Classification of Compact Homogeneous Manifolds with Pseudo-kählerian Structures

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Dans cette note, nous appliquons un théorème de modification pour des “solv-variétés” compactes et homogènes à des variétés compactes complexes équipées d’une structure pseudo-kählérienne. Nous obtenons une classification de ces variétés compactes pseudo-kählériennes sous la forme de certains produits d’espaces projectifs rationnels et homogènes, de tores, et d’espaces pseudo-kählériens réduits et primitifs simples ou doubles.

In this note we apply a modification theorem for compact homogeneous solvmanifolds to compact complex homogeneous manifolds with pseudo-kählerian structures. We are then finally able to classify these compact pseudo-kählerian manifolds, as certain products of projective rational homogeneous spaces, tori, and some simple and double reduced primitive pseudo-kähler spaces.


Key words and phrases: Solvmanifolds, cohomology, invariant structure, homogeneous space, product, fiber bundles, symplectic manifolds, splittings, decompositions, modification, Lie group, compact manifolds, uniform discrete subgroups, pseudo-kählerian structures.
Compact complex homogeneous spaces with equivariant Kähler structures were classified by Matsushima in [Mt]. Compact complex homogeneous spaces with Kähler structures (not necessarily equivariant) were classified by Borel and Remmert in [BR]. Compact complex homogeneous spaces with equivariant pseudo-kählerian structures were classified by Dorfmeister and the author in [DG]. A possible classification of compact complex homogeneous spaces with pseudo-kählerian structures was proposed by the author in [Gu1,2]. However, the classification turns out to be much more complicated than suggested therein. See [Gu4], [Ym1, 2], and [Ha]. In this note, we shall give a complete classification. I received a reprint of [Ym2] only after submitting [Gu4], and a preprint of [Ha] only after completing [Gu5].

In this paper, we shall provide the last piece of this puzzle, thereby completely solving this problem. What was missing in [Gu1, 2] is the calculation of the cohomology group of a compact solvmanifold. That problem was resolved in [Gu4], using techniques in [Gu3].

The complete proof, which is quite technical, is given in [Gu5].

We already reduced the classification to the case of a solvable parallelizable action in [Gu4], following the plan proposed in [Gu1, 2].

An application from the modification argument in [Gu4] is the so-called complex-parallelizable-right-invariant-pseudo-kählerian algebra (cf. [Gu4] Corollary 1). This made our classification possible.

Moreover, at the group level, we proved in [Gu5] that the given group $G$ can be decomposed as $G = AN$ with two abelian subgroups $A$ and $N$ (for

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1 The proof in [Gu1] works for the special cases of the manifolds being Kählerian or holomorphic symplectic. [Gu2] gave the splitting theorem up to the Mostow condition for which I mistakenly applied Iwamoto’s result from Osaka Journal of Math. and we then gave a complete proof in [Gu4].

2 One could see from our statements that even in the complex dimension three, our results are much stronger than that in [Ha].

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partial results, see also [Gu4] Corollary 2 and [Ym2], [DG] p.509 Corollary 1), and the action of $A$ on $N$ is an algebraic group action of a product of $C^*$’s.

If the Lie algebra of $A$ acts on $N$ with only one pair of eigenvalue functions $k_1$ and $k_2 = -k_1$, we shall call the given compact complex parallelizable homogeneous manifold with a pseudo-kählerian structure a primary pseudo-kählerian manifold.

**Theorem A.** Every compact complex parallelizable homogeneous manifold with a pseudo-kählerian structure up to a finite covering is a torus bundle over a pseudo-kählerian torus which as a torus bundle is a product of primary pseudo-kählerian manifolds.

The product in the Theorem A is the fiberwise product. More detail structure can be found in [Gu5].

We shall call a primary pseudo-kählerian manifold a reduced primary pseudo-kählerian manifold if the action of $A$ on $N$ is effective. Notice that we may always change a given invariant form $\omega_1$ on $A$ to any other such form. As a consequence:

**Theorem B.** After the changing of $\omega_1$, any primary pseudo-kählerian manifold, up to a finite covering and as a torus bundle, is the product of a torus and a reduced primary pseudo-kählerian manifold. Moreover, $\dim_C A = 1$ and $\dim_C N = 2m$ with $m$ the complex dimension of the eigenspaces for a reduced primary pseudo-kählerian manifold. In particular, the index of the pseudo-kählerian structure for a reduced primary pseudo-kähler space is either 1 or $-1$.

For any primary space, we have $k_1(z) = z$ and $k_2(z) = -z$. The fiber torus up to a finite covering can be split into complex irreducible ones with
respect to the $A$ action. For a primary pseudo-kählerian space, if the fiber is also an irreducible complex torus with respect to the $A$ action, we shall call it a \textit{primitive space}. If the $A$ action is also effective, we shall call it a \textit{reduced primitive space}. By changing $\omega_1$ on $A$ we can always obtain any reduced primary space, up to a finite covering, from a torus bundle product of primitive ones.

For any primitive pseudo-kählerian space, the rational module generated by the discrete subgroup $N\mathbb{Z}$ of $N$, as a rational representation of $\mathbb{Z} = \Gamma N/N$, can be split into two-dimensional representations, but $m$ in this case, can be any positive integer.

\textbf{Theorem C.} Any compact complex parallelizable homogeneous space with a pseudo-kähler structure as a torus bundle over $A$ is, up to a finite covering, the product of several primitive spaces. Moreover, any primitive space as a torus bundle is, up to a finite covering, the product of a torus and a reduced primitive space. For a primitive pseudo-kählerian manifold, $m$ can be any given positive integer.

We call a primitive pseudo-kähler space a \textit{simple primitive pseudo-kähler space} if $N/N\mathbb{Z}$ is a simple complex torus.

We call a primitive pseudo-kähler space a \textit{double primitive pseudo-kähler space} if $N/N\mathbb{Z}$ is isogenous to the product of two identical complex tori.

The example in [Ym1] is a double reduced primitive space.

\textbf{Theorem D.} Let $M$ be a reduced primitive pseudo-kähler space. If $N$ is not simple, then $M$ must be a double reduced primitive pseudo-kähler space defined above.

\textbf{Theorem E.} The generic reduced primitive pseudo-kähler spaces are simple primitive spaces.
Theorem F. Every compact complex homogeneous manifold with a pseudo-kählerian structure is the product of a projective rational homogeneous space and a solvable compact parallelizable pseudo-kähler space. Moreover, any compact parallelizable pseudo-kähler space $M$ is a torus bundle over a torus $T$ such that $M$, up to a finite covering, as a torus bundle is the product of a torus, several simple primitive pseudo-kähler spaces and some double primitive pseudo-kähler spaces.

References


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