UPPER-THRESHOLDS FOR SHOCK FORMATION IN
TWO-DIMENSIONAL WEAKLY RESTRICTED EULER-POISSON
EQUATIONS

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Abstract. The multi-dimensional Euler-Poisson system describes the dynamic behavior of many important physical flows, yet as a hyperbolic system its solution can blow up for some initial configurations. This paper strives to advance our understanding on the critical threshold phenomena through the study of a two-dimensional weakly restricted Euler-Poisson (WREP) system. This system can be viewed as an improved model upon the restricted Euler-Poisson (REP) system introduced in [H. Liu and E. Tadmor, Comm. Math. Phys. 228 (2002), 435-466]. We identify upper-thresholds for finite time blow up of solutions for WREP equations with attractive/repulsive forcing. It is shown that the thresholds depend on the size of the initial density relative to the initial velocity gradient through both trace and a nonlinear quantity.

1. Introduction and statement of main results

We are concerned with the threshold phenomenon in two dimensional Euler-Poisson equations. The pressure-less Euler-Poisson (EP) equations in multidimension are

\begin{align}
\rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= k \nabla \Delta^{-1}(\rho - c_b),
\end{align}

which are the usual statements of the conservation of mass, Newton’s second law. Here $k$ is a physical constant which parameterizes the repulsive $k > 0$ or attractive $k < 0$ forcing. Also, $c_b$ denotes the constant background state. The local density $\rho = \rho(t, x) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^+$ and the velocity field $\mathbf{u}(t, x) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ are the unknowns. This hyperbolic system (1.1) with non-local forcing describes the dynamic behavior of many important physical flows, including semi-conductors and plasma physics ($k > 0$) and the collapse of stars due to self gravitation ($k < 0$) [10, 19, 21, 1, 2, 6].

There is a considerable amount of literature relevant to the solution behavior of Euler-Poisson equations. Let us mention the study of steady-state solutions [19, 8] and the global existence of weak solutions [20]. Global existence due to damping relaxation and with non-zero background can be found in [22]. Construction of a global smooth irrotational solution in three and two dimensional space can be found in [9, 11].

To address the fundamental question of the persistence of $C^1$ regularity for solutions of the Euler-Poisson system and related models, the concept of Critical Threshold (CT) is originated and developed in a series of papers by Engelberg, Liu and Tadmor [7, 15, 16, 17, 18] and more. The critical threshold in [7] describes the conditional stability of the 1D Euler-Poisson system, where the answer to the question of global vs. local existence

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depends on whether the initial data crosses a critical threshold. Following [7], critical thresholds have been identified for several one dimensional models, including $2 \times 2$ quasi-linear hyperbolic relaxation systems [14], Euler equations with non-local interaction and alignment forces [3], and traffic flow models [13].

Moving to the multi-dimensional Euler-Poisson system, one has to identify the proper quantities to describe the critical threshold phenomenon. Liu and Tadmor introduce in [15] the method of spectral dynamics which relies on the dynamical system governing eigenvalues of the velocity gradient matrix, $M := \nabla u$, along particle paths.

We follow their approach and in order to trace the evolution of $M$, we take gradient of $(1.1b)$ to find

\begin{equation}
\partial_t M + u \cdot \nabla M + M^2 = k \nabla \otimes \nabla \Delta^{-1} [\rho - c_b] = k R [\rho - c_b]
\end{equation}

where $R$ is the Riesz matrix operator, defined as

$$ R[f] := \nabla \otimes \nabla \Delta^{-1} [f] = \mathcal{F}^{-1} \left\{ \frac{\xi_j \xi_k}{|\xi|^2} \hat{f}(\xi) \right\}_{j,k=1,2}. $$

Now, Euler-Poisson equations are recast into the coupled system

\begin{align}
\frac{d}{dt} M + M^2 &= kR[\rho - c_b], \\
\frac{d}{dt} \rho + \rho \text{tr} M &= 0,
\end{align}

with $\frac{d}{dt}$ standing for the usual convective derivative, $\partial_t + u \cdot \nabla$. It is the non-local forcing, $kR[\rho - c_b]$, which presents the main obstacle to studying the critical threshold phenomenon of the multidimensional Euler-Poisson equations.

To gain better understanding of the dynamics of velocity gradient $M$ governed by (1.3), Liu and Tadmor introduce in [15] the restricted Euler-Poisson (REP) system (1.4), which is obtained from (1.3) by restricting attention to the local isotropic trace $k_n (\rho - c_b) I_{n\times n}$ of the global coupling term $kR[\rho - c_b]$, namely,

\begin{align}
\frac{d}{dt} M + M^2 &= \frac{k}{n} (\rho - c_b) I_{n\times n}, \\
\frac{d}{dt} \rho + \rho \text{tr} M &= 0.
\end{align}

Replacing the nonlocal forcing term by a local one, it was shown that in the repulsive case, the restricted $2D$ REP model admits two-sided critical threshold [16]. For arbitrary $n \geq 3$ dimension REP model, the author and Liu identified both upper-thresholds for finite time blow up of solutions and sub-thresholds for global existence of solutions [12].

In this work, we propose the weakly restricted Euler-Poisson (WREP) system as a semi-localized alternative to (1.3). Specifically, we consider a gradient flow $M$, governed by

\begin{align}
\frac{d}{dt} M + M^2 &= \left( \frac{k}{2} (\rho - c_b) \begin{pmatrix} R_{12} \\ R_{21} \end{pmatrix} \right), \\
\frac{d}{dt} \rho + \rho \text{tr} M &= 0,
\end{align}
subject to initial data
\[(M, \rho)(0, \cdot) = (M_0, \rho_0)\].

We then investigate threshold conditions on the initial data that lead to the finite time blow up of \(M\).

To state our main results, we introduce several quantities with which we characterize the behavior of the velocity gradient tensor \(M\). The first one is the trace \(d := \text{tr} M = \nabla \cdot u\),
and the second one is a nonlinear quantity defined by
\[(1.6) \quad B = \eta^2 - \omega^2,\]
where \(\omega := M_{21} - M_{12}\) and \(\eta := M_{11} - M_{22}\). We use the notation \(B_0 = \eta_0^2 - \omega_0^2\) and \(d_0 = \text{tr}(M_0)\).

**Theorem 1.1. (Repulsive forcing, \(k > 0\))** Consider the 2-dimensional, repulsive weakly restricted Euler-Poisson system (1.5). Then, the solution of the 2D WREP system will blow up in finite time if the initial data \((\rho_0, M_0)\) lies in one of the following three regions,
\[(\rho_0, d_0, B_0) \in S_1 \cup S_2 \cup S_3:\]
(i) \((\rho_0, d_0, B_0) \in S_1,
S_1 := \left\{ (\rho, d, B) \bigg| \rho > 0, \quad B > \frac{k \rho^2}{2 c_b} \right\}.
(ii) \((\rho_0, d_0, B_0) \in S_2,
S_2 := \left\{ (\rho, d, B) \bigg| \rho > 0, \quad B = \frac{k \rho^2}{2 c_b} \quad \text{and} \quad (\rho, d) \neq (2 c_b, 0) \right\}.
(iii) \((\rho_0, d_0, B_0) \in S_3
S_3 := \left\{ (\rho, d, B) \bigg| \rho > 0, \quad 0 < B < \frac{k \rho^2}{2 c_b} \quad \text{and either} \quad G(\rho, B) \leq 0 \quad \text{or} \quad d < \text{sign}(F) \sqrt{\rho G(\rho, B)} \right\},
where
\[F = \rho - 2 c_b - \sqrt{\rho^2 - 2 c_b B/k},\]
\[G = \frac{B - 2 k c_b}{\rho} - 2 \sqrt{k^2 - 2 k c_b B} - 2 k \log \left( \frac{\rho - \sqrt{\rho^2 - 2 c_b B/k}}{2 c_b} \right).\]

**Theorem 1.2. (Attractive forcing, \(k < 0\))** Consider the 2-dimensional, attractive weakly restricted Euler-Poisson system (1.5). Then, the solution of the 2D WREP system will blow up in finite time if the initial data \((\rho_0, M_0)\) lies in the following region,
\[(\rho_0, d_0, B_0) \in S:\]
\[S := \left\{ (\rho, d, B) \bigg| \rho > 0, \quad B > 0 \quad \text{and} \quad d < \text{sign}(F) \sqrt{\rho G(\rho, B)} \right\},
where \(F\) and \(G\) are same as in Theorem 1.2.

The following lemma is crucial in our proofs of main theorems.

**Lemma 1.3.** From the 2D WREP system
\[
\frac{d}{dt} M + M^2 = \left( \begin{array}{cc}
\frac{k}{2} (\rho - c_b) & R_{12} \\
R_{21} & \frac{k}{2} (\rho - c_b)
\end{array} \right), \quad \frac{d}{dt} \rho + \rho \text{tr} M = 0,
\]
we can derive the following closed ordinary differential inequality (ODI) system:

\begin{equation}
\begin{aligned}
\frac{d}{dt} &\leq -\frac{1}{2} \left( \frac{\omega_0}{\rho_0} \right)^2 - \frac{1}{2} \left( \frac{\eta_0}{\rho_0} \right)^2 \right) \rho^2 + k(\rho - c_b), \\
\rho' &:= \frac{d}{dt} \rho.
\end{aligned}
\end{equation}

Proof. From the matrix equation (1.5), or

\[
\frac{d}{dt} M + \left( \begin{array}{cc} M_{11}^2 + M_{12} M_{21} & \frac{dM_{12}}{dt} \\ \frac{dM_{21}}{dt} & M_{12} M_{21} + M_{22}^2 \end{array} \right) = \left( \begin{array}{cc} \frac{k}{2}(\rho - c_b) & R_{12} \\ \frac{k}{2}(\rho - c_b) & R_{21} \end{array} \right),
\]

we obtain

\[
\begin{aligned}
\eta' + \eta d &= 0, \\
\omega' + \omega d &= R_{12} - R_{21} = 0, \\
\rho' + \rho d &= 0.
\end{aligned}
\]

Hence

\[
\frac{\eta}{\eta_0} = \frac{\omega}{\omega_0} = \frac{\rho}{\rho_0}.
\]

Thus,

\[
\begin{aligned}
d' &= -(M_{11}^2 + M_{22}^2) - 2M_{12} M_{21} + k(\rho - c_b) \\
&= -\left\{ M_{11}^2 + M_{22}^2 + \frac{(M_{12} + M_{21})^2}{2} \right\} + \frac{(M_{12} - M_{21})^2}{2} + k(\rho - c_b) \\
&\leq -\frac{(M_{11} + M_{22})^2 + (M_{11} - M_{22})^2}{2} + \frac{1}{2} \omega^2 + k(\rho - c_b) \\
&= -\frac{1}{2} \left( \frac{\omega_0}{\rho_0} \right)^2 - \frac{1}{2} \left( \frac{\eta_0}{\rho_0} \right)^2 \right) \rho^2 + k(\rho - c_b).
\end{aligned}
\]

\[(1.9)\]

Several remarks are in order.

1. We compare the current blow-up results with those in [16]. The critical threshold results in REP[16](restricted Euler-Poisson system) were formulated in terms of the spectral gap. That is, in Theorem 1.2 [16], REP model’s sub-critical data (if and only if global existence condition) is expressed in terms of \((\rho_0, d_0, \Gamma_0)\). Here, \(\Gamma_0\) is the initial spectral gap, i.e., \(\Gamma_0 = (\lambda_2(0) - \lambda_1(0))^2\). In Theorem 1.1, WREP model’s super-threshold data (blow-up condition) is expressed in terms of \((\rho_0, d_0, B_0)\), where \(B_0 = (M_{11}(0) - M_{22}(0))^2 - (M_{21}(0) - M_{12}(0))^2 =: \eta_0^2 - \omega_0^2\). One can easily derive that \(\Gamma_0 = B_0 + (M_{21}(0) + M_{12}(0))^2\).

For comparison purpose, REP equations’ \(d\) dynamics equation in [16] can be re-written as follows:

\[
\begin{aligned}
d' &= -\frac{1}{2} \left( \frac{\omega_0}{\rho_0} \right)^2 - \frac{1}{2} \left( \frac{\eta_0}{\rho_0} \right)^2 - (M_{12} + M_{21})^2 \right) \rho^2 + k(\rho - c).
\end{aligned}
\]

One can notice that the coefficient of \(\rho^2\) in the above equation is more negative relative to that in (1.7) due to the presence of \(-(M_{12} + M_{21})^2\) term. Therefore, we can expect that the blow-up initial configuration set \((M_0, \rho_0)\) of WREP is contained in that of REP.
Figure 1. Blow-up initial configuration of WREP and global existence initial configuration of REP

More precisely, in Figure 1 (A) and (B), blow-up initial configuration for WREP, and global existence configuration for REP are plotted in $[-10,10]^3$ cubes with $k = c = 0.5$. In the last figure, it is observed that when $(M_{21}(0) + M_{12}(0))^2 = 0$ (which in turn implies $\Gamma_0 = B_0$), the WREP blow-up set and REP global existence set fit together nicely without any intersection.

2. Chae, Cheng and Tadmor [4, 5] obtain the blow up result for $n$-dimensional full Euler-Poisson systems (1.3) with attractive forcing $k < 0$. For proofs of the results in [4, 5], the vanishing initial vorticity condition (which ensures $\nabla \times u = 0$ for all time) is essential to ensure the following Ricatti-type key inequality

\[
(1.10) \quad d' \leq -\frac{1}{n}d^2 + k(\rho - c_b).
\]

Our Lemma 1.3 tells us that one can derive the similar Ricatti-type inequality (1.7) when initial vorticity condition $\nabla \times u_0(x) \neq 0$.

3. The critical threshold in the 1D Euler-Poisson system depends only on the relative size of the initial velocity slope and the initial density [7]. In contrast to the 1D EP system, the threshold conditions in 2D REP depend on several initial quantities: density $\rho_0$, divergence $\nabla \cdot u_0$, vorticity $\nabla \times u_0$ and gap $v_{0x} - v_{0y}$.

4. The above results show that to ensure the finite time blow up, relatively small absolute value of initial vorticity $|u_{0y} - v_{0x}|$ is needed. This fact is consistent with the results in [16]: the results in [16] show that the global smooth solution is ensured if the initial velocity gradient has complex eigenvalues, which applies, for example, for a class of initial configurations with sufficiently large vorticity.

5. With relatively small initial vorticity, the finite time blow up occur if the initial divergence is below a thresholds, expressed in terms of the initial density and elements in $\nabla u_0$. So we can view our results in the perspective of the critical thresholds.
We shall conclude this section by discussing some questions for future study. For the question of extension of blow-up results in WREP, from (1.3), consider the $\eta$ ODE

$$\eta' + \eta d = k(R_{11} - R_{22}),$$

where $R$ is the Riesz matrix operator. We have no clear idea on how fast $\eta$ are changing in time because we are lack of $L^\infty$ bound of $R_{ij}$. The key assumption in our WREP model were

$$R_{11} = R_{22} = \frac{k}{2}(\rho - c_b).$$

With this assumption the above $\eta$ ODE is reduced to simple ODE, $\eta' + \eta d = 0$ in (1.8) and the ODE allows us to write $\eta$ in terms of $\rho$. To extend the blow-up result, an advanced sharp technique with an aid of harmonic analysis seems to be needed. Another effort to advance our understanding on the full Euler-Poisson equations may be made by modifying the Riesz transform. In [23], the author propose a modified Euler-Poisson equations with a modified Riesz transform where singularity at the origin is removed.

Concerning the question of possible regularity result for sub-critical data, we first notice that the derivation of Ricatti-type dynamics in Lemma 1.3 relies on the assumption $R_{11} = R_{22}$. The one sided Ricatti-type ODI(ordinary differential inequality) in (1.7) allow us to derive finite time blow up of $d$ (i.e., $d \to -\infty$) only. That is, due the the one sided inequality nature, there is no way to bound $d$ from below.

In order to derive a regularity result for sub-critical data, one may need to bound $|d|$ and $\rho$ for all time. This may be possible provided that;

i) If one can find a closed system of ODE not ODI. For example, in [16], Liu-Tadmor considered the 2D REP(restricted Euler-Poission equation), and derived the following ODE system

$$d' + \frac{d^2 + \beta \rho^2}{2} = k\rho, \quad \rho' = -\rho d,$$

and studied the dynamics of $(\rho, d)$ parametrized by $\beta$(where $\beta$ is the ratio between the initial spectral gap and the initial density). This ODE system enable a complete complete description of the critical threshold phenomenon for the 2D REP to be delivered.

Or,

ii) If one can find a two sided differential inequality of $\rho$ or $d$. For example, in [12], the author and Liu considered the $n$-dimensional REP, and the following two sided ODI was derivied:

$$-n\rho \lambda_n \leq \rho' \leq -n\rho \lambda_1.$$

(here $\lambda_i$ is the eigenvalue of $\nabla u$) This two sided differential inequality leded to the desired thresholds for both global existence and the finite time blow-up.

With this weakly restricted Euler-Poisson equations, The author was unable to obtain any two sided ODI nor closed ODE system which may bound $|d|$ or $\rho$. Therefore, a regularity result for sub-critical data for WREP is still yet to be found.
2. Proof of Theorem 1.1.

In this section we show the existence of an upper threshold for the 2D WREP with \( k > 0 \). Let \( A := \frac{B_0}{2\rho_0^2} = -\frac{1}{2} \left\{ \left( \frac{\omega_0}{\rho_0} \right)^2 - \left( \frac{\eta_0}{\rho_0} \right)^2 \right\} \), then the ODI system (1.7) is rewritten as

\[
(2.1a) \quad \frac{d}{dt} \leq -\frac{1}{2} d^2 - A\rho^2 + k(\rho - c_b),
\]

\[
(2.1b) \quad \rho' = -d\rho.
\]

2.1. Cases (i) and (ii): We write (2.1a) as

\[
(2.2) \quad \frac{d}{dt} \leq -\frac{1}{2} d^2 - A \left( \rho - \frac{k}{2A} \right)^2 + \frac{k^2}{4A} - k c_b.
\]

If \( B_0 > \frac{k\rho_0^2}{2c_b} \) (i.e., \( k < 4Ac_b \)), we have \( A > 0 \) so the second term in the right hand side of (2.2) is non-positive. Therefore, it follows that

\[
\frac{d}{dt} \leq -\frac{1}{2} d^2 + \frac{k^2}{4A} - k c_b = -\frac{1}{2} d^2 + k \left( \frac{k}{4A} - c_b \right).
\]

We see that \( k \left( \frac{k}{4A} - c_b \right) < 0 \), therefore the above inequality ensures that \( d \) blow up in finite time for any choice of \( d_0 \).

Also, if \( B_0 = \frac{k\rho_0^2}{2c_b} \), i.e., \( k = 4Ac_b \), then (2.2) is reduced to

\[
\frac{d}{dt} \leq -\frac{1}{2} d^2 - A(\rho - 2c_b)^2.
\]

Thus, \( d \) blow up in finite time for all initial data except \( (\rho_0, d_0) = (2c_b, 0) \).

2.2. Case (iii): We assume that \( 0 < B_0 < \frac{k\rho_0^2}{2c_b} \) (i.e., \( 0 < A < \frac{k}{4c_b} \)).

For simplicity we set \( c_b = 1 \) and compare the ODI (2.1) with the following ODE system by still using the original variables.

\[
(2.3) \quad \frac{d}{dt} = -\frac{1}{2} d^2 - A\rho^2 + k(\rho - 1),
\]

\[
\rho' = -d\rho.
\]

Here we note that \( k > 4Ac_b = 4A \cdot 1 \) because of the assumption in this subsection. Also note that the ODE system admits two distinct critical points, i.e. \( (\rho, d) = (\alpha_1, 0) \) where,

\[
(\alpha_1, 0) := \left( \frac{k + \sqrt{k^2 - 4kA}}{2A}, 0 \right), \quad (\alpha_2, 0) := \left( \frac{k - \sqrt{k^2 - 4kA}}{2A}, 0 \right)
\]

and that \( (\rho, d) = (\alpha_1, 0) \) is a saddle and \( (\rho, d) = (\alpha_2, 0) \) is a spiral. We shall use the above facts to construct the threshold via the phase plane analysis. Following the same \( q \)-transformation as that employed in [16], we set \( q := d^2 \) and differentiate along the particle path \( \{(t, X(t, a)) \mid X(t, a) = u(t, X(t, a)), X(t = 0, a) = a\} \), we get

\[
\frac{dq}{d\rho} = 2d \frac{d'}{d\rho} = \frac{q}{\rho} + 2A\rho - 2k\left(1 - \frac{1}{\rho}\right),
\]
which yields

\[ \frac{d}{d\rho} \left( \frac{q}{\rho} \right) = 2A - 2k \left( \frac{1}{\rho} - \frac{1}{\rho^2} \right). \tag{2.4} \]

Integration leads to a global invariant

\[ \frac{d^2}{\rho} - \frac{d^2}{\rho_*} = -2 \int_\rho^{\rho_*} \frac{-A s^2 + k (s - 1)}{s^2} \, ds, \tag{2.5} \]

where \(d_*\) and \(\rho_*\) are some constants. By setting \((\rho_*, d_*) = (\alpha_1, 0)\), we find the separatrix curve passing \((\alpha_1, 0)\),

\[ \frac{d^2}{\rho} = -2 \int_{\alpha_1}^{\rho} \frac{-A s^2 + k (s - 1)}{s^2} \, ds. \tag{2.6} \]

The above curve has two \(x\) intercepts. One is \((\alpha_1, 0)\) and the other is denoted by \((\alpha_3, 0)\) with \(0 < \alpha_3 < \alpha_2\). In fact, consider

\[ \int_{\rho}^{\alpha_1} \frac{-A s^2 + k (s - 1)}{s^2} \, ds = \int_{\rho}^{\alpha_2} \frac{-A s^2 + k (s - 1)}{s^2} \, ds + \int_{\alpha_2}^{\alpha_1} \frac{-A s^2 + k (s - 1)}{s^2} \, ds. \]

Note that \(-A s^2 + k (s - 1) \geq 0\) , \(\forall s \in [\alpha_2, \alpha_1]\) and \(\lim_{\rho \to 0^+} \int_{\rho}^{\alpha_2} \frac{-A s^2 + k (s - 1)}{s^2} \, ds \to -\infty\). This proves the existence of intercept \((\alpha_3, 0)\) and the following identity,

\[ \int_{\alpha_3}^{\alpha_1} \frac{-A s^2 + k (s - 1)}{s^2} \, ds = 0. \tag{2.7} \]

Back to ODI system (2.1), the same \(q\)-transformation gives us

\[ \frac{d}{d\rho} \left( \frac{q}{\rho} \right) \geq 2A - 2k \left( \frac{1}{\rho} - \frac{1}{\rho^2} \right). \tag{2.8} \]

We now discuss subcases distinguished by the location of initial points; see Figure 2.

•

\((\rho_0, d_0) \in \Omega_1 := \left\{ (\rho, d) \mid \alpha_3 \leq \rho \leq \alpha_1, \ d < -\sqrt{2 \rho \int_{\rho}^{\alpha_1} \frac{-A s^2 + k (s - 1)}{s^2} \, ds} \right\} \]

First, we show no orbit of the ODI touches the lower left arc of separatrix curve (2.6): (2.8) leads to

\[ \frac{d^2}{\rho} - \frac{d^2_0}{\rho_0} \geq -2 \int_{\rho_0}^{\rho} \frac{-A s^2 + k (s - 1)}{s^2} \, ds. \tag{2.9} \]

Now, consider a point \((\rho_0, d_*)\) on the separatrix curve, which above \((\rho_0, d_0)\), i.e.,

\[ \frac{d^2_*}{\rho_0} = -2 \int_{\alpha_1}^{\rho_0} \frac{-A s^2 + k (s - 1)}{s^2} \, ds. \]
Since $d_0 < d_* < 0$ we have
\[
\frac{d^2}{\rho} \geq \frac{d_0^2}{\rho_0} - 2 \int_{\rho_0}^{\rho} \frac{-A s^2 + k(s - 1)}{s^2} \, ds
\]
\[
> \frac{d^2}{\rho_0} - 2 \int_{\rho_0}^{\rho} \frac{-A s^2 + k(s - 1)}{s^2} \, ds
\]
\[
= -2 \int_{\alpha_1}^{\rho} \frac{-A s^2 + k(s - 1)}{s^2} \, ds.
\]
This relation shows that, if $(\rho_0, d_0) \in \Omega_1$, then no $(\rho(t), d(t))$ crosses separatrix curve (2.6).

Next, we show that if $(\rho_0, d_0) \in \Omega_1$ then $(\rho(t), d(t)) \nrightarrow (\alpha_1, 0)$ as $t \to \infty$. Suppose $\rho(t) \not\rightarrow \alpha_1$ and $d(t) \not\rightarrow 0$ as $t \to \infty$. Then as $t \to \infty$, (2.9) leads to
\[
-\frac{d_0^2}{\rho_0} \geq -2 \int_{\rho_0}^{\alpha_1} \frac{-A s^2 + k(s - 1)}{s^2} \, ds.
\]
But this contradicts to the fact that $(\rho_0, d_0) \in \Omega_1$.

Finally, we show that if $(\rho_0, d_0) \in \Omega_1$ then $\exists t^* < \infty$ such that $\rho(t^*) > \alpha_1$. Suppose $\alpha_3 \leq \rho(t) \leq \alpha_1, \forall t > 0$. Then, since $d(t) < 0$ for all time, from $\rho(t) = \rho_0 \exp(-\int_0^t d(s) ds)$, $d(t)$ must go to 0. Since no orbit can touch the lower left arc, we are left with only one possibility $(\rho(t), d(t)) \rightarrow (\alpha_1, 0)$. But this contradicts to the second argument. Hence $\rho(t) > \alpha_1$ in finite time $t^*$.  

**Figure 2.** The blow up region of $k > 4 Ac_0$ case.
\[(\rho_0, d_0) \in \Omega_2 := \{(\rho, d)|0 < \rho < \alpha_3, \ d < 0\} \]. We will show that if \((\rho_0, d_0) \in \Omega_2\), then \((\rho(t), d(t)) \in \Omega_1\) in finite time. Suppose not, i.e., suppose \(\rho(t) < \alpha_3, \ \forall t > 0\).

Then from (2.1a),

\[
  d' < -\frac{1}{2}d^2 - A(\rho - \alpha_1)(\rho - \alpha_2) < -\frac{1}{2}d^2 - M, \ \forall t > 0,
\]

where \(M := A(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) > 0\). That is, \(d' < -M, \ \forall t > 0\), which upon integration over \([0, t]\) gives \(d \leq d_0 - Mt\). This tells us that

\[
  -\int_0^t d(s)ds \geq -d_0t + \frac{Mt^2}{2}
\]

and hence

\[
  \rho(t) = \rho_0 \exp\left(-\int_0^t d(s)ds\right) \geq \rho_0 \exp\left(-d_0t + \frac{Mt^2}{2}\right).
\]

But, since \(d_0 < 0\) and \(M > 0\), \(\rho(t) \geq \alpha_3\) in finite time. Therefore, we get the contradiction.

\[(\rho_0, d_0) \in \Omega_3 := \{(\rho, d)|0 < \rho \leq \alpha_3 \text{ and } d \geq 0\}\]. As long as \((\rho(t), d(t)) \in \Omega_3\), \(\rho\) is decreasing and \(d' < -\frac{1}{2}d^2 - M \leq -M\) where \(M := A(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)\).

Therefore, \((\rho_0, d_0) \in \Omega_3\) implies \((\rho(t), d(t)) \in \Omega_2\) in finite time.

\[(\rho_0, d_0) \in \Omega_4 := \{(\rho, d)|\ \rho > \alpha_1, \ 0 \leq d < \sqrt{-\frac{2\rho}{\alpha_1} - \frac{As^2 + k(s - 1)}{s^2}} ds\}\]

First, we show no orbit of the ODI touches the upper right branch of the separatrix curve: Note that in \(\Omega_4\), since \(\rho, d > 0\), we have \(\rho' \leq 0\). Thus (2.8) leads to

\[
  \frac{d^2}{\rho} - \frac{d_0^2}{\rho_0} \leq -2 \int_{\alpha_1}^{\rho_0} \frac{-As^2 + k(s - 1)}{s^2} ds.
\]

Now, consider a point \((\rho_0, d_*)\) on the invariant, i.e.,

\[
  \frac{d_*^2}{\rho_0} = -2 \int_{\alpha_1}^{\rho_0} \frac{-As^2 + k(s - 1)}{s^2} ds.
\]

Since \(0 < d_0 \leq d_*\) we have

\[
  \frac{d^2}{\rho} \leq \frac{d_0^2}{\rho_0} - 2 \int_{\rho_0}^{\rho} \frac{-As^2 + k(s - 1)}{s^2} ds
\]

\[
  < \frac{d_*^2}{\rho_0} - 2 \int_{\rho_0}^{\rho} \frac{-As^2 + k(s - 1)}{s^2} ds
\]

\[
  = -2 \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - 1)}{s^2} ds
\]

Hence, as long as \((\rho_0, d_0) \in \Omega_4\), no orbit of the ODI touches the upper right branch of the separatrix curve.
Next, we show that if \((\rho_0, d_0) \in \Omega_4\) then \((\rho(t), d(t)) \to (\alpha_1, 0)\) as \(t \to \infty\). Suppose \(\rho(t) \searrow \alpha_1\) and \(d(t) \searrow 0\) as \(t \to \infty\). Then as \(t \to \infty\), (2.10) leads to
\[
-\frac{d_0^2}{\rho_0} \leq -2 \int_{\rho_0}^{\alpha_1} \frac{-As^2 + k(s - 1)}{s^2} \, ds.
\]
But this contradicts to the fact that \((\rho_0, d_0) \in \Omega_1\).

Finally, due to non-touching result and the fact that \(\lim_{t \to \infty}(\rho, d) \neq (\alpha_1, 0)\), any orbit starting from within \(\Omega_4\) must enter \(\Omega_5 := \{(\rho, d) | \rho > \alpha_1\text{ and }d < 0\}\) through \(d = 0\) and \(\rho > \alpha_1\). Assume such a crossing time is \(T^*\), then
\[
d' \leq -A(\rho(T^*) - \alpha_1)(\rho(T^*) - \alpha_2),
\]
which upon integration over \([0, T^*]\) gives
\[
T^* \leq \frac{d_0}{A(\rho(T^*) - \alpha_1)(\rho(T^*) - \alpha_2)}
\]
which implies that \(T^*\) must be finite.

To sum up, we arrive at the following observation.

**Lemma 2.1.** If \((\rho_0, d_0) \in \bigcup_{i=1}^{4} \Omega_i\), then \((\rho(t), d(t)) \in \Omega_5\) in finite time, where
\[
\Omega_5 := \{(\rho, d) | \alpha_1 < \rho \text{ and } d < 0\}.
\]

Now in \(\Omega_5\), we shall carry out the blow up analysis of
\[
\begin{align*}
  d' &\leq -\frac{1}{2}d^2 - Ap^2 + k(\rho - c_b) = -\frac{1}{2}d^2 - A(\rho - \alpha_1)(\rho - \alpha_2) \\
  \rho' &= -d\rho
\end{align*}
\]
through a comparison with the corresponding ODE system
\[
\begin{align*}
e' &= -\frac{1}{2}e^2 - A\zeta^2 + k(\zeta - c_b) = -\frac{1}{2}e^2 - A(\zeta - \alpha_1)(\zeta - \alpha_2) \\
\zeta' &= -e\zeta.
\end{align*}
\]

The following lemma shows the monotonicity relation between the ODE and the ODI in \(\Omega_5\).

**Lemma 2.2.** \[
\begin{align*}
d(0) < e(0) < 0 \\
\zeta(0) < \rho(0),
\end{align*}
\]
implies \[
\begin{align*}
d(t) &< e(t) < 0 \\
\zeta(t) &< \rho(t),
\end{align*}
\]
\(\forall t \geq 0\), as long as \(\zeta(t) > \alpha_1\), \(\forall t \geq 0\).

**Proof.** It can be proved by contradiction. Suppose \(t_1\) is the earliest time when the above assertion is violated. Then
\[
\zeta(t_1) = \zeta(0)e^{-\int_{0}^{t_1} e(t) \, dt} < \rho(0)e^{-\int_{0}^{t_1} d(t) \, dt} = \rho(t_1).
\]
Therefore, we are left with only one possibility \(e(t_1) = d(t_1)\).

From (2.11) and (2.12),
\[
(e - d)' \geq -\frac{1}{2}(e^2 - d^2) - A\{(\zeta - \alpha_1)(\zeta - \alpha_2) - (\rho - \alpha_1)(\rho - \alpha_2)\}.
\]
Since \(e(t) - d(t) > 0\) for \(t < t_1\) and \(e(t_1) - d(t_1) = 0\), hence at \(t = t_1\) we have
\[
(e(t_1) - d(t_1))' \leq 0.
\]
But, since $\rho(t_1) > \zeta(t_1)$, when (2.13) is evaluated at $t = t_1$ gives

$$-A\{\zeta(t_1) - \alpha_1)(\zeta(t_1) - \alpha_2) - (\rho(t_1) - \alpha_1)(\rho(t_1) - \alpha_2)\} \geq 0.$$ 

This leads to a contradiction, as needed. □

The following lemma provides the blow up conditions of the modified system in (2.12), which in turn, will lead to the blow up of the original system in (2.11).

**Lemma 2.3.** Consider the modified system (2.12), equipped with initial data $(\zeta_0, e_0)$. If $(\zeta_0, e_0) \in \Omega_5$, then $e \to -\infty$, $\zeta \to \infty$ at a finite time.

**Proof.** Consider

$$\begin{cases}
  e' = -\frac{1}{2}e^2 - A(\zeta - \alpha_1)(\zeta - \alpha_2), \\
  \zeta' = -e\zeta.
\end{cases}$$

Note that if $(\zeta_0, e_0) \in \Omega_5$, then $\zeta(t)$ is increasing $\forall t$. Thus, $\zeta(t) > \alpha_1, \forall t$. This implies $e' < -\frac{1}{2}e^2$, which upon integration yields

$$e(t) < \frac{2e_0}{te_0 + 2}.$$ 

This implies that

$$e(t) \to -\infty \text{ and } \zeta(t) = \zeta_0 \exp\left(-\int_0^t e(s)ds\right) \to \infty \text{ as } t \to t^*$$

with the blow up time $t^* < -\frac{2}{e_0}$. □

Now we are ready for the last step of proving Theorem 1.1. We combine the monotonicity relation in Lemma 2.2 with Lemma 2.1 and Lemma 2.3. Consider given any initial data $(\rho_0, d_0) \in \Omega_5$ for the ODI. Since $\Omega_5$ is an open set, so we can find $\epsilon > 0$ such that $(\rho_0 - \epsilon, d_0 + \epsilon) \in \Omega_5$. We set this latter data as an initial data of the ODE for the comparison purpose. This latter initial data will lead to finite time blow up of the ODE and thus initial data $(\rho_0, d_0) \in \Omega_5$ will lead to finite time blow up of the ODI. Furthermore, by Lemma 2.2, we know that if an initial data of the ODI is contained in $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, then $(\rho(t), d(t)) \in \Omega_5$ in finite time. Hence, initial data $(\rho_0, d_0) \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$ will lead to finite time blow up of the original ODI.

To sum up, the above arguments give us the upper thresholds which lead to finite-time breakdown of the WREP equation. The threshold curve can be expressed as union of two sets: One is half straight line

$$\{(\rho,d) \mid \rho = \alpha_3, \ d > 0\},$$

and the other is a union of the lower-left arc and upper-right branches of the separatrix curve $\frac{d^2}{\rho} = -2 \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s-1)}{s^2} ds$, i.e.,

$$\left\{(\rho,d) \mid \rho \geq \alpha_3, \ d = \text{sgn}(\rho - \alpha_1)\sqrt{-2\rho \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s-1)}{s^2} ds}\right\}.$$ 

We recover normalized constant $c_b$, then expanding the above integral and using the identity in (2.7), recalling that $B = 2A\rho^2$, complete the proof of Theorem 1.1.
3. Proof of Theorem 1.2.

In this section we prove the existence of a one-sided threshold condition which leads to finite-time breakdown of the 2D WREP with attractive forcing ($k < 0$). We shall carry out the blow up analysis of

\[ d' \leq -\frac{1}{2}d^2 + \frac{1}{2}\left\{ \left( \frac{\omega_0}{\rho_0} \right)^2 - \left( \frac{\eta_0}{\rho_0} \right)^2 \right\} \rho^2 + k(\rho - c_b), \]

\[ \rho' = -d\rho, \]

through a comparison with the corresponding ODE system

\[ e' = -\frac{1}{2}e^2 + \frac{1}{2}\left\{ \left( \frac{\omega_0}{\rho_0} \right)^2 - \left( \frac{\eta_0}{\rho_0} \right)^2 \right\} \zeta^2 + k(\zeta - c_b), \]

\[ \zeta' = -e\zeta. \]

As we did before, let $A := -\frac{1}{2}\left\{ \left( \frac{\omega_0}{\rho_0} \right)^2 - \left( \frac{\eta_0}{\rho_0} \right)^2 \right\} > 0$ and for simplicity we set $c_b = 1$. The following lemma shows the monotonicity relation between (3.1) and (3.2). The proof is similar to that in [5], so details are omitted.

**Lemma 3.1.** \( \begin{cases} d(0) < e(0) \\ 0 < \zeta(0) < \rho(0) \end{cases} \) implies \( \begin{cases} d(t) < e(t) \\ 0 < \zeta(t) < \rho(t) \end{cases} \) for $t \geq 0$, as long as all solution remain finite on time interval $[0, t]$.

![Figure 3. The blow up region of $k < 0$ case.](image-url)
System (3.2) admits three distinct critical points:

\[(\rho^\pm, d^\pm) := (0, \pm \sqrt{-2k}), \quad (\alpha_1, 0) := \left(\frac{k + \sqrt{k^2 - 4Ak}}{2A}, 0\right)\]

and that \((\rho^+, d^+)\) is a nodal sink, \((\rho^-, d^-)\) is a nodal source and \((\alpha_1, 0)\) is a saddle point. Also, as we did in the previous section, the separatrix curve passing \((\alpha_1, 0)\) is given by

\[
\frac{d^2}{\rho} = -2 \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - 1)}{s^2} ds.
\]

Note that the comparison principle in Lemma 3.1 applies in \(\Omega_1 := \{ (\rho, d) | \rho > \alpha_1 \text{ and } d < 0 \}\). We now discuss subcases distinguished by the location of initial points;

- \((\rho_0, d_0) \in \Omega_2 := \left\{ (\rho, d) | \alpha_1 < \rho, \ 0 \leq d < \sqrt{-2\rho \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - 1)}{s^2} ds} \right\} \)

Due to non-touching result we showed in the previous section and the fact that \(\lim_{t \to \infty}(\rho, d) \not\to (\alpha_1, 0)\), we know that if \((\rho_0, d_0) \in \Omega_2\), then \((\rho(t), d(t)) \in \Omega_1\) in finite time. The proof of this is the same as in \(\Omega_4\) of the repulsive case.

- \((\rho_0, d_0) \in \Omega_3 := \left\{ (\rho, d) | 0 < \rho \leq \alpha_1, \ d < -\sqrt{-2\rho \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - 1)}{s^2} ds} \right\} \)

As we did in repulsive case, the non-touching result and the fact that \(\lim_{t \to \infty}(\rho, d) \not\to (\alpha_1, 0)\) can be applied here too. We know that \(\lim_{t \to \infty} d(t) \not\to 0\). Thus \(\rho(t) = \rho_0 \exp(-\int_0^t d(s) ds) > \alpha_1\) in finite time.

To sum up, we arrive at the following observation.

**Lemma 3.2.** If \((\rho_0, d_0) \in \Omega_2 \cup \Omega_3\), then \((\rho(t), d(t)) \in \Omega_1\) in finite time.

The following lemma provides the blow up conditions of the modified system in (3.2), which in turn, will lead to the blow up of the original system in (3.1).

**Lemma 3.3.** Consider the modified system (3.2), equipped with initial data \((\zeta_0, e_0)\). If \((\zeta_0, e_0) \in \Omega_1\), then \(e \to -\infty, \zeta \to \infty\) at a finite time.

**Proof.** Consider

\[
\begin{cases}
    e' = -\frac{1}{2}e^2 - A(\zeta - \alpha_1)(\zeta - \beta), \\
    \zeta' = -e\zeta.
\end{cases}
\]

where \(\beta = \frac{k - \sqrt{k^2 - 4Ak}}{2A} < 0\). Note that if \((\zeta_0, e_0) \in \Omega_1\), then \(\zeta(t)\) is increasing in \(t\). Thus, \(\zeta(t) > \alpha_1, \forall t\). This implies \(e' < -\frac{1}{2}e^2\), which upon integration yields

\[
e(t) < \frac{2e_0}{te_0 + 2}.
\]

Hence, the blow up time \(t^*\) of \(e(t)\) must satisfy

\[
t^* < -\frac{2}{e_0}.
\]
Also, \( e \rightarrow -\infty \) and \( \zeta = \zeta_0 \exp(-\int_0^t e(s) \, ds) \rightarrow \infty. \)

Now we are ready for the last step of proving Theorem 1.2. We combine the monotonicity relation in Lemma 3.1 with Lemma 3.2 and 3.3. Consider given any initial data \((\rho_0, d_0) \in \Omega_1\) for the ODI. Since \( \Omega_1 \) is an open set, we can find \( \epsilon > 0 \) such that \((\rho_0 - \epsilon, d_0 + \epsilon) \in \Omega_1\). We set this latter data as an initial data of the ODE for the comparison purpose. This latter initial data will lead to finite time blow up of the ODE and thus initial data \((\rho_0, d_0) \in \Omega_1\) will lead to finite time blow up of the ODI. Furthermore, by Lemma 3.2, we know that if an initial data of the ODI is contained in \( \Omega_2 \cup \Omega_3 \), then \((\rho(t), d(t)) \in \Omega_1\) in finite time. Hence, initial data \((\rho_0, d_0) \in \Omega_1 \cup \Omega_2 \cup \Omega_3\) will lead to finite time blow up of the original ODI.

We close this section by stating the upper thresholds which determine the blow up region of the WREP equation. The upper right and lower left branches of

\[
\frac{d^2}{\rho} = -2 \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - 1)}{s^2} \, ds
\]

are the critical thresholds. So the upper thresholds can be expressed as

\[
\left\{(\rho, d) \mid \rho > 0, \quad d = \text{sgn}(\rho - \alpha_1) \sqrt{-2\rho \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s - 1)}{s^2} \, ds}\right\}.
\]

Changing \( A \) back to \( B = 2A\rho^2 \) we complete the proof of Theorem 1.2.

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**References**


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