersects l_t transversely only once for each $t \in \{1, ..., r\}$ but A does not intersect o_i. Then we take β as the intersecting point of A and $\underrightarrow{p_3p_4}_{u_j}$ and denote the arc of A from α to β by $\overline{\alpha\beta}$. Similarly,

let q_1, q_2, q_3, q_4 be the first, the second, the third, the fourth points from s_i in $Bd(V_{u_j,\epsilon}) \cap (l^1 \cup l^s)$, respectively, and let γ be a point on $\overrightarrow{q_1q_2}^{o_i}$ between q_1 and x_1. Draw an arc B in $V_{u_j,\epsilon}$ starting at γ so that B intersects l^t transversely only once for each $t \in \{1, ..., s\}$ but B does not intersect u_j. Then we take δ as the intersecting point of B and $\overrightarrow{q_3q_4}^{o_i}$ and denote the arc of B from γ to δ by $\overline{\gamma\delta}$.

From now on, D is the diagram of K again, i.e., D is not a regular projection of a knot but a knot diagram. Let $\underrightarrow{\alpha\beta}$ be a regularly projecting arc in R^3 whose regular projection is $\overline{\alpha\beta}$ such that $\underrightarrow{\alpha\beta}$ has no overcrossing, and let $\overrightarrow{\gamma\delta}$ be a regularly projecting arc in R^3 whose regular projection is $\overline{\gamma\delta}$ such that $\overrightarrow{\gamma\delta}$ has no undercrossing. Then

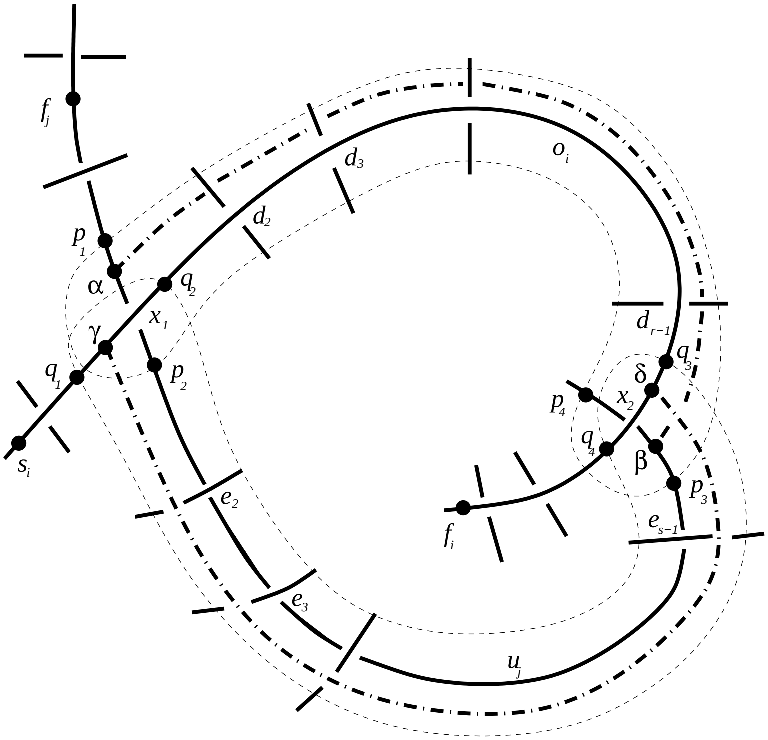


Figure 1. Proof of Lemma 2.5: Case 1.(1) and Case 2.(2) when $y_1 = x_1$, $y_2 = x_2$.

Case 1. $r \leq s$: Let $D' = (D - \underrightarrow{\alpha\beta}_{u_j}) \cup \underrightarrow{\alpha\beta}$. Then D' is a diagram of K.

(1) If $\underrightarrow{p_1p_2}_{u_j}$ and $\underrightarrow{p_3p_4}_{u_j}$ make the same sign with o_i, then
$c(D') = c(D) - (s-1) + (r-2)$, hence, $c(D) = c(D') + (s-r) + 1 > c(D')$;
(2) If $\underrightarrow{p_1p_2}_{u_j}$ and $\underrightarrow{p_3p_4}_{u_j}$ make opposite signs with o_i, then
$c(D') = c(D) - s + (r-2)$, hence, $c(D) = c(D') + (s-r) + 2 > c(D')$.
Case 2. $r > s$: Let $D' = (D - \overrightarrow{\gamma\delta}^{o_i}) \cup \overrightarrow{\gamma\delta}$. Then D' is a diagram of K.
(1) If $\overrightarrow{q_1q_2}^{o_i}$ and $\overrightarrow{q_3q_4}^{o_i}$ make the same sign with u_j, then
$c(D') = c(D) - (r-1) + (s-2)$, hence, $c(D) = c(D') + (r-s) + 1 > c(D')$;
(2) If $\overrightarrow{q_1q_2}^{o_i}$ and $\overrightarrow{q_3q_4}^{o_i}$ make opposite signs with u_j, then
$c(D') = c(D) - r + (s-2)$, hence, $c(D) = c(D') + (r-s) + 2 > c(D')$.
Therefore, we have a diagram D' of K such that $c(D') < c(D)$. This is a contradiction to $c(D) = c(K)$. □

Now, we prove the second key lemma for Theorem 2.9 by a similar idea as the one used in the proof of Lemma 2.5. By this lemma, if an overpass of a knot diagram D with respect to an over-underpass sequence of D crosses an adjacent underpass or an underpass crosses an adjacent overpass, then the number of crossings of D is not minimal.

Lemma 2.6. *If D is a minimal diagram of a knot K with respect to crossings, $b(D) = k \geq 2$, and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of D, then no overpass of D with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses its adjacent underpasses and no underpass crosses its adjacent overpasses, where the adjacent underpasses of the i-th overpass o_i are the $(i-1)$-th and the i-th underpasses u_{i-1} and u_i for each $i \in \{2, ..., k\}$; the adjacent underpasses of o_1 are u_k and u_1; the adjacent overpasses of u_i are o_i and o_{i+1} for each $i \in \{1, ..., k-1\}$; the adjacent overpasses of u_k are o_k and o_1.*

Proof. We may assume that each overpass of D with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses each underpass at most once by Lemma 2.5.

Suppose that there is $i \in \{1, ..., k\}$ such that the i-th overpass $o_i = [s_i, f_i]$ of D with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses the i-th underpass $u_i = [f_i, s_{i+1}]$, with the subscript counted modulo k. Let x be the crossing of D between o_i and u_i. Let y be the crossing of D such that y_+ is the overcrossing of o_i just before f_i, and let z be the crossing of D such that z_- is the undercrossing of u_i just after f_i. Note that x, y, z need not be distinct. However, if some of them are identical, then, by Reidemeister moves, we can reduce the crossing easily. Let r be the number of overcrossings on the arc $\overrightarrow{x_+y_+}^{o_i}$ of o_i from x_+ to y_+, and let s be the number of undercrossings on the arc $\underrightarrow{z_-x_-}_{u_i}$ of u_i from z_- to x_-. Temporarily, we regard the knot diagram D as its regular projection as we did in the proof of Lemma 2.5. Let us take an ϵ-neighborhood $U_{o_i,\epsilon}$ of $\overrightarrow{xy}^{o_i}$ and an ϵ-neighborhood $V_{u_i,\epsilon}$ of $\underrightarrow{zx}_{u_i}$ for a sufficiently small positive real number

ϵ as described in the proof of Lemma 2.5. Let a_1 and a_2 be the first and the second points from s_i in $Bd(U_{o_i,\epsilon/2}) \cap o_i$, respectively, and let a_3 and a_4 be the first and the second points from s_i in $Bd(U_{o_i,\epsilon/2}) \cap u_i$, respectively. Similarly, let b_1 and b_2 be the first and the second points from s_i in $Bd(V_{u_i,\epsilon/2}) \cap o_i$, respectively, and let b_3 and b_4 be the first and the second points from s_i in $Bd(V_{u_i,\epsilon/2}) \cap u_i$, respectively. Let $\overline{a_2a_4}$ be the arc of $Bd(U_{o_i,\epsilon/2})$ from a_2 to a_4 such that $|\overline{a_2a_4} \cap D| = (r-1)+2$, and let $\overline{b_1b_3}$ be the arc of $Bd(V_{u_i,\epsilon/2})$ from b_1 to b_3 such that $|\overline{b_1b_3} \cap D| = (s-1)+2$.

From now on, D is the diagram of K again. Let $\underrightarrow{a_2a_4}$ be a regularly projecting arc in R^3 whose regular projection is $\overline{a_2a_4}$ such that $\underrightarrow{a_2a_4}$ has no overcrossing, and let $\overrightarrow{b_1b_3}$ be a regularly projecting arc in R^3 whose regular projection is $\overline{b_1b_3}$ such that $\overrightarrow{b_1b_3}$ has no undercrossing. Then

Case 1. $r \leq s$: Let $D' = (D - (\overrightarrow{a_2f_i}^{o_i} \cup \underrightarrow{f_ia_4}_{u_i})) \cup \underrightarrow{a_2a_4}$. Then D' is a diagram of K and $c(D') = c(D) - s + (r-1)$, hence, $c(D) = c(D') + (s-r) + 1 > c(D')$.

Case 2. $r > s$: Let $D' = (D - (\overrightarrow{b_1f_i}^{o_i} \cup \underrightarrow{f_ib_3}_{u_i})) \cup \overrightarrow{b_1b_3}$. Then D' is a diagram of K and $c(D') = c(D) - r + (s-1)$, hence, $c(D) = c(D') + (r-s) + 1 > c(D')$.

Therefore, we have a diagram D' of K such that $c(D') < c(D)$. This is a contradiction to $c(D) = c(K)$.

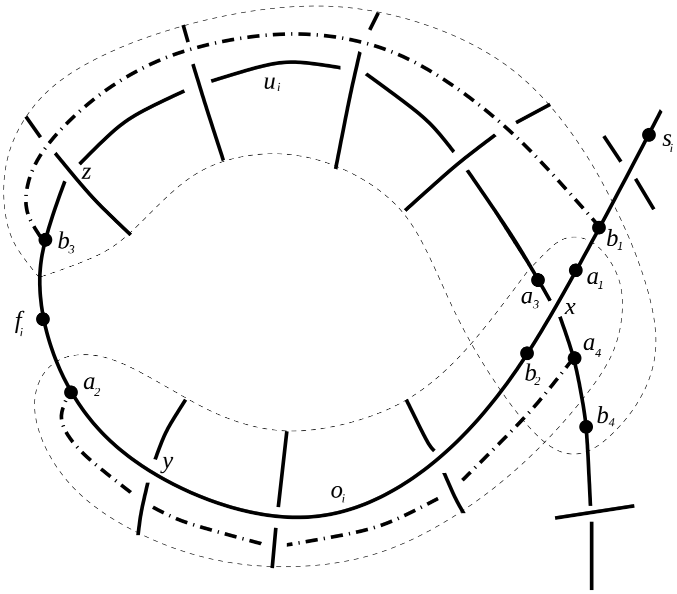


Figure 2. Proof of Lemma 2.6.

As the other case, if there is $i \in \{1, ..., k\}$ such that the i-th underpass $u_i = [f_i, s_{i+1}]$ of D with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses the $(i+1)$-th overpass $o_{i+1} = [s_{i+1}, f_{i+1}]$, with the subscript counted modulo k, then we can construct a diagram D' of K such that $c(D') < c(D)$ by the same argument above. This proves the lemma. □

The following two lemmas are immediate consequences of Lemma 2.6 and the other keys for Theorem 2.9.

Lemma 2.7. *If D is a minimal diagram of a knot K with respect to crossings, $b(D)$ is a positive integer k, and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of D, then there is no overpass of D with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ which crosses all underpasses.*

Proof. Suppose that an overpass of D with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses all underpasses. Then the overpass crosses its adjacent underpasses. This is a contradiction to $c(D) = c(K)$ by Lemma 2.6. □

Lemma 2.8. *If D is a minimal diagram of a knot K with respect to crossings, $b(D) = k \geq 2$, and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of D, then every overpass of D with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses at most $k-2$ underpasses.*

Proof. Suppose that an overpass o of D with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses $k-1$ underpasses. Then, by Lemma 2.5, o crosses each of the $k-1$ underpasses exactly once. Since every overpass has two adjacent underpasses and o crosses $k-1$ underpasses, o must cross at least one of its adjacent underpasses. This is a contradiction to $c(D) = c(K)$ by Lemma 2.6. □

Theorem 2.9. *If D is a minimal diagram of a knot K with respect to crossings, then $b(D) \leq c(D) \leq b(D)(b(D) - 2)$.*

Proof. Let D be a diagram of a knot K such that $c(D) = c(K)$. If K is a trivial knot, then D is a simple closed curve on the plane, hence, $c(D) = b(D) = 0$. Remark that $b(D)$ can not be 1. If $b(D) = 1$, then, by Corollary 2.2, D represents a trivial knot, i.e., K is a trivial knot, hence, $c(D) = c(K) = 0$. This contradicts to Lemma 2.4. Next, we claim that $b(D)$ can not be 2 if D is a minimal diagram of a nontrivial knot K with respect to crossings. This follows from the same argument in the proof of Lemma 2.8. Suppose that K is a nontrivial knot, $c(D) = c(K)$, $b(D) = 2$, and s_1, f_1, s_2, f_2 is an over-underpass sequence of D. Then each overpass of D with respect to s_1, f_1, s_2, f_2 must cross one of the underpasses. However, all underpasses are adjacent to each overpass. Suppose that $b(D) = k \geq 3$ and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of D. We may assume that the overpasses $[s_1, f_1], [s_2, f_2], ..., [s_{k-1}, f_{k-1}], [s_k, f_k]$ are disjoint closed arcs on the projection plane of D and the underpasses $[f_1, s_2], [f_2, s_3], ..., [f_{k-1}, s_k], [f_k, s_1]$ are also disjoint closed arcs on it. Remark that the number of crossings of D is the number of intersections of projections of overpasses and underpasses of D. Since we have k overpasses and k underpasses, by Lemma 2.5 and Lemma 2.8, there are at most $k(k-2)$ crossings among overpasses and underpasses. □

Notice that $c(K) \leq b(K)(b(K)-2)$ does not hold in general. An example can be given with K being the $(2,p)$-torus knot. We have $b(K)=2$ in this case.

The following theorem is an immediate consequence of Theorem 2.9 if K is nontrivial. It means that the number of overpasses of a minimal knot diagram with respect to crossings can also be estimated by the number of crossings. Once more, in this case, the number of overpasses needs not be minimal and is at least 3 as shown in the proof of Theorem 2.9.

Theorem 2.10. *If K is a nontrivial knot and D is a minimal diagram of K with respect to crossings, then* $1+\sqrt{1+c(D)} \leq b(D) \leq c(D)$.

3. The most nonalternating knots are $(k-1, \pm k)$-torus knots

Theorem 2.10 provides us the most optimal lower bound for the number of overpasses of D when D is a minimal knot diagram with respect to crossings. Here, an interesting problem occurs. In Lemma 2.4, we showed that a knot diagram D is alternating if and only if $c(D)=b(D)$. Also, by Kauffman [3], an alternating knot has a minimal knot diagram with respect to crossings which is alternating. Hence, the second inequality of Theorem 2.10 becomes an equality only when the knot is an alternating knot. On the other hand, as shown by Murasugi [4], the standard diagram D of a $(k-1, \pm k)$-torus knot is a minimal knot diagram with respect to crossings and $c(D)=k(k-2)$. So the first inequality of Theorem 2.10 becomes an equality for the standard $(k-1, \pm k)$-torus knot diagram with $b(D)=k$. Is the converse true under the condition that D is a minimal knot diagram with respect to crossings? We prove this as the last theorem of this paper.

Theorem 3.1. *If D is a minimal diagram of a nontrivial knot K with respect to crossings and $c(D)=b(D)(b(D)-2)$, then D is the standard diagram of either the $(b(D)-1, b(D))$-torus knot or the $(b(D)-1, -b(D))$-torus knot. Hence, K is the $(b(D)-1, \pm b(D))$-torus knot.*

Proof. Let D be a minimal diagram of a nontrivial knot K with respect to crossings such that $c(D)=b(D)(b(D)-2)$. Suppose that $b(D)=k$. Then $k \geq 3$ (See the proof of Theorem 2.9). In order to prove this theorem, we will consider all possible knot diagrams satisfying the hypothesis and show that they can only be the standard diagrams of either the $(k-1, k)$-torus knot or the $(k-1, -k)$-torus knot depending on signs of crossings. For convenience, we regard the knot diagram D as its regular projection here. Also, to visualize overpasses and underpasses clearly, we imagine blue and red colors for overpasses and underpasses, respectively. In this sense, crossings are only the intersections of blue arcs and red arcs on the plane, i.e., crossings are purple!

Now, let us start drawing all possible knot diagrams satisfying the hypothesis. Notice that such knot diagrams must satisfy the following 3 rules:

Rule 1: Every overpass and every underpass intersect each underpass and each overpass at most once, respectively (By Lemma 2.5);

Rule 2: No overpass and no underpass intersect its adjacent underpasses and its adjacent overpasses, respectively (By Lemma 2.6);

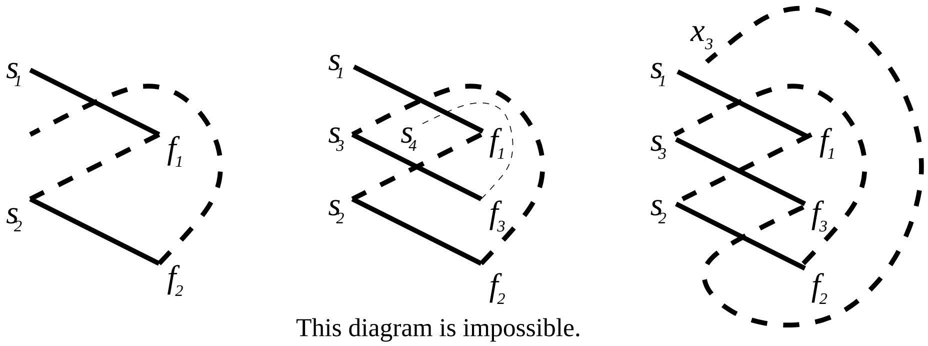
Rule 3: Every overpass and every underpass intersect $k-2$ underpasses and $k-2$ overpasses, respectively (By Lemma 2.8).

If some overpass has less than $k-2$ overcrossings, then at least one overpass has more than $k-2$ overcrossings; similarly, if some underpass has less than $k-2$ undercrossings, then at least one underpass has more than $k-2$ undercrossings. This is a contradiction to Lemma 2.8. From now on, we describe drawing the knot diagrams. Let us take a point s_1 on the plane and draw the first overpass $o_1 = [s_1, f_1]$. Draw the first underpass $u_1 = [f_1, s_2]$ and the second overpass $o_2 = [s_2, f_2]$. Until now, we can not have any crossing by Rule 2. When we draw the second underpass u_2 from f_2, we must make u_2 intersect o_1 first by Rule 2,3. Notice that we have only 2 cases that u_2 intersects o_1 first as follows.

Case 1. The sign $sign(u_2, o_1)$ of crossing between u_2 and o_1 is -1,

where $sign(u_2, o_1) = +1 = sign(o_1, u_2)$ if o_1 intersects u_2 from left to right and $sign(u_2, o_1) = -1 = sign(o_1, u_2)$ if o_1 intersects u_2 from right to left when we look u_2 as a line segment passing through the crossing upward. Notice that, by Rule 1, we can define the sign of each crossing by this way.

After u_2 intersects o_1 first with $sign(u_2, o_1) = -1$, we must change color from red to blue by Rule 1,2. Take a point s_3 so that $u_2 = [f_2, s_3]$ has only one undercrossing. Then the third overpass o_3 must intersect u_1 first with $sign(o_3, u_1) = -1$ by Rule 2,3. After o_3 intersects u_1 first, we must change color from blue to red by Rule 1,2. Take a point f_3 so that $o_3 = [s_3, f_3]$ has only one overcrossing. Then the third underpass u_3 can intersect only either o_1 first with $sign(u_3, o_1) = -1$ or o_2 first with $sign(u_3, o_2) = -1$. We claim that u_3 must intersect o_2 first with $sign(u_3, o_2) = -1$. Suppose that u_3 intersects o_1 first with $sign(o_3, u_1) = -1$. Then, after u_3 intersects o_1 first, we must change color from red to blue by Rule 1,2. Take a point s_4 so that $u_3 = [f_3, s_4]$ has only one undercrossing. In this case, we can not draw any knot diagram such that u_3 intersects o_2. This is a contradiction to Rule 2,3. Hence, u_3 intersects o_2 first with $sign(u_3, o_2) = -1$. After u_3 intersects o_2 first, take a point x_3 which is neither s_1 nor a crossing so that the arc $[f_3, x_3]$ of u_3 has only one undercrossing. Notice that we can connect x_3 and s_1 by an arc without crossing.



This diagram is impossible.

Figure 3. The first three overpasses and underpasses, $sign(u_2, o_1) = -1$.

If we have only 3 overpasses and 3 underpasses, we must connect x_3 and s_1 by an arc without crossing. This is the only way to draw the knot diagrams satisfying Rule 1,2,3 and gives us the standard diagram of the $(2, -3)$-torus knot.

See Figure 3, where solid arcs are supposed to be of the blue color (overpasses) and dashed arcs of the red color (underpasses).

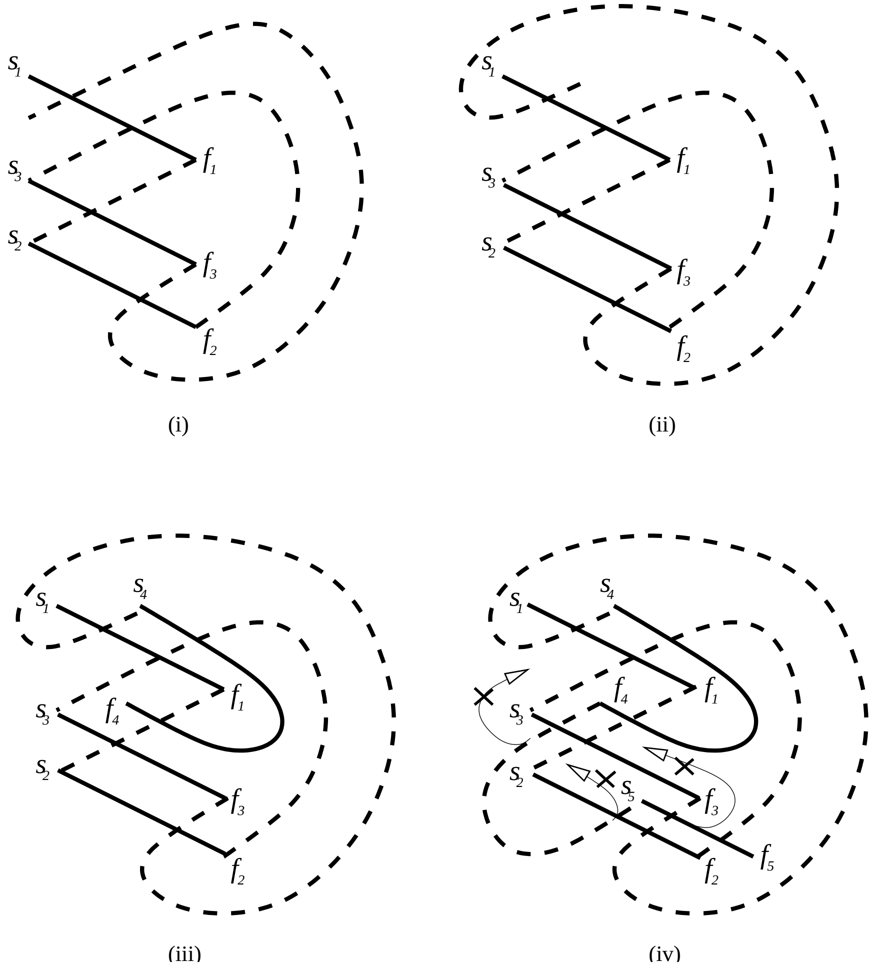
Suppose that we have more than 3 overpasses, i.e., $k > 3$. Then the third underpass u_3 must intersect o_1 second by Rule 1,2,3 and we have only 2 cases that u_3 intersects o_1 second.

Claim: u_3 must intersect o_1 second with $sign(u_3, o_1) = -1$. See Figure 4(i).

Proof of the Claim: Suppose that u_3 intersects o_1 second with $sign(u_3, o_1) = +1$. See Figure 4(ii). Then, after u_3 intersects o_1 second, we must change color from red to blue by Rule 1. Take a point s_4 so that $u_3 = [f_3, s_4]$ has only two undercrossings. Then o_4 must intersect u_2 first with $sign(o_4, u_2) = -1$ and u_1 second with $sign(o_4, u_1) = +1$ by Rule 1,2,3. See Figure 4(iii).

After o_4 intersects u_1 second, we must change color from blue to red by Rule 1. Take a point f_4 so that $o_4 = [s_4, f_4]$ has only two overcrossings. Then u_4 can intersect only either o_1 first with $sign(u_4, o_1) = +1$ or o_3 first with $sign(u_4, o_3) = -1$. However, u_4 can not intersect o_1 first. If u_4 intersects o_1 first with $sign(u_4, o_1) = +1$, we must change color from red to blue by Rule 1,2. By a similar argument as the one used before, in this case, we can not draw any knot diagram such that u_4 intersects o_2. This is a contradiction to Rule 2,3. Hence, u_4 must intersect o_3 first with $sign(u_4, o_3) = -1$.

After u_4 intersects o_3 first, u_4 can intersect only either o_1 second with $sign(u_4, o_1) = +1$ or o_2 second with $sign(u_4, o_2) = +1$. However, u_4 can not intersect o_1 second. If u_4 intersects o_1 second with $sign(u_4, o_1) = +1$, after u_4 intersects o_1 second, we must change color from red to blue by Rule 1,2. In this case, we can not draw any knot diagram such that u_4 intersects o_2. This is a contradiction to Rule 2,3. Hence, the only way to draw u_4 is that u_4 intersects o_3 first with $sign(u_4, o_3) = -1$ and o_2 second with $sign(u_4, o_2) = +1$. See Figure 4(iv).



(i) (ii)

(iii) (iv)

Figure 4. The first three overpasses and underpasses can only be drawn as in (i).

After u_4 intersects o_2 second, we must change color from red to blue by Rule 1. Take a point s_5 so that $u_4 = [f_4, s_5]$ has only two undercrossings. Then o_5 can intersect only either u_1 first with $sign(o_5, u_1) = +1$ or u_3 first with $sign(o_5, u_3) = -1$. However, o_5 can not intersect u_1 first. If o_5 intersects u_1 first with $sign(o_5, u_1) = +1$, then we must change color from blue to red by Rule 1,2. In this case, we can not draw any knot diagram such that o_5 intersects u_2. This is a contradiction to Rule 2,3. Hence, o_5 must intersect u_3 first with $sign(o_5, u_3) = -1$. Again, see Figure 4(iv).

After o_5 intersects u_3 first, o_5 can intersect only either u_1 second with $sign(o_5, u_1) = +1$ or u_2 second with $sign(o_5, u_2) = +1$. However, o_5 can not intersect u_1 second. If o_5 intersects u_1 second with $sign(o_5, u_1) = +1$, after o_5 intersects u_1 second, we must change color from blue to red by Rule 1,2. In this case, we can not draw any knot diagram such that o_5 intersects u_2. This is a contradiction to Rule 2,3. Hence, o_5 must intersect u_2 second with $sign(o_5, u_2) = +1$. Once more, see Figure 4(iv).

After o_5 intersects u_2 second, we must change color from blue to red by Rule 1. In this case, we can not draw any knot diagram such that o_5 intersects u_1. This is a contradiction to Rule 2,3. Hence, when $sign(u_3, o_1) = +1$, we can not draw any knot diagram satisfying Rule 1,2,3. Therefore, $sign(u_3, o_1)$ must be -1. This finishes the proof of the Claim.

Now, we use an induction on the order of the over-underpass sequence. The argument used here is just a generalization of the previous one. Hence, our argument will be slightly sketchy here.

Suppose, without loss of generality, that $3 \leq n < k$ and $s_1, f_1, s_2, f_2, ..., s_n, f_n$ is an over-underpass sequence of the standard diagram of the $(n-1, -n)$-torus knot.

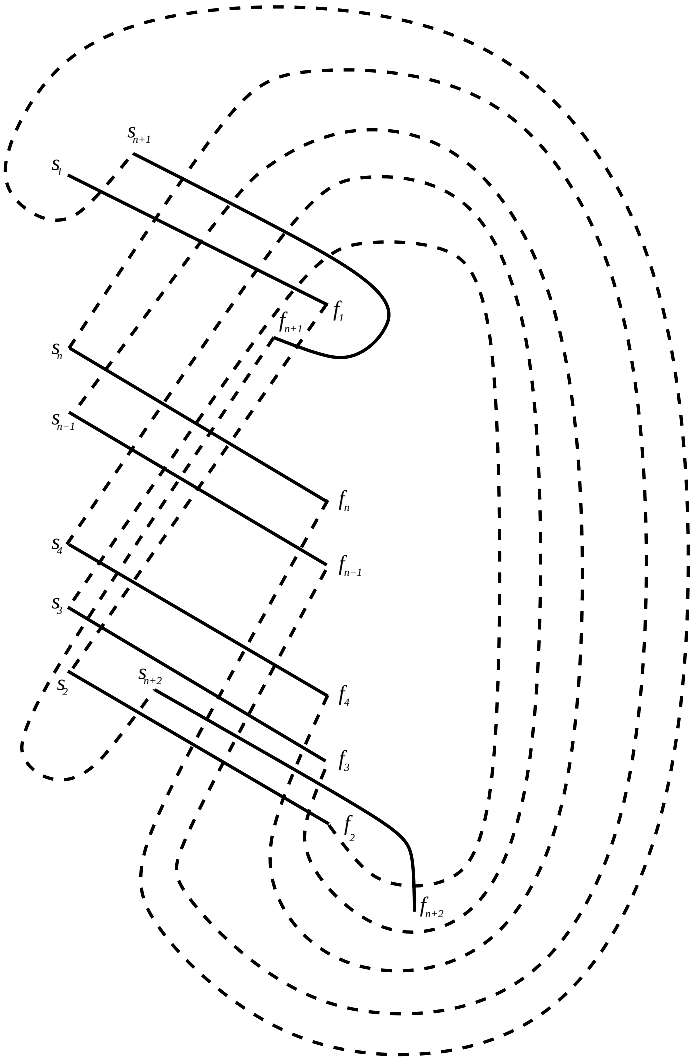


Figure 5. It is impossible for u_n to intersect o_1 with $sign(u_n, o_1) = -1$.

Let $o_i = [s_i, f_i]$ and $u_i = [f_i, s_{i+1}]$ for each $i \in \{1, ..., n-1\}$, and let $o_n = [s_n, f_n]$ and $u_n^* = [f_n, s_1]$. Take a point x_n of u_n^* between the $(n-2)$-th undercrossing of u_n^* and s_1. Then the arc $[f_n, x_n]$ of u_n^* has also $n-2$ undercrossings.

Notice that $o_1, u_1, o_2, u_2, ..., o_{n-1}, u_{n-1}, o_n$ are overpasses and underpasses of a knot diagram D satisfying Rule 1,2,3 and $[f_n, x_n]$ is an arc of the n-th underpass u_n of D. Suppose that D has more than n overpasses. Then there is an $((n-2)+1)$-th undercrossing on u_n by o_1 according to Rule 1,2,3. We claim that at this undercrossing, u_n must intersect o_1 with $sign(u_n, o_1) = -1$.

To prove this claim, suppose that u_n intersects o_1 with $sign(u_n, o_1) = +1$. See Figure 5. Then, after u_n intersects o_1 with $sign(u_n, o_1) = +1$, we must change color from red to blue by Rule 1. Take a point s_{n+1} so that $u_n = [f_n, s_{n+1}]$ has $(n-2)+1$ undercrossings. Then the only way to draw o_{n+1} is that o_{n+1} intersects u_{n-1} first, u_{n-2} second, ..., u_2 $(n-2)$-th, and u_1 $((n-2)+1)$-th so that $sign(o_{n+1}, u_{n-1}) = sign(o_{n+1}, u_{n-2}) = ... = sign(o_{n+1}, u_2) = -1$, and $sign(o_{n+1}, u_1) = +1$ by Rule 1,2,3. After o_{n+1} intersects u_1 with $sign(o_{n+1}, u_1) = +1$, we must change color from blue to red by Rule 1. Take a point f_{n+1} so that o_{n+1} has $(n-2)+1$ overcrossings. Then the only way to draw u_{n+1} is that u_{n+1} intersects o_n first, o_{n-1} second, ..., o_3 $(n-2)$-th, and o_2 $((n-2)+1)$-th so that $sign(u_{n+1}, o_n) = sign(u_{n+1}, o_{n-1}) = ... = sign(u_{n+1}, o_3) = -1$, and $sign(u_{n+1}, o_2) = +1$ by Rule 1,2,3. After u_{n+1} intersects o_2 with $sign(u_{n+1}, o_2) = +1$, we must change color from red to blue by Rule 1. Take a point s_{n+2} so that u_{n+1} has $(n-2)+1$ undercrossings. Then the only way to draw o_{n+2} is that o_{n+2} intersects u_n first, u_{n-1} second, ..., u_3 $(n-2)$-th, and u_2 $((n-2)+1)$-th so that $sign(o_{n+2}, u_n) = sign(o_{n+2}, u_{n-1}) = ... = sign(o_{n+2}, u_3) = -1$, and $sign(o_{n+2}, u_2) = +1$ by Rule 1,2,3. After o_{n+2} intersects u_2 with $sign(o_{n+2}, u_2) = +1$, we must change color from blue to red by Rule 1. However, in this case, we can not draw any knot diagram such that o_{n+2} intersects u_1. This is a contradiction to Rule 2,3. Therefore, $sign(u_n, o_1)$ can not be $+1$.

As a next step, we claim that the only way to draw o_{n+1} and u_{n+1} gives us the standard diagram of the $((n+1)-1, -(n+1))$-torus knot (see Figure 6). This will finish the induction.

After u_n intersects o_1 $((n-2)+1)$-th with $sign(u_n, o_1) = -1$, we must change color from red to blue by Rule 1,2. Take a point s_{n+1} so that $u_n = [f_n, s_{n+1}]$ has only $(n-2)+1$ undercrossings. Then the only way to draw o_{n+1} is that o_{n+1} intersects u_{n-1} first, u_{n-2} second, ..., u_1 $((n-2)+1)$-th so that $sign(o_{n+1}, u_{n-1}) = sign(o_{n+1}, u_{n-2}) = ... = sign(o_{n+1}, u_1) = -1$ by Rule 1,2,3. Suppose that o_{n+1} intersects u_i first for some $i < n-1$. Then $sign(o_{n+1}, u_i)$ must be -1 and o_{n+1} must intersect u_i first, u_{i-1} second, ..., u_1 i-th so that $sign(o_{n+1}, u_i) = sign(o_{n+1}, u_{i-1}) = ... = sign(o_{n+1}, u_1) = -1$ by Rule 1,2,3. However, after o_{n+1} intersects u_1 i-th with $sign(o_{n+1}, u_1) = -1$, o_{n+1} must stop before intersecting u_n and we must change color from blue to red by Rule 1,2. In this case, we can not draw any knot diagram such that o_{n+1} intersects u_{n-1}. This is a contradiction to Rule 2,3. Hence, o_{n+1} must intersect u_{n-1} first, u_{n-2} second, ..., u_1 $((n-2)+1)$-th so that $sign(o_{n+1}, u_{n-1}) = sign(o_{n+1}, u_{n-2}) = ... = sign(o_{n+1}, u_1) = -1$. Take a point f_{n+1} so that $o_{n+1} = [s_{n+1}, f_{n+1}]$ has only $(n-2)+1$ overcrossings. Then, by a similar argument as before, we can show that the only way to draw u_{n+1} is that u_{n+1} intersects o_n first, o_{n-1} second, ..., o_2 $((n-2)+1)$-th so that $sign(u_{n+1}, o_n) = sign(u_{n+1}, o_{n-1}) = ... = sign(u_{n+1}, o_2) = -1$ by Rule 1,2,3.

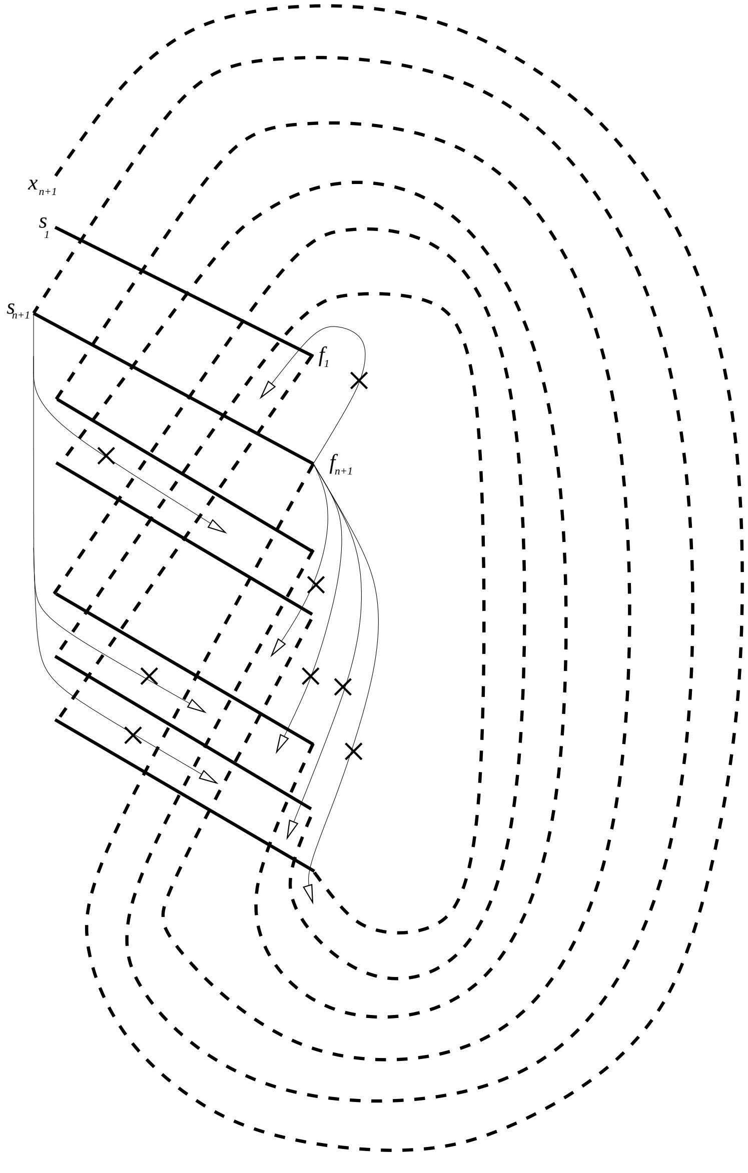


Figure 6. The only way to draw o_{n+1} and u_{n+1}.

Suppose that u_{n+1} intersects o_i first for some $i < n$. Then $sign(u_{n+1}, o_i)$ must be -1 and u_{n+1} must intersect o_i first, o_{i-1} second, ..., o_1 i-th so that $sign(u_{n+1}, o_i) = sign(u_{n+1}, o_{i-1}) = ... = sign(u_{n+1}, o_1) = -1$ by Rule 1,2,3. However, after u_{n+1} intersects o_1 i-th with $sign(u_{n+1}, o_1) = -1$, u_{n+1} must stop before intersecting o_{n+1} and we must change color from red to blue by Rule 1,2. In this case, we can not draw any knot diagram such that u_{n+1} intersects o_n. This is a contradiction to Rule 2,3. Hence, u_{n+1} must intersect o_n first, o_{n-1} second, ..., o_2 $((n-2)+1)$-th so that $sign(u_{n+1}, o_n) = sign(u_{n+1}, o_{n-1}) = ... = sign(u_{n+1}, o_2) = -1$. Take a point x_{n+1} so that the arc $[f_{n+1}, x_{n+1}]$ of u_{n+1} has only $(n-2)+1$ undercrossings. Notice that, in this case, we can connect x_{n+1} and s_1 by an arc without crossing so that we complete drawing the knot diagram.

This is the only way to draw a knot diagram with $n+1$ overpasses satisfying the Rules 1,2,3.

To complete the argument, when $n = k-1$, we can get the standard diagram D of the $(k-1, -k)$-torus knot and this is the only way to draw the knot diagrams satisfying Rule 1,2,3, and hence, the theorem is proved in Case 1.

Case 2. The sign $sign(u_2, o_1)$ of crossing between u_2 and o_1 is $+1$.

By the same argument as used in Case 1, we can get only standard diagrams of the $(b(D)-1, b(D))$-torus knot. This proves the theorem. □

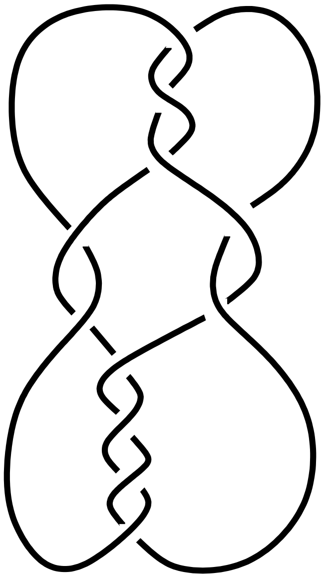


Figure 7. One of Goeritz's unknot diagrams.

4. A FINAL REMARK

One may view Lemmas 2.5 and 2.6 from another perspective. Suppose D is a knot diagram, not necessarily being minimal with respect to crossings. If either an overpass intersects an underpass more than once or an overpass intersects its adjacent underpass, then Lemmas 2.5 and 2.6 give a specific way to change the knot diagram D by isotopy with the number of crossings reduced.

Consider a knot diagram D satisfying the following two conditions:

(1) each overpass intersects each underpass at most once; and

(2) each overpass does not intersect its adjacent underpasses.

For such a knot diagram D, we have

$$b(D) \leq c(D) \leq b(D)(b(D)-2).$$

When $c(D) = b(D)$, D is an alternating knot diagram. By Theorem 3.1, when $c(D) = b(D)(b(D)-2)$, D is the standard diagram of a $(b(D)-1, \pm b(D))$ torus knot. Thus, it is natural to wonder whether conditions (1) and (2) above are sufficient for a knot diagram of a prime knot being minimal with respect to crossings. Goeritz's unknot diagrams show that the answer to this question is negative. See Figure 7. Are there any other necessary conditions on the overpasses and underpasses of a knot diagram for it being minimal with respect to crossings? This will be the topic of our further investigation.

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