

# KNOT ADJACENCY AND SATELLITES

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ABSTRACT. A knot  $K$  is called  $n$ -adjacent to the unknot, if  $K$  admits a projection containing  $n$  *generalized crossings* such that changing any  $0 < m \leq n$  of them yields a projection of the unknot. We show that a non-trivial satellite knot  $K$  is  $n$ -adjacent to the unknot, for some  $n > 0$ , if and only if it is  $n$ -adjacent to the unknot in any companion solid torus. In particular, every *model knot* of  $K$  is  $n$ -adjacent to the unknot. Along the way of proving these results, we also show that 2-bridge knots of the form  $K_{p/q}$ , where  $p/q = [2q_1, 2q_2]$  for some  $q_1, q_2 \in \mathbf{Z}$ , are precisely those knots that have genus one and are 2-adjacent to the unknot.

*Keywords:* companion torus, model knot,  $n$ -adjacent to the unknot,  $n$ -trivializer.

*Mathematics Subject Classification:* 57M25, 57N10.

## 1. INTRODUCTION

The development of the theory of finite type knot invariants has led to the notion of  $n$ -triviality which is a multiplex unknotting operation. This notion was introduced independently by Gussarov and Ohyaama ([G], [O]). Roughly speaking, a knot is  $n$ -trivial if it can be unknotted in  $2^n - 1$  different ways by *multiple crossing changes*. The research in this paper is motivated by the following question: If a non-trivial satellite knot  $K$  is  $n$ -trivial is there a companion torus of  $K$  that is disjoint from all the crossing changes that exhibit  $K$  as  $n$ -trivial? In this paper, we are concerned with a stronger version of  $n$ -triviality where each set of multiple crossing changes is taken to be a set of *twist crossings* on two strings of the knot. A knot with this stronger  $n$ -triviality is called  $n$ -adjacent to the unknot. For knots which are  $n$ -adjacent to the unknot, using results of Lackenby ([La]) and Scharlemann-Thompson ([ST1], [ST2]), we obtain an affirmative answer to the aforementioned question. In fact, we show that the generalized crossings involved can be taken to be disjoint from any companion torus of  $K$ . As a consequence, we obtain that if a non-trivial satellite knot  $K$  is  $n$ -adjacent to the unknot then it is  $n$ -adjacent to the unknot in any companion solid torus. In particular, any *model*

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The research of the first author is partially supported by the NSF through grant DMS-0104000. The research of the second author is partially supported by the Overseas Youth Cooperation Research Fund of NSFC and by the NSF through grant DMS-0102231.

knot of  $K$  is  $n$ -adjacent to the unknot. Along the way of proving these results, we also characterize 2-bridge knots of the form  $K_{p/q}$ , where  $p/q = [2q_1, 2q_2]$  for some  $q_1, q_2 \in \mathbf{Z}$ , precisely as those knots that have genus one and are 2-adjacent to the unknot.

A generalized crossing of order  $q \in \mathbf{Z}$  on an embedding of a knot  $K$  is a set  $C$  of  $|q|$  twist crossings on two strings that inherit opposite orientations from any orientation of  $K$ . If  $K'$  is obtained from  $K$  by changing all the crossings in  $C$  simultaneously, we will say that  $K'$  is obtained from  $K$  by a generalized crossing change (see Figure 1). In particular, if  $|q| = 1$ ,  $K$  and  $K'$  differ by an ordinary crossing change while if  $q = 0$  we have  $K = K'$ . Note that a generalized crossing change can be achieved by  $\frac{1}{q}$ -surgery on a crossing circle, which is an unknotted curve that bounds an embedded disc  $D \subset S^3$  such that  $K$  intersects  $\text{int}(D)$  exactly twice with zero algebraic intersection number.

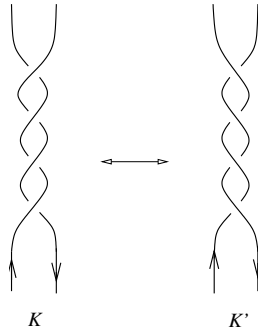


FIGURE 1. The knots  $K$  and  $K'$  differ by a generalized crossing change of order  $q = -4$ .

**Definition 1.1.** We will say that  $K$  is  $n$ -adjacent to the unknot, for some  $n \in \mathbf{N}$ , if  $K$  admits an embedding containing  $n$  generalized crossings such that changing any  $0 < m \leq n$  of them yields an embedding of the unknot. A collection of crossing circles corresponding to these crossings is called an  $n$ -trivializer. If all the generalized crossings used have order  $+1$  or  $-1$  (i.e. they are ordinary crossings), we will say that  $K$  is simply  $n$ -adjacent to the unknot. An  $n$ -trivializer that shows  $K$  to be simply  $n$ -adjacent to the unknot will be called a simple  $n$ -trivializer.

**Remark 1.2.** Let  $V$  be a solid torus in  $S^3$  and suppose that a knot  $K$  is embedded in  $V$ . Throughout the paper, we will use the term “ $K$  is  $n$ -adjacent to the unknot in  $V$ ” to mean the following: There exists an embedding of  $K$  in  $V$  that contains  $n$  generalized crossings such that changing any  $0 < m \leq n$  of them unknots  $K$  in  $V$ .

To state our result recall that if  $K$  is a non-trivial satellite with *companion knot*  $\hat{K}$  and *model knot*  $P$  then: i)  $\hat{K}$  is non-trivial; ii)  $P$  is geometrically essential in a standardly embedded solid torus  $V_1 \subset S^3$ ; and iii) there is a homeomorphism  $h : V_1 \rightarrow V := h(V_1)$ , such that  $h(P) = K$  and  $\hat{K}$  is the core of  $V$ .

**Theorem 1.3.** *Let  $K$  be a non-trivial satellite knot and let  $V$  be any companion solid torus of  $K$ . Then,  $K$  is  $n$ -adjacent to the unknot, for some  $n > 0$ , if and only if it is  $n$ -adjacent to the unknot in  $V$ .*

As a consequence of Theorem 1.3 we have the following:

**Corollary 1.4.** *A non-trivial satellite knot  $K$  is  $n$ -adjacent to the unknot, for some  $n > 0$ , if and only if any model knot of  $K$  is  $n$ -adjacent to the unknot in the standard solid torus  $V_1$ .*

The paper is organized as follows: In Section 2 we summarize some results that are used in the proof of Theorem 1.3. In Section 3 we study satellite knots of winding number zero that are  $n$ -adjacent to the unknot. In Section 4 we prove Theorem 1.3 and work out some corollaries. In Section 5, we show that a knot  $K$  of genus one is 2-adjacent to the unknot iff it is a 2-bridge knot of the form  $[2q_1, 2q_2]$  for some  $q_1, q_2 \in \mathbf{Z}$ .

Note that a weaker version of Theorem 1.3 is generalized to a broader class of  $n$ -trivial knots in [K].

**Acknowledgments:** We would like to thank the referee and Ying-Qing Wu for their careful reading and critical remarks of the previous version of this paper.

## 2. PRELIMINARIES

In this section we summarize some results that will be used in the proof of Theorem 1.3. We begin with the following theorem that summarizes some results from [ST1] and [La]. Part a) of the theorem is stated as Corollary 3.2 in [ST1]. Part b) is stated as Corollary 4.4 of [ST1] and also follows from Theorem 1.4(b) and Proposition 2.5 of [La].

**Theorem 2.1.** *Let  $K, K'$  be knots that differ by a generalized crossing change of order  $q \in \mathbf{Z}$ . Let  $L$  be a crossing circle for  $K$  corresponding to this generalized crossing and that  $K$  is a non-trivial satellite.*

a) *If  $|q| > 1$  and  $\text{genus}(K') \leq \text{genus}(K) - 1$  then any companion torus of  $K$  can be isotoped in  $S^3 \setminus K$  to be disjoint from  $L$ .*

b) *If  $|q| = 1$  and  $\text{genus}(K') \leq \text{genus}(K) - 2$  then any companion torus of  $K$  can be isotoped in  $S^3 \setminus K$  to be disjoint from  $L$ .*

To continue, note that if  $L$  is an  $n$ -trivializer for a knot  $K$ , since the linking number of  $K$  with each component of  $L$  is zero,  $K$  bounds a Seifert surface in the complement of  $L$ . We will need the following lemma:

**Lemma 2.2.** *Let  $L$  be an  $n$ -trivializer of a knot  $K$ . Suppose that  $S$  is a Seifert surface bounded by  $K$  in the complement of  $L$  and such that among all such surfaces  $S$  has minimum genus. Then,  $\text{genus}(S) = \text{genus}(K)$ .*

**Proof:** For simple  $n$ -trivializers the lemma is stated as Theorem 4.1 in [HL]; the proof relies on a result of Gabai (Corollary 2.4 of [Ga]). The argument for general  $n$ -trivializer's is essentially the same. The details are given in the proof of Theorem 3.1 of [KL].  $\square$

We will apply Theorem 2.1 to the case when  $K'$  is the unknot. In this case, when  $\text{genus}(K) \geq 2$ , for every  $q$ , we can isotopy a companion torus of  $K$  to be disjoint from the crossing circle  $L$ . When  $\text{genus}(K) = 1$  and  $|q| = 1$ , Theorem 2.1 can not be used anymore. Nevertheless, the following lemma guarantees that the same conclusion still holds in this situation.

**Lemma 2.3.** *Let  $K, K'$  be knots that differ by a generalized crossing of order  $q \in \mathbf{Z}$ . Let  $L$  be the corresponding crossing circle. Suppose that  $\text{genus}(K') < \text{genus}(K)$  and let  $K_0$  denote the 2-component link obtained by smoothing  $C$  in a way consistent with the orientation of  $K$ . Then, there exist Seifert surfaces  $\Sigma$  and  $\Sigma_0$  of maximal Euler characteristic for  $K$  and  $K_0$  respectively,  $\Sigma \cap L = \emptyset$ , such that  $\Sigma$  is obtained from  $\Sigma_0$  by plumbing on an unknotted annulus with a  $(2, 2q)$ -torus link as its boundary and  $L$  as one of its small linking circles.*

Here by a *small linking circle* of an annulus  $A$  embedded in  $S^3$ , we mean an unknot which bounds a disk  $D$  such that  $D \cap A$  is a proper non-boundary parallel arc in  $A$ .

**Proof:** For  $|q| = 1$  the lemma is stated as Proposition 3.1 in [ST2]. The proof of that proposition uses Theorem 1.4 (of [ST2]) that gives a relation between the Euler characteristics of the triple  $(K, K', K_0)$ . Theorem 6.4.3 of [Ka] states that the same relation holds when  $|q| > 1$ . Using this, the arguments used in the proof of Proposition 3.1 in [ST2] go through to give the lemma in the case that  $|q| > 1$  (see also the proof of Theorem 6.4.2 of [Ka]).  $\square$

**Corollary 2.4.** *Let  $K$  be a non-trivial satellite knot which can be unknotted by a single generalized crossing change, and  $L$  be the corresponding crossing circle. Then any companion torus of  $K$  can be isotoped in  $S^3 \setminus K$  to be disjoint from  $L$ .*

**Proof:** The case when  $\text{genus}(K) \geq 2$  is covered by Theorem 2.1. So we assume  $\text{genus}(K) = 1$ . Let  $\Sigma$  be the genus one Seifert surface of  $K$  claimed to exist in Lemma 2.3. Then  $\Sigma$  is the plumbing of two annuli  $A_1$  and  $A_2$ . One of them, say  $A_1$ , is unknotted with  $L$  as its small linking circle. Thus,  $K$  is contained in a torus  $T = \partial N$ , where  $N$  is a tubular neighborhood of  $A_2$ . Since  $A_1$  is unknotted, we may assume that  $A_1 \subset N$ . It is not hard to see that  $T$  is the innermost companion torus of  $K$ : every other companion torus of  $K$  can be isotoped to contain the solid torus  $N$  in one side. Thus, we can isotope every companion torus of  $K$  to be disjoint from  $L$ .  $\square$

### 3. SATELLITE KNOTS WITH ZERO WINDING NUMBER

Throughout this section, we suppose that  $K$  is a non-trivial satellite knot and  $V$  is a companion solid torus of  $K$ , such that the winding number of  $K$  in  $V$  is zero.

**3.1. A technical lemma.** In this subsection we prove a technical lemma which will play a key role in our discussion in the next subsection.

Let  $S$  be a minimal genus Seifert surface of  $K$ . We assume that the intersection of  $S$  and  $T = \partial V$  is transverse and the number of components of  $S \cap T$  is minimal. Denote by  $M_1$  and  $M_2$  the closures of components of  $(S^3 \setminus K) \setminus T$  in  $S^3 \setminus K$ , respectively, with  $M_2$  a compact 3-manifold. Let  $\alpha$  be a proper arc on  $S$ .

**Lemma 3.1.** *Suppose there is an isotopy of  $S^3$ , fixing  $K$  pointwise, which brings  $\alpha$  to an arc  $\alpha'$  in  $V$ , then we can isotopy  $\alpha$  on the surface  $S$ , relative to  $\partial S = K$ , to a proper arc  $\alpha''$  in  $V$ .*

**Proof:** We may assume that

- (1)  $S \cap T$  is a collection of disjoint parallel copies of an essential simple closed curve on  $T$ ;
- (2) every component of  $S \cap M_i$  is incompressible and boundary incompressible in  $M_i$ .

Since these points follow from well known facts in 3-dimensional manifold topology, we only give a brief explanation. Point (1) follows immediately from the fact that  $T$  is incompressible in the complement of  $K$ . To see (2), first by the incompressibility of  $S$  and  $T$  in  $S^3 \setminus K$ , it is easy to deduce that each component of  $S \cap M_i$  is incompressible in  $M_i$ . If there is an essential boundary compressing disk  $D$  for a component of  $S \cap M_i$  in  $M_i$ , then  $D \cap T$  must be an arc whose end points lie on different components of  $S \cap T$ . Thus we may isotopy  $S$  to reduce  $|S \cap T|$ , which would contradict the assumption that  $|S \cap T|$  is minimal.

Now let  $\alpha$  and  $\alpha'$  be as in the lemma. Up to isotopy on  $S$ , relative to  $\partial S = K$ , we may assume that  $\alpha$  intersects each component of  $S \cap T$  in essential arcs. We will show that this assumption will force  $\alpha$  to be disjoint from  $T$ . Let  $f : D^2 \rightarrow S^3$  be a path homotopy from  $\alpha$  to  $\alpha'$  with  $f(\text{Int}(D))$  disjoint from  $K$ . Since  $T$  is incompressible, we may assume that  $f^{-1}(T)$  is a set of proper arcs on  $D^2$ . Note that all endpoints of  $f^{-1}(T)$  are on  $\alpha$  because  $\alpha'$  is disjoint from  $T$ . Thus we can choose a component  $\beta$  of  $f^{-1}(T)$  which is outermost in the sense that it cuts off a subdisk  $D_1$  in  $D^2$  whose interior is disjoint from  $f^{-1}(T)$ , and  $\gamma = f(\partial D_1 \setminus \text{Int}(\beta))$  is a subarc of  $\alpha$ . Since the interior of  $D_1$  is disjoint from  $f^{-1}(T)$ ,  $\gamma$  is an proper arc on a component  $A$  of  $S \cap M_i$ , which is essential by the above assumption. However, this contradicts the following lemma and the fact that  $A$  is boundary incompressible in  $M_i$ . So we conclude that  $\alpha$  can be isotoped on  $S$ , relative to  $\partial S = K$ , to be disjoint from  $T$ . This proves the lemma.  $\square$

**Lemma 3.2.** *Let  $F$  be an incompressible and boundary incompressible surface in a 3-manifold  $M$  with  $\partial M$  incompressible. Then there is no non-closed proper essential curve  $a$  on  $F$  that is homotopic to a curve  $b$  on  $\partial M$  relative to  $\partial a$ .*

**Proof:** Consider the double of  $F$  in the double of  $M$ , denoted by  $\hat{F}$  and  $\hat{M}$ , respectively. By an innermost-circle outermost-arc argument one can easily show that  $\hat{F}$  is incompressible in  $\hat{M}$ . On the other hand, the double of a homotopy from  $a$  to  $b$  would give rise to a null homotopy disk for the double of  $a$ . Since the double of  $a$  is an essential curve on  $F$ , this contradicts the fact that an incompressible surface is  $\pi_1$ -injective.  $\square$

**3.2. Finding an  $n$ -trivializer in a companion solid torus.** We can now have the the following lemma, which will allows us to find an  $n$ -trivializer for  $K$  in any companion solid torus.

**Lemma 3.3.** *Let  $K$  be a non-trivial satellite and let  $V$  be a companion solid torus of  $K$ . Suppose that the winding number of  $K$  in  $V$  is zero. If  $K$  is  $n$ -adjacent to the unknot, for some  $n > 0$ , then, there exists an  $n$ -trivializer for  $K$  that lies in  $V$ .*

**Proof:** Let  $L := \cup_{i=1}^n L_i$  be an  $n$ -trivializer of  $K$  and let  $D_1, \dots, D_n$  be crossing discs bounded by  $L_1, \dots, L_n$ , respectively. Let  $S$  be a Seifert surface for  $K$  in the complement of  $L$  that has minimum genus. Then by Lemma 2.2,  $S$  is also a minimal genus Seifert surface of  $K$ . We may isotope  $S$  so that each  $S \cap \text{int}(D_i)$  is the union of an arc  $\alpha_i$  and several closed components. The arc  $\alpha_i$  is properly embedded on  $S$ . Since  $S$  is incompressible in the complement of  $L$ , after an isotopy we can arrange so that  $S \cap D_i$  contains no closed curves that are inessential on  $D_i$ . Thus each closed component of  $S \cap D_i$  has to be parallel to  $L_i$  on  $D_i$ . By replacing

$L_i$  with the closed component of  $S \cap D_i$  that is innermost on  $D_i$ , we may assume that  $S \cap D_i = \alpha_i$ . Since twisting along  $\alpha_i$  unknots  $K$ , it follows that  $\alpha_i$  must be essential on  $S$ . Furthermore, the arcs  $\alpha_1, \dots, \alpha_n$  are disjoint from each other.

By Corollary 2.4, for each  $L_i$ , we can isotope the torus  $T = \partial V$  in the complement of  $K$  to  $T'$  such that  $T' \cap L_i = \emptyset$ . Assume that  $T'$  intersects the disk  $D_i$  transversely. Since  $T'$  is disjoint from  $L_i = \partial D_i$ , each component of  $T' \cap D_i$  is a simple closed curve in  $D_i$ . If a component of  $T' \cap D_i$  bounds a disk in  $D_i$  which contains only one point in  $K \cap D_i$ , we would have the winding number of  $K$  in  $V$  to be  $\pm 1$ . So every component of  $T' \cap D_i$  either bounds a disk in  $D_i$  which is disjoint from  $K \cap D_i$  or bounds a disk in  $D_i$  which contains  $K \cap D_i$ . In either cases, a further isotopy of  $T'$  in the complement of  $K$  will remove this component of  $T' \cap D_i$ . The reversed isotopy in the complement of  $K$  from this  $T'$  to  $T$  then will bring the arc  $\alpha_i$  into  $V$ . Thus, by Lemma 3.1, we can isotope each  $\alpha_i$  on the minimal genus Seifert surface  $S$ , relative to  $\partial S = K$ , to a proper arc  $\alpha'_i$  in  $V$ .

On  $S$ , let  $\alpha$  and  $\beta$  be two proper 1-submanifolds whose intersection is transverse. Suppose that there is an isotopy of  $S$  that reduces the geometric intersection  $|\alpha \cap \beta|$ . Then there will be a disk  $D$  on  $S$  such that  $D \cap (\alpha \cup \beta) = \partial D$ ,  $D \cap \alpha$  and  $D \cap \beta$  are subarcs in the interior of  $\alpha$  and  $\beta$ , respectively. This is a well-known fact (see, for example, Proposition 3.10 in [FLP] or Lemma 3.1 in [HS]).

We apply this fact to  $\{\alpha_1, \dots, \alpha_n\}$  and  $S \cap T = C_1 \sqcup \dots \sqcup C_r$  (see the proof of Lemma 3.1). Since  $\alpha_1$  can be made disjoint from  $S \cap T$  by an isotopy of  $S$  relative to  $\partial S$ , we find a disk  $D$  between  $\alpha_1$  and  $S \cap T$  as described above. We then use this  $D$  to define an isotopy of  $S$  relative to  $\partial S$  to remove a pair of intersection points of  $\alpha_1$  and  $S \cap T$ . This isotopy will not increase the intersection points of the other  $\alpha_i$ 's with  $S \cap T$ . And it will also keep  $\alpha_i$ 's disjoint. So inductively, we have an isotopy on  $S$  relative to  $\partial S$ , which brings the entire disjoint collection of proper arcs  $\{\alpha_1, \dots, \alpha_n\}$  to a disjoint collection of proper arcs  $\{\alpha'_1, \dots, \alpha'_n\}$  in  $V$ .

Finally, we construct a small disk  $D'_i$  in  $V$  whose intersection with  $S$  is  $\alpha'_i$ , for each  $i$ , and they are disjoint from each other. Let  $L'_i = \partial D'_i \subset V$ . Since  $L' = \cup_{i=1}^n L'_i$  is isotopic to  $L$  in the complement of  $K$ , it is an  $n$ -trivializer for  $K$  that lies in  $V$ .  $\square$

#### 4. THE PROOF OF THE MAIN RESULT

Here we finish the proof of Theorem 1.3 and Corollary 1.4.

**Proof of Theorem 1.3:** The “if” direction of the statement is clear. To prove the “only if” direction, suppose that  $K$  is a non-trivial satellite that is  $n$ -adjacent

to the unknot. Let  $V$  be any companion solid torus of  $K$ . Let  $\hat{K}$  denote the core of  $V$  and set  $T := \partial V$ . We need the following:

*Claim.* The winding number of  $K$  in  $V$  is zero.

*Proof of Claim.* By Corollary 2.4, there exists a component  $L_1 \subset L$  that can be isotoped to be disjoint from  $T$ . Let  $D_1$  be a crossing disc bounded by  $L_1$ . After an isotopy in the complement of  $K$ ,  $D_1 \cap T$  will consist of a collection of curves, none of which bounds a disc in  $D_1$  in the complement of  $K$ . Let  $C$  be a component of  $D_1 \cap T$ . If  $C$  is boundary parallel on  $D_1$  then it can be eliminated by an isotopy in the complement of  $K$  so that  $L_1$  is still disjoint from  $T$ . If all components of  $D_1 \cap T$  are boundary parallel, we will have  $D_1$  disjoint from  $T$  after an isotopy in the complement of  $K$ . Then  $D_1$  is contained in  $V$ . Since a satellite with non-zero winding number cannot be unknotted by crossing changes in  $\text{Int}(V)$ , we conclude every component of  $D_1 \cap T$  bounds a disc on  $D_1$  that contains exactly one point of  $D_1 \cap K$ . Since  $K$  was assumed to be a non-trivial satellite we conclude that  $K$  is a composite knot and  $T$  is the follow-swallow torus. But then the crossing change realized by  $L_1$  occurs within a summand of  $K$  and it cannot unknot  $K$ . This contradicts the fact that  $L_1$  is part of an  $n$ -trivializer and it finishes the proof of the claim.

Let us now finish the proof of the theorem. The claim above allows us to assume that the winding number of  $K$  in  $V$  is zero. By Lemma 3.3,  $K$  admits an  $n$ -trivializer  $L'$  in  $V$ . Now each of the surgeries along the sublinks of  $L'$  that unknot  $K$  must turn it into a knot that is isotopically trivial in  $\text{Int}(V)$ . For, otherwise the knot obtained from  $K$  after any of these surgeries will still have  $\hat{K}$  as a companion and it can't be the unknot. Thus  $K$  is  $n$ -adjacent to the unknot in  $V$ .  $\square$

**Proof of Corollary 1.4:** Let  $P$  be any model of  $K$  in a standard solid torus  $V_1 \subset S^3$  and let  $h : \rightarrow S^3$  the satellite embedding. If  $P$  is  $n$ -adjacent to the unknot in  $V_1$  and  $L \subset V_1$  is an  $n$ -trivializer then  $h(L)$  is an  $n$ -trivializer for  $K$  in  $V$ . Conversely, by Theorem 1.3 and its proof, if  $K$  is  $n$ -adjacent to the unknot then any  $n$ -trivializer, say  $L$ , can be isotoped into  $V := h(V_1)$  as an  $n$ -trivializer of  $K$  in  $V$ . But then the crossing circles  $h^{-1}(L_1), \dots, h^{-1}(L_n)$  form an  $n$ -trivializer for  $P$  in  $V_1$ .  $\square$

There exist many criteria in terms of the finite type knot invariants or polynomial invariants that detect  $n$ -adjacency to the unknot. For example, in [AK] it is shown that if a knot is  $n$ -adjacent to the unknot, for some  $n \geq 3$ , then all the finite type invariants of order  $< 2n - 1$  and the Alexander polynomial are trivial. More recently, criteria that detect simple 2-adjacency to the unknot were obtained by

N. Askitas and A. Stoimenow ([AS]) in terms of the HOMFLY polynomial, and by the second named author of this paper and Z. Tao in terms of the Kauffman polynomial. Due to the computational complexity of the invariants involved, these criteria become harder to test for knots that are non-trivial satellites. The results of this paper, reduce the problem of deciding whether a non-trivial satellite  $K$  is  $n$ -adjacent to the unknot to deciding the same problem for a model knot of  $K$ . In particular, we have the following:

**Corollary 4.1.** *Suppose that  $P$  is a knot that is not  $n$ -adjacent to the unknot. Then, no satellite that is modeled on  $P$  is  $n$ -adjacent to the unknot.*

**Proof:** This follows from Theorem 1.3 and the fact that if  $P$  is  $n$ -adjacent to the unknot in the solid torus then it is  $n$ -adjacent to the unknot in  $S^3$ .  $\square$

## 5. KNOTS OF GENUS ONE

We finish this paper by taking a look at genus one knots that are  $n$ -adjacent to the unknot, for  $n > 1$ . In fact, we will obtain a characterization of knots of genus one which are 2-adjacent to the unknot.

Consider 2-bridge knots of the form  $K_{p/q}$ , where  $p/q = [2q_1, 2q_2]$  in Conway's notation (see, for example, [BZ]). Such a knot is formed by plumbing two unknotted,  $2q_1$  and  $2q_2$  twisted annuli, and taking the boundary of the resulting genus one surface. Obviously, this is a genus one knot which is 2-adjacent to the unknot. The orders of the two generalized crossing changes are  $q_1$  and  $q_2$ , respectively.

**Theorem 5.1.** *A genus one knot  $K$  is 2-adjacent to the unknot if and only if  $K = K_{p/q}$ ,  $p/q = [2q_1, 2q_2]$ , for some integers  $q_1, q_2$ .*

It is clear that we only need to prove the “only if” part. So we suppose  $K$  is a genus one knot and it is 2-adjacent to the unknot. Let  $L = L_1 \cup L_2$  be a 2-trivializer of  $K$  of order  $(q_1, q_2)$ . By Lemma 2.2, we have a Seifert surface  $S$  of  $K$  in the complement of  $L$  and the genus of  $S$  is one. We may assume that the crossing disks  $D_1, D_2$ , with  $\partial D_1 = L_1$  and  $\partial D_2 = L_2$ , intersect  $S$  along essential proper arcs  $\alpha_1, \alpha_2$ , respectively.

**Lemma 5.2.** (See also [HL].) *The arcs  $\alpha_1, \alpha_2$  are not parallel on  $S$ .*

**Proof:** If  $\alpha_1$  and  $\alpha_2$  were parallel to each other on  $S$ ,  $L_1$  and  $L_2$  would cobound an annulus in the complement of  $K$ . Then perform both  $1/q_1$ -surgery on  $L_1$  and  $1/q_2$  surgery on  $L_2$  would be the same as doing  $1/(q_1 + q_2)$ -surgery on  $L_1$  or  $L_2$ . Since  $K$  is nontrivial, we would have two distinct surgeries on  $L_1$  under which  $S$  does not remain of minimal genus.

Since a twist along  $L_1$  unknots  $K$ , it follows that the 3-manifold  $M := S^3 \setminus \eta(K \cup L_1)$  is irreducible.  $S$  gives rise to a properly embedded surface in  $M$  that minimizes the Thurston norm in its homology class. Corollary 2.4 of [Ga], applied to  $M$  and  $T := \partial\eta(L_1)$ , implies that there can be at most one Dehn filling of  $T$  (or equivalently at most one surgery along  $L_1$ ) under which  $S$  doesn't remain a minimum genus surface for  $K$ . This contradiction finishes the proof of the lemma.  $\square$

Next we use Lemma 2.3 to  $K$  and  $L_1$ . This lemma gives us another genus one Seifert surface  $\Sigma$  of  $K$  in the form of the plumbing of annuli  $A_1$  and  $A_2$ , such that  $A_1$  is unknotted,  $2q_1$ -twisted, and its small linking circle is  $L_1$ . Without loss of generality, we may assume that  $D_1 \cap \Sigma = \alpha_1$ . So we may cut open both  $S$  and  $\Sigma$  along  $\alpha_1$ . For  $\Sigma$ , we get the annulus  $A_2$  from this surgery. For  $S$ , we get another annulus  $A'_2$  from this surgery. The annuli  $A_2$  and  $A'_2$  have the same boundary  $K_0$ , and we may assume that they are disjoint. Thus  $T = A_2 \cup A'_2$  is a torus which bounds a solid torus in  $S^3$ . Assume that the core circle of  $A_2$  (and the core circle of  $A'_2$ ) is a  $(m, l)$  curve on  $T$ , where  $l \geq 0$  is the winding number in the longitude direction and  $m$  is the winding number in the meridian direction on  $T$ .

The arc  $\alpha_2$  on  $A'_2$  should have its end points on different boundary components of  $A'_2$  by Lemma 5.2. We may pick a possible  $\alpha_2$  and all other possible  $\alpha_2$ 's are obtained by Dehn twist along  $A'_2$ . See Figure 2.

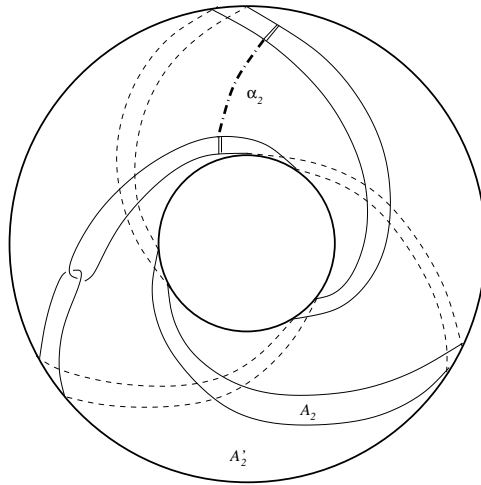


FIGURE 2. A possible position for the arc  $\alpha_2$  on the annulus  $A'_2$ .

Let us perform a generalized crossing change at  $L_1$ , which will unhook the clasp seen on  $A_2$  in Figure 2. We then shrink the two separated clasp ends along  $A_2$  until

they meet the ends of  $\alpha_2$  (the places on  $A_2$  marked by double lines in Figure 2). Denote the subarc of the core circle of  $A_2$  between the double line marks, which does not run through the clasp, by  $\beta$ . Then we may get a simple closed curve  $J = \alpha_2 \cup \beta$ . The curve  $J$  intersects the core circle of  $A'_2$  only once. Let  $J$  be a  $(a, b)$  curve on  $T$ . We have  $|mb - la| = 1$ .

If we perform generalized crossing changes at both  $L_1$  and  $L_2$ ,  $K$  will be changed to the unknot. This is possible only when  $(a, b) = (0, \pm 1)$  or  $(a, b) = (\pm 1, 0)$ . Otherwise,  $J$  would have a non-zero framing in the solid torus  $V$  bounded by  $T$  and generalized crossing changes at both  $L_1$  and  $L_2$  would change  $K$  into a  $[2r, 2s]$  knot in  $V$  for  $rs \neq 0$ . Such a knot can not be unknotted in  $S^3$ .

When  $(a, b) = (0, \pm 1)$ ,  $V$  has to be unknotted in  $S^3$ . Notice that we must have  $m = \pm 1$ . If  $l = 0$ , the knot  $K$  would be trivial. So we may assume that  $l \geq 1$ . Thus, we can have one possible choice of  $\alpha_2$  as shown in Figure 3. Any other choices of  $\alpha_2$  are obtained by applying a power of the Dehn twist of along the core of  $A'_2$  to this particular  $\alpha_2$ . From this fact, we see that this particular  $\alpha_2$  is the only one with  $(a, b) = (0, \pm 1)$ . In Figure 3, we can see that the arc  $\alpha_2$  can be isotoped to the arc  $\gamma$  in  $A'_2$ . Furthermore,  $\gamma$  and an arc on  $A_2$ , which is the intersection of  $A_2$  with its small linking disk, cobound a disk whose interior is disjoint from  $T$ . Thus the generalized crossing change at  $L_2$  is the same as a generalized crossing change at a small linking circle of  $A_2$ . This implies that  $K$  is a 2-bridge knot of the form  $[2q_1, 2q_2]$ .

When  $(a, b) = (\pm 1, 0)$ , we have  $l = 1$  and  $m \neq 0$ . We can argue as in the previous case: First, we find a unique choice of  $\alpha_2$ . This choice of  $\alpha_2$  will force  $V$  to be unknotted in  $S^3$ . And then the knot  $K$  will be a 2-bridge knot of the form  $[2q_1, 2q_2]$ . This finishes the proof of Theorem 5.1.  $\square$

**Corollary 5.3.** *The only genus one knots that are simply 2-adjacent to the unknot are the two trefoils and the figure eight.*

**Proof:** This corollary corresponds to the case of  $q_1 = \pm 1$  and  $q_2 = \pm 1$  of Theorem 5.1.  $\square$

Note that, as observed by T. Stanford, if a knot  $K$  is simply 2-adjacent to the unknot, then we have  $a_2(K) = 0$  or  $\pm 1$ , where  $a_2$  is the second coefficient of the Alexander-Conway polynomial. Using this observation, we see that the knot  $5_2$  is not simply 2-adjacent to the unknot since  $a_2(5_2) = 2$ . By Theorem 5.1,  $5_2$  is not 2-adjacent to the unknot since it is the 2-bridge knot  $[2, 3]$ . Apparently, no method is known to detect this using knot invariants.

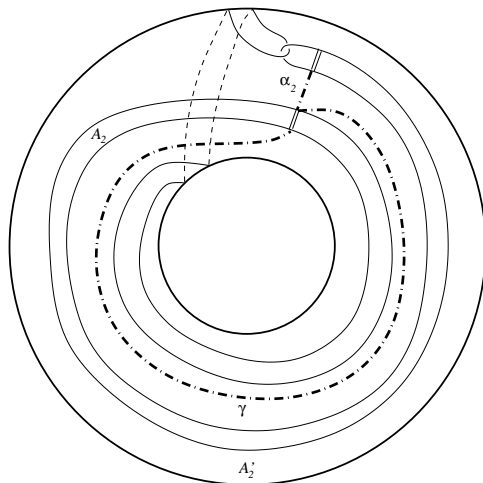


FIGURE 3. This is a knot of the form  $[2q_1, 2q_2]$ .

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