L^2-Alexander invariants

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Fuglede-Kadison Determinant

• For a discrete group Γ, $l^2(\Gamma)$ is the Hilbert space of square summable complex valued functions on Γ. Or equivalently, $l^2(\Gamma)$ consists of elements

$$f = \sum_{\gamma \in \Gamma} a_\gamma \gamma, \quad a_\gamma \in \mathbb{C}, \quad \sum_{\gamma \in \Gamma} |a_\gamma|^2 < \infty.$$

• The inner product of $f = \sum a_\gamma \gamma$ and $g = \sum b_\gamma \gamma$ is

$$\langle f, g \rangle = \sum a_\gamma \bar{b}_\gamma.$$

• The von Neumann algebra $\mathcal{N}\Gamma$ is the algebra of Γ-equivalent bounded linear operators on $l^2(\Gamma)$:

$$\mathcal{N}\Gamma = \mathcal{B}(l^2(\Gamma))^\Gamma$$

where γ acts on $l^2(\Gamma)$ isometrically by left multiplication.

• For $A \in \mathcal{N}\Gamma$, the von Neumann trace of A is

$$\mathrm{tr}_\tau(A) = \langle A(e), e \rangle, \quad \text{where } e \text{ is the unit of } \Gamma.$$

• For a positive integer n, set

$$l^2(\Gamma)^{[n]} = \underbrace{l^2(\Gamma) \oplus \cdots \oplus l^2(\Gamma)}_{n}$$

be a free $\mathcal{N}\Gamma$-Hilbert module of rank n. For $A \in \mathcal{B}(l^2(\Gamma)^{[n]})^\Gamma$, the von Neumann trace if

$$\mathrm{tr}_\tau(A) = \sum_{i=1}^{n} \langle A(e_i), e_i \rangle$$

where e_i is the unit of the i-th copy of $l^2(\Gamma)$.

(1) If $A \in \mathcal{B}(l^2(\Gamma)^{[n]})^\Gamma$ is invertible and A^* is the adjoint of A, then the Fuglede-Kadison determinant is

$$\boxed{\mathrm{Det}_\tau(A) = \exp\left(\frac{1}{2}\,\mathrm{tr}_\tau(\log A^* A)\right)}$$

(2) if $A \in \mathcal{B}(l^2(\Gamma)^{[n]})^\Gamma$ is injective, then

$$\boxed{\mathrm{Det}_\tau(A) = \lim_{\epsilon \to 0^+} \sqrt{\mathrm{Det}_\tau(A^* A + \epsilon)}}$$

• The Fuglede-Kadison determinant is closely related with the Mahler measure of a polynomial.

Let $\Gamma = \mathbb{Z}^n$. For an element $f \in \mathbb{C}\mathbb{Z}^n$, we may think of it as an element in $\mathcal{N}\mathbb{Z}^n$ by right multiplication. We may also think of it as a Laurent polynomial of n variables. Then we have:

Proposition (Lück) *For an element $f \in \mathbb{C}[\mathbb{Z}^n]$, the following conditions are equivalent:*

(a) f is invertible in $\mathcal{N}\mathbb{Z}^n$;

(b) f viewed as a Laurent polynomial does not vanish at any point of the torus T^n.

Under one of these conditions, we have

$$\boxed{\mathrm{Det}_\tau(f) = \exp \int_{T^n} \log |f(z)| \, d\mu(z)}$$

where μ is the Haar measure of T^n.

This result has generalizations to other amenable groups, where the right hand side is replaced by the entropy of some dynamical systems (Deninger).

The Work of Li and Zhang

• Let K be a knot in S^3 and $\Gamma = \pi_1(S^3 \setminus K)$. We have a Wirtinger presentation of Γ:

$$\Gamma \cong \langle x_1, x_2, \ldots, x_k \,|\, r_1, r_2, \ldots, r_{k-1} \rangle$$

Define α to be a $\mathbb{C}^*$ representation of Γ:

$$\alpha : \Gamma \longrightarrow \mathbb{C}^*, \quad \alpha(x_i) = t \in \mathbb{C}^* \subset \mathbb{C}$$

The right multiplication of Γ on $l^2(\Gamma)$ gives rise to a representation

$$\rho : \mathbb{Z}\Gamma \longrightarrow \mathcal{N}\Gamma$$

We also have the tensor representation

$$\Psi = \rho \otimes \alpha : \mathbb{Z}\Gamma \longrightarrow \mathcal{N}\Gamma \otimes \mathbb{Z}[t^{\pm 1}] \subset \mathcal{N}\Gamma$$

Apply the Fox free derivatives, we define

$$A_{ij}(t) = \Psi\left(\frac{\partial r_i}{\partial x_j}\right) \in \mathcal{N}\Gamma$$

• For $m = 1, 2, \ldots, k$, let $A^{(m)}(t)$ be obtained from the matrix $(A_{ij}(t))_{(k-1)\times k}$ with the m-th column deleted. We may think of $A^{(m)}(t)$ as a bounded linear operator on $l^2(\Gamma)^{[k-1]}$ by right matrix multiplication:

$$A^{(m)}(t) \in \mathcal{B}(l^2(\Gamma)^{[k-1]})^{\Gamma}$$

Theorem (Li-Zhang) *If one of the operators* $A^{(m)}(t)$, $m = 1, 2, \ldots, k$ *is injective, then other operations are all injective and we have*

$$\mathrm{Det}_\tau(A^{(m)}(t)) = \mathrm{Det}_\tau(A^{(m')}(t))$$

Furthermore, we have

(1) $\mathrm{Det}_\tau(A^{(m)}(t))$ *depends only on* $|t|$*;*

(2) $\mathrm{Det}_\tau(A^{(m)}(t))$*, up to a factor of* $|t|^p$*,* $p \in \mathbb{Z}$*, is invariant under Reidermeister moves; and*

(3) $\mathrm{Det}_\tau(A^{(m)}(t)) = 1$ *for* $|t|$ *small or large enough.*

• Using a theorem of Lück on a criterion of deciding if an operator is injective, and following the same line of argument in Li-Zhang for the case of $|t| = 1$, we have established the following theorem.

Theorem. *$A^{(m)}(t)$ is injective for all t. Thus $\mathrm{Det}_\tau(A^{(m)}(t))$ is defined for all $t \in \mathbb{C}^*$.*

• Li and Zhang call the quantitiy

$$\boxed{\Delta_K^{(2)}(t) :\sim \mathrm{Det}_\tau(A^{(m)}(t))}$$

by the name of L^2**-Alexander invariants of** K. Here $\sim$ means equal up to a factor of $|t|^p$, $p \in \mathbb{Z}$.

• Lück defined $\Delta_K^{(2)}(1)$ as the L^2-Reidermeister torsion of the universal covering of the knot complement $S^3 \setminus K$. The L^2-Alexander invariants of Li and Zhang give us a positive real parameter deformation of Lück's L^2-Reidermeister torsion.

The Volume Conjecture

• We have the following theorem relating the hyperbolic volume with the L^2-Alexander invariant at $t = 1$.

Theorem. (Lück, et al.) *Let K be a hyperbolic knot, and* $\mathrm{vol}(K)$ *be the hyperbolic volume of the knot complement $S^3 \setminus K$. Then*

$$\log \Delta_K^{(2)}(1) = \frac{1}{6\pi}\,\mathrm{vol}(K)$$

Thus the Volume Conjecture can be reformulated as follows:

Conjecture. (Kashaev, Murakai-Murakami)

$$\boxed{\lim_{N\to+\infty} \left| J_K\left(N, \exp\left(\frac{2\pi i}{N}\right)\right)\right|^{\frac{1}{3N}} = \Delta_K^{(2)}(1)}$$

where $J_K(N, q)$ is the normalized colored Jones polynomial of K.

• We formulate the following Generalized Volume Conjecture.

Generalized Volume Conjecture.[a] For every $t \in \mathbb{C}^*$,

$$\boxed{\lim_{N\to+\infty} \left| J_K \left(N, \exp\left(\frac{2\pi i t}{N}\right)\right)\right|^{\frac{1}{3N}} = \Delta_K^{(2)}(t)}$$

• Note that Gukov has another version of the Generalized Volume Conjecture, whose right hand side involves quantities coming from the $\mathrm{SL}(2,\mathbb{C})$ representation variety of $\pi_1(S^3 \setminus K)$.

• Note that $\Delta_K^{(2)}(t) = 1$ for $|t|$ small or large enough. Therefore, our Generalized Volume Conjecture is consistent with the recent result of Garoufalidis and Le regarding the left hand side of the formula above for "small angles".

[a]The final form of this conjecture is subject to modification.