Affine Compact Almost-Homogeneous Manifolds of Cohomogeneity One

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Abstract: This paper is one in a series generalizing our results in [GC, Gu4, 5, 8] on the existence of extremal metrics to the general almost-homogeneous manifolds of cohomogeneity one. In this paper, we deal with the affine cases with hypersurface ends. In particular, we study the existence of Kähler-Einstein metrics on these manifolds and obtain new Kähler-Einstein manifolds as well as Fano manifolds without Kähler-Einstein metrics. As a consequence of our study, we also give a solution of the problem posted by Ahiezer on the nonhomogeneity of compact almost-homogeneous manifolds of cohomogeneity one; this clarifies the classification of these manifolds as complex manifolds. We also deal with Fano properties of the affine compact manifolds.

1 Introduction

The theory of simply connected compact Kähler homogeneous manifolds has applications in many branches of mathematics and physics. These complex manifolds possess significant properties: they are projective, Fano, Kähler-Einstein, rational, etc..

One class of more general Kähler manifolds which would be useful is the class of almost compact Kähler manifolds with two orbits. Especially those manifolds of cohomogeneity one.

If we assume that they are simply connected, then they are automatically projective. One of many interesting questions of them is when they are Fano, that is, with a positive first Chern class, and therefore more interestingly when they are Kähler-Einstein. Other questions might be: What is the

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biholomorphic classification of them? What are the automorphism groups of them? When are they actually homogeneous?

This paper is one of a series of papers in which we answer above questions and we finished the project of the existence of Calabi extremal metrics in any Kähler class on any compact almost-homogeneous manifolds of cohomogeneity one.

There are three types of these kind of manifolds. We refer the readers to the next section for the details. The type III compact complex almost homogeneous manifolds of real cohomogeneity one were dealt in [Gu2] more than fifteen years ago. There is no much stability involved there.

We shall deal with the type I case in [Gu9] and the type II case in this paper and [Gu12]. This is the first class of manifolds for which the existence is completely understood and it is equivalent to the geodesic stability.

The purpose of this paper is to prove that there is a Kähler metric of constant scalar curvature on the affine almost-homogeneous manifold of cohomogeneity one if the generalized Futaki invariant is positive, i.e., (10) holds (Theorem 9). We shall prove the converse in [Gu6]. In [GC] and [Gu4,5,8] we dealt with some examples.

We should mention that our concept of generalized Futaki invariant might not be the same as the one in [DT] although it might appear to be similar for our case. A very interesting question is to find a degeneration such that Ding-Tian's idea might apply to our case here. It is related to the normal line bundle of the exceptional divisor, but it is not from the projective normal line bundle. The generalized Futaki invariant in our case comes from some kind of combination of the generalized Futaki invariants along the maximal geodesic rays in the moduli space of Kähler metric but does not necessarily come directly from any one of them as we have described and observed in [Gu5,8].

In [Gu8], we only dealt with one manifold which is the example (3) in [Ak p.68]. In this separate paper, we deal with the other two essential cases there, which might cause some difficulties, since the manifolds there are quite unfamiliar.

These two essential cases will be given in section 3. But for the convenience to the readers, I will give a short description here: Let M be a compact complex almost homogeneous manifold, G be the group action such that $M = O \cup D$ with O an open orbit and D a closed orbit. M is called affine if O is a G equivariant $\mathbf{C}^n$ bundle (not necessary a vector bundle) over a compact complex homogeneous manifold. In our cases, M is a G equivariant fiber bundle over a manifold C such that $C = G/P$ for a parabolic subgroup $P = SS_1R$ of G with S, S_1 semisimple and R the radical, S_1R acts on the fiber F trivially. Moreover, F itself is an affine compact complex almost homogeneous manifold of $\mathbf{C}P^n$ bundles with an open orbit S/U. We have:

For the first case in [Ak p.68] $S = B_n$ $n \geq 2$, the Lie algebra of U is generated by a Cartan subalgebra and the complex root vectors of $\pm(e_i - e_j)$; $e_i + e_j$ $1 \leq i < j \leq n$ (no positive roots e_i, which correspond to the $\mathbf{C}^n$). F is a $\mathbf{C}P^n$ bundle (see [Ak p.68, 73]). We denote F by $F(B_n)$.

For the second case in [Ak p.68] $S = C_n$ $n \geq 3$, the Lie algebra of U is generated by a Cartan subalgebra and the complex root vectors of $\pm(e_i \pm e_j); \pm 2e_i$ $2 \leq i < j \leq n$; and $2e_1$ (no positive roots $e_1 \pm e_j$, which correspond to the $\mathbf{C}^{2n}$). F is a $\mathbf{C}P^{2n}$ bundle (see [Ak p.68, 73]). We denote F by $F(C_n)$.

Being different from the third case in [Ak p.68] we dealt in [Gu8], in which the manifold can only be a blow up of a homogeneous space, in these two cases F are homogeneous (see [Ak p.69]).

In the same time, we also treated the manifolds which are fiber bundles with typical fibers of the first and fifth cases in [Ak p.73] as one situation in [Gu12]. Although the fiber of the last case is just $\mathbf{C}P^n \times (\mathbf{C}P^n)^*$, it is still in the case of affine type. Therefore, to finish the affine case we have to deal with that case also. We originally wrote a paper for all the type II cases. But it was too long for publication. Therefore, we finally separated it into this paper and [Gu12]. Conceptually, this paper is much more difficult and original than [Gu12], but technically [Gu12] is more difficult and includes more cases.

As in [Gu8], we take our original method in [Gu4,5]. From Lie group point of view our method can be regarded as a nilpotent path method, i. e., we consider a path, starting from the singular real orbit, generated by the action of a 1-parameter subgroup generated by a nilpotent element. One could also consider the path as a path generated by a semisimple element H_α, where α is the root which generates the $sl(2)$ Lie subalgebra $\mathcal{A}^{\mathbf{C}}$.

In this paper, we first give a preliminary on compact almost homogeneous manifolds of cohomogeneity one in the second section, and look back to what we did in [GC], [Gu4, 5] from a Lie group point of view in the third section. Then we apply the same argument in the third section of [Gu8] to the affine case. We found that the same method works for the complex structure of both the affine and the type II cases. We deal with two cases we mentioned above.

In the fourth section, we found that the same argument works for the Kähler structure. This is a section in which we deal with many different possibilities of the pairs of groups (S, G). This also shows that the affine classes are very big and are not extraordinary at all (see also the proof of the Lemma 6 and [Gu12] for a huge amount of this kind of manifolds). A new ingredient is the appearing of the 3-strings, i.e., 3 dimensionl irreducible representations of $\mathcal{A}^{\mathbf{C}}$. It is quite different from the situation in [Gu8]. Fortunately, the determinants of 3-strings are linear functions of the energy norm function $\mathcal{U}$.

The fifth section is one of our major input in this research. To calculate the Ricci curvature we apply a modified Koszul's trick which was motivated by [Ks p.567–570] as we did in [Gu8]. This is a difficult part and was missing in [Si]. It turns out that both our earlier works in holomorphic symplectic manifolds [Gu3] and homogeneous spaces [Gu7], [DG1,2] help us go through this research. The formula we used from [DG1 4.11] is due to Professor Dorfmeister.

We calculate the scalar curvature in the sixth section and setting up the equations in the seventh section. The pattern of these equations make it possible to reduce a fourth order ODE to a second order ODE.

We finally prove our Theorem 9 in the eighth section. One consequence of our argument is that the manifolds we considered in this paper are all Fano (Corollary 1). This is not true for the case in which $S = A_n$ ([Gu12]).

We then treat the Kähler Einstein case in the ninth section. The pattern of the examples seems quite bizarre in the ninth section if the asymptotic Mumford weakly stability is the same as geodesic stability or weakly K-stability. It is also clear to us that our geodesic stability is stronger than the weakly K-stability. The weakly K-stability should correspond to the nonpositivity of our integral. It is more like a semistability. So far we still can not find an example with a zero integral for the Ricci class. Otherwise, it should become a candidate which is weakly K-stable but not geodesically stable.

In the last section, we obtain a result on these manifolds. We solve a problem on the nonhomogeneous property of compact almost-homogeneous manifolds of cohomogeneity one and with a hypersurface end. This is important for our new Kähler-Einstein manifolds since we need to know that they are not homogeneous and therefore are new. This is also a question raised by Ahiezer. I later found that he also obtained a solution but with a different proof (in Russian only). In our proof we actually prove that if M is not homogeneous, then the group is actually the identity component of the automorphism group and the manifolds are different from each other. This gives a complete classification of compact almost homogeneous manifolds of cohomogeneity one and with a hypersurface end. They are either homogeneous or nonhomogeneous completions of $\mathbf{C}^*$ bundles, or nonhomogeneous almost-homogeneous manifolds of cohomogeneity one with semisimple group action and a hypersurface end. The first and the second classes in this classification are well understood for many years. Our result clarify the third class. Then we calculate concretly that for the homogeneous cases, our condition (10) holds. This of course should be true, but we just use it as examples.

In this paper, as in [Gu8] we also have three natural variables: t the nipotent time, θ the phase angle, τ the micro time. They help us understand the equation very much. The choice $\theta = \frac{t^2}{d+t^2}$ make the equation much simpler. We avoided another natural variable s the semisimple time which was in [Gu5], but it will eventually appear in [Gu6]. As in [Gu8], the energy norm function $\mathcal{U}$ and the Ricci mixed energy norm function $\mathcal{U}_\rho$ in the sections 4 and 6 are seemly God given, which are the reasons that we can solve this probem.

By taking the advantage of the solution for higher codimensional ends in [Gu10], we also checked the possibility of blowing down of our manifolds. In all our calculations we also need to take care carefully of the change of the invariant inner products when we restrict our calculation to a typical subgroup S in G.

2 Preliminaries

Here, we summarize some known results about compact complex almost homogeneous manifolds of cohomogeneity one. In this paper, we only consider manifolds with a Kähler structure. For earlier results on this subject, we refer the readers to [Ak] and [HS].

We call a compact complex manifold an almost-homogeneous manifold if its complex automorphism group has an open orbit. We say that a manifold is of cohomogeneity one if the maximal compact subgroup has a (real) hypersurface orbit. In [GC] and [Gu5], we reduced compact complex almost homogeneous manifolds of cohomogeneity one into three types of manifolds.

We denote the manifold by M and let G be a complex subgroup of its automorphism group which has an open orbit on M.

Let us assume first that M is simply connected. Let the open orbit be

G/U, K be the maximal connected compact subgroup of G, L be the generic isotropic subgroup of K, i.e., K/L be a generic K orbit. We have that (see [GC Theorem 1]):

Proposition 1. *If G is not semisimple, then M is a completion of a $\mathbf{C}^*$ bundle over a projective rational homogeneous space.*

For the structure of the projective rational homogeneous spaces, we refer the reader for the detailed discussion in [Gu7]. Here, we just recall some results which will be used in this paper.

A projective rational homogeneous space is a quotient of a complex semisimple Lie group G over a parabolic subgroup P. Let Δ be a root system of G. A subgroup P is a parabolic subgroup if its Lie algebra contains all the roots and the positive root vectors.

If a compact almost-homogeneous Kähler manifold is a completion of a $\mathbf{C}^*$ bundle over a product of a torus and a projective rational homogeneous space, we call it a *manifold of type III*. We have dealt with this kind of manifolds in our dissertation [Gu1,2]. There always exists an extremal metric in any Kähler class. Recently, we generalized this existence result to a family of metrics, which connects the extremal metric [Gu2] and the generalized quasi-einstein metric [Gu10], called the extremal-soliton metrics in [Gu11]. The existence of the extremal-soliton is the same as the geodesic stability with respect to a generalized Mabuchi functional.

In general, if M is a compact almost homogeneous Kähler manifold and O is the open orbit, then $D = M - O$ is a proper closed submanifold. Moreover, D has at most two connected components. We call each component of D an end. If D has two components (or one component), we say that M is an almost homogeneous manifold with two ends (or one end). We have (see [HS Theorem 3.2]):

Proposition 2. *If M is a compact almost-homogeneous Kähler manifolds with two ends, then M is a manifold of type III.*

Therefore, we only need to deal with the case with one end. Again, in the case of M being simply connected, we only need to take care of the case in which G is semisimple. If G is semisimple and M has two G orbits, one open and one closed, and moreover if the closed orbit is a complex hypersurface, there are two possibilities. Let $\mathcal{K}, \mathcal{L}$ be the Lie algebras of K, L. Then the centralizer of $\mathcal{L}$ in $\mathcal{K}$ is a direct sum of $\mathcal{L}$ and a Lie subalgebra $\mathcal{A}$ which is either one dimensional or the 3-dimensional Lie algebra $su(2)$. If $\mathcal{A}$ is one dimensional, we call M a *manifold of type I*. If $\mathcal{A}$ is $su(2)$, we call M a *manifold of type II*. We also denote the complexification of $\mathcal{A}$ by $\mathcal{A}^{\mathbf{C}}$.

In general, if the closed orbit has a higher codimension, we can always blow up the closed orbit to obtain a manifold $\tilde{M}$ with a hypersurface end. We call the manifold M a *manifold of type I (or II)* if $\tilde{M}$ is of type I (or II).

There is a special case of the type II manifolds. If the open orbit is a $\mathbf{C}^k$ bundle over a projective rational homogeneous manifold, we call M an *affine type manifold* (not to be confused with the closed complex submanifolds of $\mathbf{C}^m$). We note that in our case, the $\mathbf{C}^k$ bundle is not a complex holomorphic vector bundle.

Let $Aut_0(M)$ be the identity component of the complex automorphism of M, then any compact almost-homogeneous manifold is either homogeneous or almost-homogeneous of cohomogeneity one with the $Aut_0(M)$ action. The homogeneous ones are well understood. Therefore, we are only interested in those manifolds which are almost-homogenoues of cohomogeneity one with $Aut_0(M)$ action. We have (see [Gu5 section 12]):

Proposition 3. *Any compact almost-homogeneous Kähler manifold M of cohomogeneity one is an $Aut_0(M)$ equivariant fibration over a product of a rational projective homogeneous manifold Q and a complex torus T with a fiber F. Therefore, M can be regarded as a fiber bundle over T with a simply connected fiber M_1. One of following holds:*

(i) M is a manifold of type III.

(ii) M_1 is of type II but not affine.

(iii) M_1 is affine.

(iv) M_1 is of type I.

We say that M is a *manifold of type I (or type II, affine)* if M_1 is a manifold of type I (or type II, affine).

We actually can also obtain a structure of the M_1 bundle over T from [HS]. We only need to understand the bundle structure for the open orbit. By [HS Corollary 4.4], we have that the bundle structure is a product unless when we apply Proposition 3 to $\tilde{M}$ we have that $F = Q^k$ is a k-dimensional hyperquadric. In the latter case, there is an unbranched double covering $\bar{M}$ of M such that the bundle structure is a product. We have that:

Proposition 4. *The M_1 bundle over T is a product except in the case with which the open orbit is an F_0 bundle over $Q \times T$ such that F_0 is in either the second, or the sixth, or the eighth case in [Ak p.67]. In the latter cases the M_1 bundle has an unbranched double covering which is a product of M_1 and T.*

In [Gu8, 12] and this paper, we dealt with the simply connected affine and the type II cases with a hypersurface end. In [Gu9], we shall deal with the simply connected type I cases with a hypersurface end. Then, we shall deal with the simply connected cases with a higher codimensional end in [Gu9 section 11], and the general case in [Gu9 section 12].

3 The complex structures of the affine almost homogeneous manifolds

In this section we will deal with the complex structure J of the affine almost homogeneous manifolds. Let us recall some basic notations of the Lie algebras.

To make the things simpler we look at two special cases [Ak p.68] first. We let G and U be the corresponding complex Lie groups and $O = G/U$ be the open orbit. $U \subset G$ is always a subgroup containing a maximal torus and:

(1) $F(B_2)$ $G = B_2$, roots of U are $\pm(e_1 - e_2), e_1 + e_2$.

(2) $F(C_3)$ $G = C_3$, roots of U are $\pm(e_2 \pm e_3), \pm 2e_2, \pm 2e_3, 2e_1$.

In the case (1), the roots of the affine space are e_1 and e_2. The long root $\alpha_1 = e_1 - e_2$ and the short root $\alpha_2 = e_2$ consist a fundamental root system of this Lie algebra. B_2 has other positive roots $\alpha_1 + \alpha_2 = e_1$, $\alpha_1 + 2\alpha_2 = e_1 + e_2$. B_2 has a Cartan subalgebra

$$\mathcal{H} = \left\{ \begin{bmatrix} 0 & -a_1 i & 0 & 0 & 0 \\ a_1 i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 i & 0 \\ 0 & 0 & a_2 i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} |_{a_1, a_2 \in \mathbf{C}} \right\}.$$

The vector e_1 corresponds to $(a_1, a_2) = (1, 0)$ and e_2 corresponds to $(a_1, a_2) = (0, 1)$. The open orbit is generated by the combined action of B_2 on

$$\mathbf{A} = [0, 0, 0, 0, 1]^T$$

which represents a 4 dimensional complex subspace $\pi = \ker \mathbf{A}^T$ of $\mathbf{C}^5$ and

$$\mathbf{B} = \begin{bmatrix} 1 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \end{bmatrix}^T$$

which represents a 2 dimensional complex subspace $l \subset \pi$ generated by the column vectors of $\mathbf{B}$. We let

$$E_{\pm e_1} = \begin{bmatrix} 0_{2\times 2} & 0_{2\times 2} & B^T \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 1} \\ -B & 0_{1\times 2} & 0 \end{bmatrix}$$

with $B = \frac{1}{\sqrt{2}}[\pm 1, i]$,

$$E_{e_1 \pm e_2} = \begin{bmatrix} 0_{2\times 2} & A & 0_{2\times 1} \\ -A^T & 0_{2\times 2} & 0_{2\times 1} \\ 0_{1\times 2} & 0_{1\times 2} & 0 \end{bmatrix}$$

for

$$A = \frac{1}{2}\begin{bmatrix} 1 & \pm i \\ i & \mp 1 \end{bmatrix},$$

$$E_{-\alpha} = \bar{E}_\alpha^T.$$

$$F_\alpha = E_\alpha - E_{-\alpha}, G_\alpha = i(E_\alpha + E_{-\alpha}),$$

then

$$[F_\alpha, G_\alpha] = 2H_\alpha$$

and

$$[H_\alpha, F_\alpha] = i(H_\alpha, H_\alpha)_0 E_\alpha,$$

where $(\ ,\ )_0$ is the standard inner product such that $(e_i, e_i)_0 = 1$.

We also have that

$$[E_{\pm e_i}, E_{\pm(e_j - e_i)}] = \mp E_{\pm e_j},$$

$$[E_{e_i}, E_{\pm e_j}] = \mp E_{e_i \pm e_j},$$

$$[E_{-e_i}, E_{\pm e_j}] = \mp E_{-e_i \pm e_j}$$

and

$$[E_{\pm e_i}, E_{\pm(e_i + e_j)}] = \pm E_{\mp e_j}.$$

The tangent space is generated by E_α's with

$$\alpha = \pm(\alpha_1 + \alpha_2), -(\alpha_1 + 2\alpha_2), \pm(\alpha_2).$$

The affine space $\mathbf{C}^2$ is generated by the root vectors with

$$\alpha_1 + \alpha_2 = e_1,\ \ \alpha_2 = e_2.$$

As in the case of [Gu4], we consider the nilponent orbit generated by $E_{\alpha_1+\alpha_2}$.

Now,

$$p_t = \exp(tE_{\alpha_1+\alpha_2})\left([0,0,0,0,1]^T \times \begin{bmatrix} 1 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \end{bmatrix}^T\right)$$
$$= \left[\frac{t}{\sqrt{2}}, \frac{it}{\sqrt{2}}, 0, 0, 1\right]^T \times \begin{bmatrix} 1 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \end{bmatrix}^T,$$

$$p_\infty = [1, i, 0, 0, 0]^T \times \begin{bmatrix} 1 & i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \end{bmatrix}^T.$$

Let

$$F = E_{\alpha_1+\alpha_2} - E_{-\alpha_1-\alpha_2}, G = i(E_{\alpha_1+\alpha_2} + E_{-\alpha_1-\alpha_2}), H = H_{\alpha_1+\alpha_2},$$

then as before

$$JF = -G + \frac{2H}{t}.$$

Let T be the tangent vector of the curve p_t, then

$$JH = -tT.$$

Similarly, $JF_{\alpha_1} = -G_{\alpha_1}$, $JF_{\alpha_1+2\alpha_2} = -G_{\alpha_1+2\alpha_2}$, $JF_{\alpha_2} = -G_{\alpha_2} + \frac{2G_{\alpha_1}}{t}$. In particular, at P_∞ we have $JF_\alpha = -G_\alpha$.

Similarly, we consider $F(B_n)$, then the roots of U are

$$\pm(e_i - e_j), e_i + e_j$$

with $1 \leq i < j \leq n$. The open orbit is a combination of the B_n action on

$$[0, \cdots, 0, 1]^T_{1\times(2n+1)} \times \begin{bmatrix} 1 & i & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & i & \cdots & 0 & 0 & 0 \\ & & & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & i & 0 \end{bmatrix}^T.$$

For the complex Lie group B_n, we have $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = e_n$. Therefore, $e_i = \sum_{j=i}^n \alpha_j$, $e_i - e_k = \sum_{j=i}^{k-1} \alpha_j$, $e_i + e_k = \sum_{j=i}^{k-1} \alpha_j + 2\sum_{j=k}^n \alpha_j$. In particular $e_1 = \sum_1^n \alpha_j$. Therefore, similarly we have that:

The vector space $\mathbf{C}^n$ is generated by root vectors with e_i.

Proposition 5. *For* $F(B_n)$*,*

$$JF_{e_1} = -G_{e_1} + \frac{2H}{t},$$

$$JF_{e_i+e_k} = -G_{e_i+e_k},$$

$$JF_{e_i} = -G_{e_i} + \frac{2G_{e_1-e_i}}{t}$$

and

$$JF_{e_1-e_i} = -G_{e_1-e_i}.$$

We also have

$$F_{e_i-e_k} = G_{e_i-e_k} = 0$$

for $i > 1$*. In particular, at* p_∞*,* $JF_\alpha = -G_\alpha$ *for* $\alpha \neq e_i - e_k$ *with* $1 < i < k$*.*

In $F(C_3)$ of the case (2), the roots of the affine space are $e_1 \pm e_2$ and $e_1 \pm e_3$. The two short roots $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3$ and the long root $\alpha_3 = 2e_3$ consist a fundamental root system of the Lie algebra. C_3 has other positive roots $\alpha_1+\alpha_2 = e_1-e_3$, $\alpha_1+\alpha_2+\alpha_3 = e_1+e_3$, $\alpha_1+2\alpha_2+\alpha_3 = e_1+e_2$, $\alpha_2+\alpha_3 = e_2+e_3$, $2\alpha_2+\alpha_3 = 2e_2$ and $2\alpha_1+2\alpha_2+\alpha_3 = 2e_1$.

The complex Lie group C_3 has a Cartan subalgebra

$$\mathcal{H} = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_3 \end{bmatrix} |_{a_1,a_2,a_3 \in \mathbf{C}} \right\}.$$

The vector e_1 corresponds to $(a_1, a_2, a_3) = (1, 0, 0)$, e_2 to $(0, 1, 0)$, e_3 to $(0, 0, 1)$. The open orbit is generated by the combined C_3 action on $\mathbf{A} = [1, 0, 0, 0, 0, 0]^T$ which represents a complex 1 dimensional subspace l of $\mathbf{C}^6$ generated by $\mathbf{A}$ and

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$$

which represents the complex 2 dimensional column space π of $\mathbf{B}$. We have $l \subset \pi$.

We let

$$E_\alpha = \begin{bmatrix} A_\alpha & 0 \\ 0 & -A_\alpha^T \end{bmatrix}$$

with

$$A_{e_1-e_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$A_{-e_1+e_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We let

$$E_\beta = \begin{bmatrix} 0 & B_\beta \\ 0 & 0 \end{bmatrix}$$

with

$$B_{2e_1} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B_{e_1+e_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We also let $E_{-\beta} = E_\beta^T$.

We have that $[F_\alpha, G_\alpha] = 2H_\alpha$ and $[H_\alpha, E_\alpha] = i(H_\alpha, H_\alpha)_0 E_\alpha$, where $(\ ,\ )_0$ is the standard inner product such that $(e_1 - e_2, e_1 - e_2)_0 = 2$,

$$[E_{\pm 2e_i}, E_{\mp(e_i+e_j)}] = \pm\sqrt{2}E_{\pm(e_i-e_j)}, [E_{\pm(e_i-e_j)}, E_{\pm 2e_j}] = \pm\sqrt{2}E_{\pm(e_i+e_j)},$$

$$[E_{\pm(e_i-e_j)}, E_{\pm(e_i+e_j)}] = \pm\sqrt{2}E_{\pm 2e_i}, [E_{e_i-e_j}, E_{e_j-e_k}] = E_{e_i-e_k},$$

$$[E_{\pm(e_i-e_j)}, E_{\pm(e_j+e_k)}] = \pm E_{\pm(e_i+e_k)}.$$

The tangent space is generated by E_α's with

$$\alpha = \pm(e_1 \pm e_j), -2e_1.$$

The affine space $\mathbf{C}^4$ is generated by

$$e_1 \pm e_j.$$

As above, we consider the nilponent orbit generated by E_{α_1}.

Now,

$$p_t = \exp(tE_{\alpha_1})\left([1,0,0,0,0,0]^T \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T\right)$$

$$= \quad [1,0,0,0,0,0]^T \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t & 0 \end{bmatrix}^T,$$

$$p_\infty = [1,0,0,0,0,0]^T \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T.$$

Let

$$F = E_{\alpha_1} - E_{-\alpha_1}, G = i(E_{\alpha_1} + E_{-\alpha_1}),$$

then as above we have that

$$JF = -G + \frac{H}{t}.$$

Let T be the tangent vector of the curve p_t, then

$$JH = -2tT.$$

Similarly,

$$JF_{\alpha_2} = G_{\alpha_2}, F_{\alpha_3} = G_{\alpha_3} = 0,$$

$$JF_{\alpha_2+\alpha_3} = G_{\alpha_2+\alpha_3}, JF_{2\alpha_2+\alpha_3} = G_{2\alpha_2+\alpha_3},$$

$$JF_{2\alpha_1+2\alpha_2+\alpha_3} = -G_{2\alpha_1+2\alpha_2+\alpha_3},$$

$$JF_{\alpha_1+2\alpha_2+\alpha_3} = -G_{\alpha_1+2\alpha_2+\alpha_3} - \frac{\sqrt{2}G_{2\alpha_2+\alpha_3}}{t},$$

$$JF_{\alpha_1+\alpha_2+\alpha_3} = -G_{\alpha_1+\alpha_2+\alpha_3} - \frac{2G_{\alpha_2+\alpha_3}}{t}$$

and

$$JF_{\alpha_1+\alpha_2} = -G_{\alpha_1+\alpha_2} - \frac{2G_{\alpha_2}}{t}.$$

At p_∞, $F_{2e_3} = G_{2e_3} = 0$, $JF_{2e_2} = G_{2e_2}$, $JF_{e_2\pm e_3} = G_{e_2\pm e_3}$. For other roots α we have that $JF_\alpha = -G_\alpha$.

Similarly, we consider $F(C_n)$, then the roots of U are

$$\pm(e_i \pm e_j)$$

with $1 < i < j \leq n$ and

$$\pm 2e_i, 2e_1.$$

The open orbit is a combination of the C_n action on

$$[1, 0, \cdots, 0; 0, 0, \cdots, 0]^T \times \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T.$$

For C_n we have $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = 2e_n$, Therefore,

$$e_i - e_k = \sum_{j=i}^{k-1} \alpha_j,$$

$$e_i + e_k = \sum_{j=i}^{k-1} \alpha_j + 2\sum_{j=k}^{n-1} \alpha_j + \alpha_n,$$

$$2e_i = 2\sum_{j=i}^{n-1} \alpha_j + \alpha_n.$$

Therefore, similarly we have that:

The vector space is generated by the root vectors with $e_1 \pm e_j$.

Proposition 6. *For $F(C_n)$,*

$$JF_{\alpha_1} = -G_{\alpha_1} + \frac{H}{t},$$

$$JF_{2e_1} = -G_{2e_1},$$

$$JF_{2e_2} = G_{2e_2}.$$

We also have that

$$F_\alpha = G_\alpha = 0$$

for

$$\alpha = e_i - e_k, 2e_i, e_i + e_k$$

with $i > 2$.

And

$$JF_{e_1+e_2} = -G_{e_1+e_2} - \frac{\sqrt{2}G_{2e_2}}{t},$$

$$JF_{e_2+e_k} = G_{e_2+e_k},$$

$$JF_{e_1+e_k} = -G_{e_1+e_k} - \frac{2G_{e_2+e_k}}{t}$$

for $k > 2$.

Moreover,

$$JF_{e_2-e_k} = G_{e_2-e_k},$$

$$JF_{e_1-e_k} = -G_{e_1-e_k} - \frac{2G_{e_2-e_k}}{t}.$$

At p_∞ *we have that* $F_\alpha = G_\alpha = 0$ *if* $\alpha = 2e_i, e_i \pm e_k,\ i > 2$; $JF_{2e_2} = G_{2e_2}, JF_{e_2 \pm e_k} = G_{e_2 \pm e_k}$. *For other roots* α *we have that* $JF_\alpha = -G_\alpha$.

In general, as in [Ak] G is semisimple, U_G is the 1-subgroup. There is a parabolic subgroup $P = SS_1R$ with S, S_1 semisimple and R solvable such that $U_G = US_1R$ where U is a 1-subgroup of S. The manifold is a fibration over G/P with the completion of $P/U_G = S/U$ as the affine almost homogeneous fiber. In this case, the root system of S is a subsystem of the root system of G. In the Lie algebra of G, we also have F_α, G_α for those roots of G which are not in S. The tangent space of G/U_G along p_t is decomposed into irreducible $\mathcal{A}^{\mathbf{C}}$ representations, which we call *strings*. F_α, G_α are in the complement representation of $\mathcal{S}$. But $JF_\alpha = -G_\alpha \pmod{\mathcal{S}}$ as it is in the tangent space of G/P. Therefore, we have $JF_\alpha = -G_\alpha$ for any α which is not in the root system of S. This discussion is corresponding to the discussion in the last paragraph of the second section of [Gu8].

Proposition 7. *For affine almost homogeneous manifolds of cohomogeneity one with* S, U *in the cases (1), (2) of [Ak p.68] we have:*

$$JF_\alpha = -G_\alpha$$

if α *is not in the Lie algebra of* S*, and* JF_α *follows the same formula in Propositions 5 and 6 if* α *is in the Lie algebra of* S.

If S is B_2, the bigger complex Lie group G can be B_n, C_n, F_4. If S is B_3, G can be B_n, F_4. If S is C_3, G can be C_n, F_4. If S is B_n with $n > 3$, G can only be B_{m+n}. If S is C_n with $n > 3$, then G can be C_{m+n}.

4 The Kähler structures

In this section, we shall deal with the Kähler structures. The method is basically the same as that in the section 5 of [Gu8]. In [Gu8], we dealt with a 4-string for the case $S = G_2$, i.e., a 4 dimensional ireducible representation of the Lie subalgebra $\mathcal{A}^{\mathbf{C}}$. It happens that for the cases $S = B_n$ or C_n and the case $S = A_n$ later on in [Gu12] we have to deal with 3 dimensional irreducible representations of $\mathcal{A}^{\mathbf{C}}$. We call them 3-strings. It is a miracle that our method still works for the 3-strings.

In general, we call an irreducible $\mathcal{A}^{\mathbf{C}}$ representation V an *n-string* if $\dim_{\mathbf{C}} V = n$.

For $F(B_2)$, $G = S = B_2$, by regarding the open B_2 orbit as a homogeneous space, the vector fields which corresponding to the Lie algebra are the pushdown of the right invariant vector fields on the Lie group B_2. As we did in [Gu8], we study the corresponding left invariant vector fields on the Lie group. To make the things simpler, we still use our original notation for the left invariant vector fields. Since the Kähler form is (left)invariant under the action of the maximal compact Lie subalgebra $\mathcal{K}$ of the complex Lie algebra $\mathcal{B}_2$, the pullback of this Kähler form is left $\mathcal{K}$ invariant form on B_2. Therefore, $T(\omega(X,Y)) = -\omega(T,[X,Y])$ for any $X, Y \in \mathcal{K}$.

Now,

$$\begin{aligned} T(\omega(G,H)) &= -\omega(T,F) \\ &= -\omega(JT, JF) \\ &= -\omega(\frac{H}{t}, -G + \frac{2H}{t}) \\ &= -t^{-1}\omega(G,H), \end{aligned}$$

that is, $\omega(G,H) = Ct^{-1}$ for a constant C. Then $C = 0$, otherwise $\omega(G,H)$ is infinity at p_0. Therefore,

$$\omega(G,H) = \omega(T,F) = 0.$$

Similarly,

$$\begin{aligned} tT(\omega(H,F)) - T(\omega(F,G)) &= \omega(tT, -G + \frac{2H}{t}) \\ &= \omega(tJT, J^2F) \\ &= -\omega(H,F), \end{aligned}$$

i.e., $T(t\omega(H,F)) = T(\omega(F,G))$. Therefore,

$$\omega(F,G) = t\omega(H,F) + A$$

for some constant A.

Let $(\ ,\ )_B$ be an invariant metric on $\mathcal{K}$ such that $(H,H)_B = 1$. If there is no confusion we write $(\ ,\ ) = (\ ,\ )_B$. Then $H, \frac{G}{\sqrt{2}}, \frac{F}{\sqrt{2}}$ is an unitary basis of the Lie algebra $\mathcal{A}$. Therefore

$$[X,Y] = ([X,Y],H)H + 2^{-1}([X,Y],F)F + 2^{-1}([X,Y],G)G \\ + \quad [X,Y]_l + [X,Y]_{(\mathcal{A}+\mathcal{L})^\perp}.$$

Therefore,

$$\omega(T,[X,Y]) = ([X,Y],H)\omega(T,H) + 2^{-1}([X,Y],G)\omega(T,G) \\ + \quad \omega(T,[X,Y]_{(\mathcal{A}+\mathcal{L})^\perp}).$$

But

$$\omega(T,[X,Y]_{(\mathcal{A}+\mathcal{L})^\perp}) = \omega(tH, J([X,Y]_{(\mathcal{A}+\mathcal{L})^\perp})) = 0,$$

since $JX \in (\mathcal{A}+\mathcal{L})^\perp$ if $X \in (\mathcal{A}+\mathcal{L})^\perp$. We also have

$$\omega(X,Y) = (aH + bF + cG + I, [X,Y])$$

with I in the center of l.

$$\omega(G,H) = (aH + bF + cG + I, [G,H]) = (bF,F) = 2b = 0,$$

i.e., $b = 0$. Therefore,

$$\begin{aligned} T(\omega(X,Y)) &= (a'H + c'G + I', [X,Y]) \\ &= -\omega(T,[X,Y]) \\ &= -([X,Y], \omega(T,H)H + 2^{-1}\omega(T,G)G), \end{aligned}$$

i.e., $I' = 0$ and $a' = -\omega(T,H)$, $c' = -2^{-1}\omega(T,G)$. The last two equalities are actually already known to us. We actually obtained

$$\begin{aligned} \omega(T, -G + \frac{2H}{t}) &= 2c' - \frac{2a'}{t} \\ &= \omega(JT, J^2F) \\ &= -\omega(\frac{H}{t}, F) \\ &= -t^{-1}(aH + cG, G) \\ &= -2ct^{-1}, \end{aligned}$$

that is, $tc' + c = a'$. Therefore, $a = tc + C$. That is,

$$\omega(F,G) = 2a = 2tc + 2C = t\omega(H,F) + 2C.$$

Therefore, we already have this equality with $A = 2C$. We also see that $c(0) = 0$ since $H(0) = 0$. The first equality means that I does not depend on t, i.e.,

$$I = Bi\alpha_2$$

for some constant B. Therefore, the Kähler form is

$$\begin{aligned} \omega(X,Y) &= ((tf(t)+C)H + f(t)G + Bi\alpha_2, [X,Y]) \\ &= (H(t), [X,Y]), \end{aligned}$$

where $f(t) = c$ and $H(t) = aH + cG + I$.

As an observation, we see that if

$$V_1 = \mathbf{span}(T, F_\alpha),$$

$$V_2 = \mathbf{span}(H, G_\alpha),$$

then

$$JV_1 = V_2$$

and

$$V_1^\perp = V_2$$

with respect to ω. Moreover,

$$[V_1, V_1], [V_2, V_2] \subset V_1,$$

$$[V_1, V_2] \subset V_2.$$

Proposition 8. *For $F(B_2)$, the Kähler metric is a direct sum of its restriction on the subspaces*

$$W = \mathbf{span}(T, H, F, G),$$

$$W_1 = \mathbf{span}(E_\alpha | \alpha = \alpha_1, \alpha_2, \alpha_1 + 2\alpha_2).$$

On W, let $c = f$ then the metric is

$$\begin{aligned} \begin{bmatrix} \omega(T,JT) & \omega(T,JF) \\ \omega(F,JT) & \omega(F,JF) \end{bmatrix} &= \begin{bmatrix} \omega(T,\frac{H}{t}) & \omega(JT,-F) \\ \omega(F,\frac{H}{t}) & \omega(F,-G+\frac{2H}{t}) \end{bmatrix} \\ = & \begin{bmatrix} -\frac{tf'+f}{t} & -\frac{2f}{t} \\ -\frac{2f}{t} & -\frac{2(2+t^2)f}{t} - 2C \end{bmatrix}. \end{aligned}$$

The determinant is equal to

$$\begin{aligned}
& t^{-1}\det\left[\begin{array}{cc} \omega(T,H) & \omega(T,-G) \\ \omega(F,H) & \omega(F,-G) \end{array}\right] \\
= & -t^{-1}\det\left[\begin{array}{cc} -a' & -2c' \\ -2c & 2a \end{array}\right] \\
= & 2t^{-1}(aa'+2cc') \\
= & \frac{\mathcal{U}'}{t},
\end{aligned}$$

where $\mathcal{U}=a^2+2c^2$.

We notice that $\mathcal{U}$ is the square norm $(H(t),H(t))$ up to a constant, i.e., the energy of $H(t)$ up to a constant.

We also see that $\mathcal{U}$ is increasing. We also see that $f(0)=0, -(tf)'>0$ when $t>0$, therefore, $-f>0$ when $t>0$ and $-tf$ is increasing. We also notice that $\frac{f(-t)}{-t}=\frac{f(t)}{t}$, that is, $f(t)$ is an odd function.

On W_1 we have that:

$$\begin{aligned}
& \left[\begin{array}{ccc} \omega(F_{\alpha_1},JF_{\alpha_1}) & \omega(F_{\alpha_1},JF_{\alpha_2}) & \omega(F_{\alpha_1},JF_{\alpha_1+2\alpha_2}) \\ \omega(F_{\alpha_2},JF_{\alpha_1}) & \omega(F_{\alpha_2},JF_{\alpha_2}) & \omega(F_{\alpha_2},JF_{\alpha_1+2\alpha_2}) \\ \omega(F_{\alpha_1+2\alpha_2},JF_{\alpha_1}) & \omega(F_{\alpha_1+2\alpha_2},JF_{\alpha_2}) & \omega(F_{\alpha_1+2\alpha_2},JF_{\alpha_1+2\alpha_2}) \end{array}\right] \\
= & -2\left[\begin{array}{ccc} a-B & -c & 0 \\ -c & B+\frac{2c}{t} & c \\ 0 & c & a+B \end{array}\right].
\end{aligned}$$

The determinant is equal to

$$\det(\omega(F_{\alpha_i},-G_{\alpha_j}))=-8B(U-B^2).$$

Since $F_{\alpha_1}(0)=0$, we have that $a(0)=C=B$ and $\mathcal{U}(0)=B^2$. By $\mathcal{U}$ increasing, we have that $\mathcal{U}-B^2>0$ and therefore $-8B>0$, i. e., $-B>0$.

For $F(B_n), G=S=B_n$, we can do the same. And almost everything are the same except we have $I_n=Bi\sum_2^n e_j$ instead of $I_2=Bi\alpha_2=Bie_2$. In that case, we have $n-1$ of e_1 3-strings e_1-e_k, e_k, e_1+e_k instead of $\alpha_1,\alpha_2,\alpha_1+2\alpha_2$. That is, we have triples of positive roots such that the corresponding root vectors generates 3 dimensional irreducible representations of the $sl(2)$ Lie subalgebra $\mathcal{A}^{\mathbf{C}}$ which is generated by e_1.

In general, we say that n positive roots an *n-string* of a root α if they generate n dimensional irreducible representations of the $sl(2)$ Lie subalgebra generated by α.

The restricted metrics for these 3-strings are exactly the same as that of $\alpha_1, \alpha_2, \alpha_1 + 2\alpha_2$ for the B_2 case. However, there are also $\frac{(n-1)(n-2)}{2}$ e_1 1-strings $e_i + e_k$ for $1 < i < k \leq n$. We have

$$\omega(F_{e_i+e_k}, JF_{e_i+e_k}) = -\omega(F_{e_i+e_k}, G_{e_i+e_k}) = -2B.$$

Therefore, the volume is

$$V = \frac{\mathcal{U}'}{t}(-8B(\mathcal{U} - B^2))^{n-1}(-2B)^{\frac{(n-1)(n-2)}{2}}.$$

When $G \neq S$, we have

$$\omega(X, Y) = (aH + cG + I_S + I_P.[X, Y])$$

with $I_S \in \mathcal{L} \cap \mathcal{S}$ and I_S is in the center of $\mathcal{L}$, I_P is in the center of $\mathcal{L}$ but is perpendicular to $\mathcal{S}$. We denote $I_S + I_P$ by I_G, and if there is no confusion we write $I = I_S$.

In the case $F = F(B_n)$, i.e., $S = B_n$, $G = B_{m+n}$ and the $\mathbf{C}^n$ is generated by $e_{m+1}, \cdots, e_{m+n}$, we have other e_{m+1} 3-strings $e_l - e_{m+1}, e_l, e_l + e_{m+1}$ for $l \leq m$ and other e_{m+1} 1-strings.

$$\omega(X, Y) = (aH + cG + I + i\sum_{i=1}^{m} B_i e_i, [X, Y]).$$

The determinants of the 1-strings are constants. The restricted metrics to the subspaces generated by 3-strings $e_l - e_{m+1}, e_l, e_l + e_{m+1}$ are

$$-2\begin{bmatrix} B_l - a & -c & 0 \\ -c & B_l & c \\ 0 & c & B_l + a \end{bmatrix}.$$

Therefore, the determinant for the 3-strings is

$$-8B_l(B_l^2 - \mathcal{U}).$$

We have $B_l < B, \mathcal{U} < B_l^2$.

The volume is

$$V = M\frac{\mathcal{U}'}{t}(\mathcal{U} - B^2)^{n-1}\prod_{i=1}^{m}(B_i^2 - \mathcal{U})$$

with a constant $M > 0$.

Now, let us consider the case of $G = F_4$. According to [Hu p.64], F_4 has a root system with roots $\pm e_i$ for any $0 \leq i \leq 4$ and

$$\pm(e_i \pm e_j)\ i \neq j, \pm\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$$

with a basis

$$(e_2 - e_3, e_3 - e_4, e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)).$$

A B_2 type complex Lie subgroup is generated by $e_3 - e_4, e_4$. A B_3 type complex Lie subgroup is generated by $e_2 - e_3, e_3 - e_4, e_4$. A C_3 type complex Lie subgroup is generated by $e_3 - e_4, e_4, \alpha_4$.

If $F = F(B_2)$, there are two other e_3 3-strings $e_1 - e_3, e_1, e_1 + e_3$ and $e_2 - e_3, e_2, e_2 + e_3$. There are also four other e_3 2-strings $\frac{1}{2}(e_1 \pm e_2 - e_3 \pm e_4), \frac{1}{2}(e_1 \pm e_2 + e_3 \pm e_4)$. There are also some more other e_3 1-strings, but their determinants are constants.

$$\omega(X, Y) = (aH + cG + Bie_4 + iB_1e_1 + iB_2e_2, [X, Y]).$$

All the e_3 3-strings and e_3 are in B_3 and the restriction of the ω is

$$(aH + cG + iB_1e_1 + iB_2e_2, [X, Y]).$$

As above we have that the determinants for these 3-strings are

$$-8B_i(B_i^2 - \mathcal{U})$$

with $i = 1, 2$. We also have that $B_i < B$ and $\mathcal{U} < B_i^2$. Any e_3 2-string and e_3 generates an A_2 Lie algebra and the restriction of ω is

$$(aH + cG + \frac{i}{3}(B_1 \pm B \pm B_2)(e_1 \pm e_2 \pm e_4), [X, Y]).$$

We have that the determinants for these 2-strings are

$$(B_1 \pm B_2 \pm B)^2 - \mathcal{U}$$

and $B_1 < B_2 + 2B$,$\mathcal{U} < (B_1 - B_2 - B)^2$. The volume is

$$V = M\frac{\mathcal{U}'}{t}(\mathcal{U} - B^2)\prod_{i=1}^{2}(B_i^2 - \mathcal{U})\prod((B_1 \pm B_2 \pm B)^2 - \mathcal{U}).$$

If $F = F(B_3)$, there is another e_2 3-strings $e_1 - e_2, e_1, e_1 + e_2$ and four other e_2 2-strings $\frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4), \frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4)$. There are also some e_2 1-strings, but their determinants are constants.

$$\omega(X, Y) = (aH + cG + iB(e_3 + e_4) + iB_1 e_1, [X, Y]).$$

This e_2 3-string and e_2 is in a B_2 type complex Lie subgroup with a restricted ω of

$$(aH + cG + iB_1 e_1, [X, Y]).$$

The determinant is

$$-8B_1(B_1^2 - \mathcal{U})$$

and $B_1 < B, \mathcal{U} < B_1^2$. Any e_2 2-string and e_2 generate an A_2 Lie algebra. The restricted ω is

$$(aH + cG + \frac{i}{3}(B_1 + B(\pm 1 \pm 1))(e_1 \pm e_3 \pm e_4), [X, Y]).$$

The determinants are

$$(B_1 + B(\pm 1 \pm 1))^2 - \mathcal{U}$$

and $B_1 < 3B, \mathcal{U} < (B_1 - 2B)^2$. The volume is

$$V = M\frac{\mathcal{U}'}{t}(\mathcal{U} - B^2)^2(B_1^2 - \mathcal{U})\prod((B_1 + B(\pm 1 \pm 1))^2 - \mathcal{U}).$$

Similarly, for $F(C_3)$, i.e., $G = S = C_3$ we have

$$\begin{aligned}\omega(X, Y) &= (aH + cG + I, [X, Y]) \\ &= (H(t), [X, Y])\end{aligned}$$

with $a = tc + C = tf(t) + B$, $I = Bi(e_1 + e_2) = Bi(\alpha_1 + 2\alpha_2 + \alpha_3)$ where $(\ ,\ ) = (\ ,\ )_C$ (we omit C if there is no confusion) is the invariant form with

$$(H, H)_C = (F, F)_C = (G, G)_C = 1.$$

α_1 has only one 3-strings $2e_2, e_1 + e_2, 2e_1$. The determinant of the metric on this 3-strings is $-8B(\mathcal{U} - B^2)$ where $\mathcal{U} = a^2 + c^2$ is the norm $(H(t), H(t))$ up to a constant. The determinant of the metric on the space generated by T, H, F, G is $\frac{\mathcal{U}'}{2t}$. α_1 has two 2-strings $e_2 - e_3, e_1 - e_3$ and $e_2 + e_3, e_1 + e_3$.

The matrices of the restriction of the metric on them are identical and the determinants are $\mathcal{U}-B^2$. Therefore, the volume is

$$V=\frac{\mathcal{U}'}{2t}(-8B(\mathcal{U}-B^2)^3).$$

For the case of $F(C_n)$, $G=S=C_n$, we also have

$$\omega(X,Y)=(aH+cG+I,[X,Y])$$

with $I=Bi(e_1+e_2)$. α_1 has only one 3-string $2e_2, e_1+e_2, 2e_1$ and $2(n-2)$ 2-strings (e_2-e_k, e_1-e_k), (e_2+e_k, e_1+e_k) with $k>2$. Therefore, the volume is

$$V=\frac{\mathcal{U}'}{2t}(-8B(\mathcal{U}-B^2)^{2n-3}).$$

Here, we compare the case of $S=B_2$ and $S=C_2$. For $S=B_2$, $[H_{e_1},F_{e_1}]=G_{e_1}$ and for $S=C_2$, $[H_{e_1-e_2},F_{e_1-e_2}]=2G_{e_1-e_2}$. We can assume that $H_{e_1}=k_1H_{e_1-e_2}$ and $F_{e_1}=k_2F_{e_1-e_2}$ with positive k_1, k_2, then $k_1=\frac{1}{2}$ and $k_2=2^{-\frac{1}{2}}$. Let $(\ ,\ )_B=k_3(\ ,\ )_C$ then $(H_{e_1},H_{e_1})_B=k_3(2^{-1}H_{e_1-e_2},2^{-1}H_{e_1-e_2})_C$. Therefore, we have $k_3=4$. Let B^B and B^C be the corresponding B for the cases of $S=B_2$ and $S=C_2$, then

$$([X,Y],iB^Be_2)_B=([X,Y],iB^C(e_1+e_2))_C,$$

i.e.,

$$4([X,Y],2^{-1}iB^B(e_1+e_2))_C=([X,Y],iB^C(e_1+e_2))_C.$$

Thus, $2B^B=B^C$.

We also have that $E_{e_1}=2^{-\frac{1}{2}}E_{e_1-e_2}$ and therefore $t^B=2^{\frac{1}{2}}t^C$. From

$$([X,Y],c^B(\sqrt{2}t^C)G_{e_1})_B=([X,Y],c^C(t^C)G_{e_1-e_2})_C,$$

we have that

$$4([X,Y],2^{-\frac{1}{2}}c^B(\sqrt{2}t)G_{e_1-e_2})_C=([X,Y],c^C(t)G_{e_1-e_2})_C,$$

i.e., $2^{\frac{3}{2}}c^B(\sqrt{2}t)=c^C(t)$. We can also check that $2a^B(\sqrt{2}t)=a^C(t)$. Therefore, $4\mathcal{U}^B=4(a^B)^2+8(c^B)^2=(a^C)^2+(c^C)^2=\mathcal{U}^C$.

For the case of $G=F_4$ and $S=C_3$, C_3 is generated by α_4, e_4, e_3-e_4. The α_1 in the basis of C_3 is our α_4 here. However, $H=2\alpha_4$ and $I=$

$2iB(\alpha_4 + 2e_4 + e_3 - e_4) = iB(e_1 - e_2 + e_3 + e_4)$. Therefore, by $(H.H) = 1$ we have $(e_1, e_1) = \frac{1}{4}$.

$$\omega(X,Y) = (aH + cG + I + iB_1(e_1 + e_2), [X,Y]).$$

There are two other α_4 3-strings $e_2 + e_3, \frac{1}{2}(e_1 + e_2 + e_3 - e_4), e_1 - e_4$ and $e_2 + e_4, \frac{1}{2}(e_1 + e_2 - e_3 + e_4), e_1 - e_3$. There are two other α_4 2-strings $e_2, \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$ and $e_1, \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$. Any α_4 3-string and α_4 generate a B_2 type complex Lie subalgebra. The restricted ω is

$$(aH + cG + \frac{iB_1}{2}(e_1 + e_2 \pm (e_3 - e_4)), [X,Y]).$$

By regarding $\frac{B_1}{2}$ as the B_1 in the usual case in which $G = B_2$, we have that the determinants are

$$-8(\frac{B_1}{2})((\frac{B_1}{2})^2 - \mathcal{U})$$

and $B_1 < 2B$, $4\mathcal{U} < B_1^2$. Any α_4 2-string and α_4 generate an A_2 type of complex Lie subalgebra. For the first 2-string, the restricted ω is

$$(aH + cG + \frac{i}{3}(B_1 - B)(e_1 + 3e_2 - e_3 - e_4), [X,Y]).$$

The determinant is

$$(B_1 - B)^2 - \mathcal{U}$$

and $B_1 < 2B, \mathcal{U} < (B_1 - B)^2$. For the second 2-string, the restricted ω is

$$(aH + cG + \frac{i}{3}(B_1 + B)(3e_1 + e_2 + e_3 + e_4), [X,Y]).$$

The determinant is

$$(B_1 + B)^2 - \mathcal{U}.$$

Therefore, the volume is

$$V = M\frac{\mathcal{U}'}{t}(\mathcal{U} - B^2)^3((\frac{B_1}{2})^2 - \mathcal{U})^2 \prod((B_1 \pm B)^2 - \mathcal{U}).$$

For the case $F = F(C_n)$, i.e., $S = C_n$ and $G = C_{m+n}$, we have no other $\alpha_{m+1} = e_{m+1} - e_{m+2}$ 3-string but $2m$ other α_{m+1} 2-strings $(e_i + e_{m+2}, e_i + e_{m+1})$ and $(e_i - e_{m+2}, e_i - e_{m+1})$ with $i \leq m$.

$$\omega(X,Y) = (aH + cG + I + i\sum_{i=1}^{m} B_i e_i, [X,Y]).$$

The other α_{m+1} 2-strings have determinants $(B_i \pm B)^2 - \mathcal{U}$. Therefore, the volume is

$$V = M\frac{\mathcal{U}'}{t}(\mathcal{U} - B^2)^{2n-3}\prod_{i=1}^{m}((B_i \pm B)^2 - \mathcal{U})^2$$

and $-B_i > 0, \mathcal{U} < (B_i \pm B)^2$.

For the case $S = B_2$ and $G = C_n$, the B_2 is generated by the simple roots $e_{n-1} - e_n$ and $2e_n$. $\alpha = e_{n-1} + e_n$ is the root generated the Lie subalgebra $\mathcal{A}$. In this case, $H = \frac{1}{2}H_{e_{n-1}-e_n}$ and

$$\begin{aligned}\omega(X,Y) = (aH + cG + i\frac{B}{2}(e_{n-1} - e_n) + i\sum_{k=1}^{n-2} B_k e_k, [X,Y]) \\ = \quad (H(t), [X,Y]).\end{aligned}$$

α has $2(n-2)$ other 2-strings $e_k - e_{n-1}, e_k + e_n$ and $e_k + e_{n-1}, e_k - e_n$. As above, their determinants are $(2B_k \pm B)^2 - \mathcal{U}$ with U being the norm $(H(t), H(t))$ up to a constant and $U(0) = B^2$. $B_k < 0, \mathcal{U} < (2B_k \pm B)^2$. Therefore,

$$V = M\frac{\mathcal{U}'}{t}(\mathcal{U} - B^2)\prod_{k=1}^{n-2}((2B_k \pm B)^2 - \mathcal{U})^2.$$

We notice that all the I and therefore the coefficients B, B_i depend on the inner product (.) we choose. And, we can write the volume formula as

$$M\mathcal{U}'t^{-1}(\mathcal{U} - B^2)^{k-1}\prod(a_i^2 - \mathcal{U}).$$

For each string, by changing the sign of the eigenvalues we can exchange the eigenvectors. This induces a *mirror symmetry* of the eigenvectors. Formally, we can let $c = 0$ (and assume $a \neq 0$), then we have for each eigenvector β_i $(aH + I, \beta_i) = k_{\beta_i}(a_i \pm a)$. Therefore, we can choose $a_i = -\left|\frac{(I,\beta_i)}{(H,\beta_i)}\right|$ if $(H, \beta_i) \neq 0$. And if β_{i_1} are β_{i_2} are mirror symmetry to each other, then we have the same a_i. We say that a *mirror symmetry class* is the set $[i]$ of two different roots which are mirror symmetry to each other and denote $a_{[i]} = a_i$ for $i \in [i]$. We also let $\mathcal{I}$ be the all mirror symmetry classes.

We also have in [Gu8] that in the case of the third example in [Ak p.68] the volume is

$$V = \frac{M\mathcal{U}'}{t}(\mathcal{U} - B^2)(9B^2 - \mathcal{U}),$$

and a similar result for the case of $S = A_n$ in [Gu12].

Now, we summarize what we have in this section: Let H be the vector field as in Propositions 5 and 6. We take $(H, \frac{G}{\sqrt{2}}, \frac{F}{\sqrt{2}})$ to be the orthonomal basis of $\mathcal{A}$ if $S = B_n$ and (H, G, F) to be the orthonormal basis of $\mathcal{A}$ if $S = C_n$, $I_G = I + I_P \in \mathcal{L}$ be the constant center elements of $\mathcal{L}$ in the representation of the Kähler metrics. Let $H(t) = (tf(t) + B)H + f(t)$ and $\mathcal{U} = (H(t), H(t))$, then

Theorem 1. *For the cases in which $F = F(B_n)$ or $F(C_n)$, if we represent the Kähler metrics as*

$$\omega(X, Y) = ((tf(t) + B)H + f(t)G + I_G, [X, Y])$$

then the volume is

$$V = \frac{M\mathcal{U}'}{t}(\mathcal{U} - B^2)^{k-1} \prod_{[i]\in\mathcal{I}} (a_i^2 - \mathcal{U}) \tag{1}$$

for some positive numbers M and a_i^2 with

$$a_i = -\left|\frac{(I_G, \beta_i)}{(H, \beta_i)}\right|,$$

where k is the dimension of the affine space. $k = n$ if $F = F(B_n)$ and $k = 2n - 2$ if $F = F(C_n)$. Moreover, $\mathcal{U}(0) = B^2$ and $B^2 \leq \mathcal{U} < a_i^2$. In particular, if $G = S$, we have that $V = Mt^{-1}\mathcal{U}'(\mathcal{U} - B^2)^{k-1}$.

5 Calculating the Ricci curvature

Now, we calculate the Ricci curvature. Let α_1 be the root which generates $\mathcal{A}^{\mathbf{C}}$ and $h = \log V$. Following Koszul [Ks p.567], we have that

$$\rho(X, JY) = \frac{L_{J[X_r, JY_r]}(\omega^n)(T, JT, F, JF, F_\alpha, JF_\alpha)}{2\omega^n(T, JT, F, JF, F_\alpha, JF_\alpha)},$$

where X_r, Y_r are the corresponding right invariant vector fields and here we use F_α, JF_α to represent

$$F_{\alpha_2}, JF_{\alpha_2}, \cdots, F_{\alpha_l}, JF_{\alpha_l}$$

the array of F_α with its conjugate for positive roots α other than α_1 which have nonzero F_α and G_α.

To calculate the Ricci curvature for the case $F(B_2)$, we only need to consider X, Y for

$$F_{\alpha_2}, F_{\alpha_1+2\alpha_2}.$$

We have that

$$\begin{aligned}[F_{\alpha_1+2\alpha_2}, JF_{\alpha_1+2\alpha_2}] &= [F_{\alpha_1+2\alpha_2}, -G_{\alpha_1+2\alpha_2}] \\ &= -2H_{\alpha_1+2\alpha_2} \\ &= -2H - 2H_{\alpha_2},\end{aligned}$$

$$J[F_{\alpha_1+2\alpha_2,r}, JF_{\alpha_1+2\alpha_2,r}] = 2JH_{\alpha_1+2\alpha_2,r} = 2J(H + H_{\alpha_2}) = -2tT.$$

$$[F_{\alpha_2}, JF_{\alpha_1+2\alpha_2}] = [F_{\alpha_2}, -G_{\alpha_1+2\alpha_2}] = -G.$$

$$\begin{aligned}J[F_{\alpha_2,r}, JF_{\alpha_1+2\alpha_2,r}] &= JG \\ &= J\left(G - \frac{2H}{t} + \frac{2H}{t}\right) \\ &= F - 2T.\end{aligned}$$

Again as what happened in [Ks p.567–570], usually it is not clear how to find JX for a right invariant vector field X along p_t and to deal with the left invariant form with right invariant vector fields. Therefore, the argument in [Si] does not work as we can see for our situation. We need something similar to the Koszul's trick in [Ks p.567–570]. It turns out that all the arguments there still go through for our situation. Therefore, as above we let $h = \log V$ and have that:

$$\begin{aligned}\rho(F_{\alpha_1+2\alpha_2}, JF_{\alpha_1+2\alpha_2}) &= -th' + \frac{1}{2\omega^5(T, JT, F, JF, F_\alpha, JF_\alpha)}\cdot \\ [\quad & \omega^5([2tT, T] - J[-2H - 2H_{\alpha_2}, T], JT, F, JF, F_\alpha, JF_\alpha) \\ + \quad & \omega^5(T, [2tT, JT] - J[-2H - 2H_{\alpha_2}, JT], F, JF, F_\alpha, JF_\alpha) \\ + \quad & \omega^5(T, JT, [2tT, F] - J[-2H - 2H_{\alpha_2}, F], JF, F_\alpha, JF_\alpha) \\ + \quad & \omega^5(T, JT, F, [2tT, JF] - J[-2H - 2H_{\alpha_2}, JF], F_\alpha, JF_\alpha) \\ + \quad & \omega^5(T, JT, F, JF, [2tT, F_\alpha] - J[-2H - 2H_{\alpha_2}, F_\alpha], JF_\alpha) \\ + \quad & \omega^5(T, JT, F, JF, F_\alpha, [2tT, JF_\alpha] - J[-2H - 2H_{\alpha_2}, JF_\alpha])] \\ = \quad & -th' + 6,\end{aligned}$$

here we use the notation

$$\omega^n(\cdots,[A,F_\alpha]-J[B,F_\alpha],JF_\alpha),$$

to represent

$$\begin{aligned}&\omega^n(\cdots,[A,F_{\alpha_2}]-J[B,F_{\alpha_2}],JF_{\alpha_2},\cdots,F_{\alpha_l},JF_{\alpha_l})+\cdots\\ + \quad &\omega^n(\cdots,F_{\alpha_2},JF_{\alpha_2},\cdots,[A,F_{\alpha_l}]-J[B,F_{\alpha_l}],JF_{\alpha_l})\end{aligned}$$

which is the sum of

$$\omega^n(\cdots,F_{\alpha_2},JF_{\alpha_2},\cdots,[A,F_\alpha]-J[B,F_\alpha],JF_\alpha,\cdots,F_{\alpha_l},JF_{\alpha_l})$$

for all the positive roots α other than α_1, and we use the notation

$$\omega^n(\cdots,F_\alpha,[A,JF_\alpha]-J[B,JF_\alpha])$$

to represent a similar sum.

Another way to understand the calculation is regarding the volume tensor formally as a product of the two determinant tensors τ, τ_1 of the subspaces W, W_1 (see section 4 Proposition 8). We have the formula

$$\rho(X,JY)=\frac{1}{2}J[X_r,JY_r](h)+\frac{A_{X,Y}(\tau)}{2\tau}+\frac{A_{X,Y}(\tau_1)}{2\tau_1},$$

where

$$A_{X,Y}(\tau)=\sum_i\tau(\cdots,[J[X,JY],X_i]-J[[X,JY],X_i],\cdots).$$

Applying this formula, we have the components which come from the determinants τ and τ_1:

$$\frac{A_{F_{\alpha_1+2\alpha_2},F_{\alpha_1+2\alpha_2}}(\tau)}{2\tau}=0$$

and

$$\frac{A_{F_{\alpha_1+2\alpha_2},F_{\alpha_1+2\alpha_2}}(\tau_1)}{2\tau_1}=6.$$

Similarly, we can get that

$$\rho(F_{\alpha_2},JF_{\alpha_1+2\alpha_2})=-h'+\frac{2}{t}.$$

The components from the determinants τ and τ_1 are $0, \frac{2}{t}$.

Since the Ricci curvature is determined by the Ricci form and the Ricci form is a $(1,1)$ form, as in section 4 (e.g., Theorem 1) we only need to determine the corresponding c_ρ $(= f_\rho(t))$ and B_ρ, etc., for the Ricci form. We have that $c_\rho = \frac{1}{2}\left(h' - \frac{2}{t}\right)$, $B_\rho = -1$.

For $F(B_n)$, $S = G = B_n$, we only need to calculate the extra determinant components for the e_1 strings.

For the pair $F_{e_1+e_2}, F_{e_1+e_2}$, we have that the determinant components for the other 3-strings are 4 and 2 for the 1-strings $e_2 + e_j$, 0 for other 1-strings. Therefore

$$\rho(F_{e_1+e_2}, JF_{e_1+e_2}) = -th' + 6 + 4(n-2) + 2(n-2) = -th' + 6(n-1).$$

For the pair $F_{e_2}, F_{e_1+e_2}$, we have that the determinant components for 3-strings are $\frac{2}{t}$ and 0 for the 1-strings. Therefore,

$$\rho(F_{e_2}, JF_{e_1+e_2}) = -h' + \frac{2(n-1)}{t}.$$

We have that $c_\rho = \frac{1}{2}(h' - \frac{2(n-1)}{t})$, $B_\rho = -(n-1)$.

For the case of $G = B_{m+n}$ and $S = B_n$, the determinant components for all the extra 3-strings and 1-strings are zeros. Therefore, we have that $c_\rho = \frac{1}{2}(h' - \frac{2(n-1)}{t})$ and $B_\rho = -(n-1)$.

However, in this case we also need to calculate the $B_{\rho,l}$. We can calculate $\rho(F_{e_l}, JF_{e_l})$. We have that

$$[F_{e_l}, JF_{e_l}] = [F_{e_l}, -G_{e_l}] = -2H_{e_l},$$

$$J[F_{e_l}, JF_{e_l}] = 0.$$

Therefore, one can see easily that $e_l, e_l \pm e_k$ induce a number 2 and $e_k - e_l$ induce a number -2. If $l_1 \le l \le l_2$ induce a factor $A_{l_2-l_1}$ in S_1, then

$$\rho(F_{e_l}, JF_{e_l}) = 2(l_2 - l_1 + 1) + 4(m + n - l_2).$$

Therefore, $B_{\rho,l} = l_2 + l_1 - 1 - 2(m+n)$. Actually, one can easily see that these $B_{\rho,l}$ come from those in the Ricci curvature of the G/P. There is an explicit formula of the Ricci curvature of G/P in [DG1 (4.11.7)] (see also [DG2 3]) and [Ks p.569 (4.6)](we notice that the factors of 2 are canceled out):

$$\rho_{G/P}(X, Y) = -q_{G/P}([X, Y]),$$

where $q_{G/P} = \sum_{\alpha \in \Delta^+ - \Delta_P} \alpha$ with Δ_P the root system for the semisimple part of P, and $q_{G/P}$ is corresponding to an element in the abelian part of the reductive part of P by an invariant metric. In [DG1], one has that $JF_\alpha = G_\alpha$ instead of $JF_\alpha = -G_\alpha$ here. In particular, $q_{G/P}(\mathcal{S}) = 0$ always.

In general, if $F = F(B_n)$, all other contributions of $[2T - F, \ ]$ and $J[G, \ ]$ are zeros and $c_\rho = \frac{1}{2}(h' - \frac{2(n-1)}{t})$.

Similarly, all other contributions of $[2tT, \ ]$ are zeros. The contributions of $J[H + H_\gamma, \ ]$, where γ corresponds to e_2 in B_n, are also zeros by the property of $q_{G/P}$ above. We therefore have $B_\rho = -(n-1)$. Other coefficients come from the Ricci curvature of G/P as above.

Now, we take care of the case $F = F(C_n)$, i.e., $S = C_n$. In the case $n = 2$ and $G = S$, we have that the metric is a product of its restrictions to W and W_1, where W is generated by T, JT, F, JF and W_1 is generated by the 3-string $2e_2, e_1 + e_2, 2e_1$. As above, to calculate the Ricci curvature, we only need to deal with X, Y for

$$F_{e_1+e_2}, F_{2e_1}.$$

We have that

$$[F_{2e_1}, JF_{2e_1}] = -2H - 2H_{e_1+e_2},$$

$$J[F_{2e_1,r}, JF_{2e_1,r}] = -4tT,$$

$$[F_{e_1+e_2}, JF_{2e_1}] = -\sqrt{2}G,$$

$$J[F_{e_1+e_2}, JF_{2e_1}] = \sqrt{2}(2T - F).$$

As above, for F_{2e_1}, F_{2e_1} the W contributions are zero, the contributions from W_1 are 12 and

$$\rho(F_{2e_1}, JF_{2e_1}) = -2(th' - 6).$$

For $F_{e_1+e_2}, F_{2e_1}$ the contributions of W are 0 and the contributions from W_1 are $\frac{2\sqrt{2}}{t}$.

$$\rho(F_{e_1+e_2}, JF_{2e_1}) = \sqrt{2}(-h' + \frac{2}{t}).$$

Therefore, $c_\rho = h' - \frac{2}{t}$ and $B_\rho = -2$. This is similar to the case of B_2 above as we see before that $2B^B_\rho = B^C_\rho = -2$. We also have that

$$c^C_\rho(t) = 2^{\frac{3}{2}} c^B_\rho(\sqrt{2}t)$$

$$
\begin{aligned}
&= \sqrt{2}((h^B)'(\sqrt{2}t) - \frac{2}{\sqrt{2}t}) \\
&= \sqrt{2}\frac{dh^B}{d(\sqrt{2}t)}(\sqrt{2}t) - \frac{2}{t} \\
&= \frac{dh^B}{dt}(\sqrt{2}t) - \frac{2}{t} \\
&= \frac{dh^C}{dt}(t) - \frac{2}{t} \\
&= h' - \frac{2}{t}.
\end{aligned}
$$

If $F = F(C_3)$ and $G = S = C_3$, we have two other 2-strings $e_2 - e_3, e_1 - e_3$ and $e_2 + e_3, e_1 + e_3$. For the pair F_{2e_1} and F_{2e_1}, the contributions of each 2-string are both 4. For the pair $F_{e_1+e_2}$ and F_{2e_1}, the contributions are both $\frac{2\sqrt{2}}{t}$. Therefore, $c_\rho = h' - \frac{6}{t}$ and $B_\rho = -2$.

Similarly, if $F = F(C_n)$ and $G = S = C_n$, then $c_\rho = h' - \frac{2(2n-3)}{t}$ and $B_\rho = -2$.

As above, the other contributions of $[4tT,\ ], J[2H + 2H_\gamma,\ ]$ with γ corresponding $e_1 + e_2$ in C_n, $[\sqrt{2}(2T - F),\ ], J[\sqrt{2}G,\ ]$ are also zeros.

Similarly, we dealt in [Gu12] on the A_n action.

We have that:

Theorem 2. *If $F = F(B_n)$ or $F(C_n)$, then $c_\rho = M_\rho(h' - \frac{2(k-1)}{t})$ with a positive number M_ρ, where k is the dimension of the affine space. Moreover, the pair (M_ρ, B_ρ) are $(\frac{1}{2}, -(n-1)), (1, -2)$ for the case $S = B_n, C_n$. Other coefficients, i.e., other part of $I_{\rho,G}$, come from the Ricci curvature of G/P which is $-(q_{G/P}, [X,Y])_0$ with $q_{G/P} = \sum_{\alpha\in\Delta^+-\Delta_P} H_\alpha$ with the standard inner product.*

6 Calculating the scalar curvature

To calculate the scalar curvature we separate our subspaces into five kind of spaces. The first W is generated by T, JT, F, JF. The second , third, fourth and fifth are the subspaces of 1, 2, 3 and 4-strings. The Ricci curvature is a sum of its restriction to each subspaces $\rho = \sum_i \rho_i$. Similarly $\omega = \sum_i \omega_i$. Then, by Theorem 1 we have that

$$V = \frac{M\mathcal{U}'Q(U)}{t} = \frac{M\mathcal{U}'}{t}(\mathcal{U} - B^2)^{k-1}Q_1(\mathcal{U}),$$

$\rho \wedge \omega^{n-1} = \sum_i \Omega_i$ where $\Omega_i = \rho_i \wedge \omega^{n-1}$.

Let $\mathcal{U}_\rho = (aH + cG, a_\rho H + c_\rho G)$, then $\mathcal{U}_\rho(0) = BB_\rho$.

$$\Omega_W = (n-1)! K \mathcal{U}'_\rho Q(\mathcal{U})/t$$

if the determinant of W is $K\mathcal{U}'/t$. For 1-strings,

$$\Omega_i = K_i \mathcal{U}' Q(\mathcal{U})/t.$$

For 2-strings,

$$\Omega_i = -2(n-1)!(\mathcal{U}_\rho - a_i a_{\rho,i})\frac{V}{q_i}$$

where $q_i = a_i^2 - \mathcal{U}$ is the linear factor of Q introduced from the given 2-string. Similarly, we can see, by a direct calculation, that for a 3-string

$$\Omega_i = -(2\mathcal{U}_\rho - 2a_i a_{\rho,i} + \frac{a_{\rho,i}}{a_i}(\mathcal{U} - a_i^2))\frac{(n-1)!V}{q_i}.$$

For the case of 4-strings, it only occurs when $G = G_2$ and H correspond to the short root. In this case, we have that

$$\begin{aligned}
\Omega_1 &= \rho_1 \wedge \omega^{n-1} \\
&= -4(\mathcal{U}_\rho(5B_1^2 - \mathcal{U}) + B_1 B_{\rho,1}(5\mathcal{U} - 9B_1^2))\frac{(n-1)!V}{(B_1^2 - \mathcal{U})(9B_1^2 - \mathcal{U})} \\
&= -2[\mathcal{U}_\rho[B_1^2 - \mathcal{U}) + (9B_1^2 - \mathcal{U})] \\
&- B_1 B_{\rho,1}[9(B_1^2 - \mathcal{U}) + (9B_1^2 - \mathcal{U})]]\frac{(n-1)!V}{(B_1^2 - \mathcal{U})(9B_1^2 - \mathcal{U})} \\
&= -2(\mathcal{U}_\rho - 9B_1 B_{\rho,1})\frac{(n-1)!V}{9B_1^2 - \mathcal{U}} - 2(\mathcal{U}_\rho - B_1 B_{\rho,1})\frac{(n-1)!V}{B_1^2 - \mathcal{U}}.
\end{aligned}$$

Therefore,

$$\rho \wedge \omega^{n-1} = (n-1)! M \frac{(\mathcal{U}_\rho Q(\mathcal{U}))' + p_0(\mathcal{U})\mathcal{U}'}{t}.$$

Theorem 3. *The scalar curvature is* $\frac{2(\mathcal{U}_\rho Q)' + p\mathcal{U}'}{\mathcal{U}'Q}$ *with a polynomial* p *of* $\mathcal{U}$*. Moreover, let* k *be the same as in Theorem 1 and 2,*

$$p(\mathcal{U}) = (\mathcal{U} - B^2)^{k-2}(-2(k-1)BB_\rho Q_1(\mathcal{U}) + (\mathcal{U} - B^2)P_1(\mathcal{U})),$$

where $P_1(\mathcal{U})$ *is a polynomial of* $\mathcal{U}$ *and is a positive linear sum of (1)* Q_1 *and (2) the products of* $\deg Q_1 - 1$ *linear factors of* Q_1*. Only 1-strings and*

3-strings have contributions to (1); the contribution of each 1-string and 3-string is $\frac{c_{\rho,l}}{c_l}$ *for the* Q_1 *term, where* $c_i = \omega(F_{\alpha_i}, JF_{\alpha_i})$ *for 1-strings and* $c_i = a_i$ *for 3-strings. Only 2-strings, 3-strings and 4-strings have contributions to (2); the contribution of each 2-string and 4-string related to the products of* $\deg Q_1 - 1$ *linear factors of* Q_1 *is* $2\frac{a_{\rho,i}a_iQ_1}{q_i}$*. In particular, if* $G = S$*, we have that*

$$p(\mathcal{U}) = -2(k-1)BB_\rho(\mathcal{U} - B^2)^{k-2}.$$

7 Setting up the equations

Now, we shall set up the equations for the metrics with constant scalar curvature. Before we do that, we shall understand more about the metrics. We have that:

Theorem 4. *With the* $B, f, \mathcal{U}, a_i$ *as in Theorem 1, we have that* ω *is a metric on the open orbit if and only if* $B < 0$*,* f *is an odd function with* $f'(0) < 0$*,* $\mathcal{U}' > 0$ *and* $\mathcal{U} < a_i^2$*.*

To understand the metrics near the hypersurface orbit, we can let $\theta = \frac{t^2}{d+t^2}$ with $d = 2$ for $S = B_n$ and $d = 1$ otherwise, and we see that $\theta' = \frac{2t}{d+t^2} - \frac{2t^3}{(d+t^2)^2} = \frac{2dt}{(d+t^2)^2}$. We can also see that $\mathcal{U}_\theta(1) = \lim_{t\to+\infty} \frac{(d+t^2)^2\mathcal{U}'}{2dt} > 0$ exists. In particular, $\mathcal{U}$ is bounded, so is tf. Let $l = \lim_{t\to+\infty} tf$. We first notice that the closure D of the orbit of the complex Lie group $SL(2, \mathbf{C})$ generated by α_1 is a fiber bundle with a $\mathbf{C}P^1$ as the base and another $\mathbf{C}P^1$ as the fiber. D is $\mathbf{C}P^1 \times \mathbf{C}P^1$ by an argument as in [SP]. A calculation of the section 2 of [Gu8] gives the desired property. The restriction of the metric on D also shows that B, l are topological invariants.

Theorem 5. ω in Theorem 1 extends to a Kähler metric over the exceptional divisor if and only if $\lim_{t\to+\infty} tf = l > a_i - B$ and $\mathcal{U}_\theta(1) > 0$.

Now, for any given pair B, l with $0 > l > a_i - B$ we can check that $f(t) = \frac{lt}{d+t^2}$ satisfies Theorems 4 and 5. We shall see later on that this actually gives us the solutions of our equations for the homogeneous cases, i. e., when $G = S$. So we have that:

Theorem 6. The Kähler classes are in one to one correspondence with the elements in the set $\Gamma = \{(B, l)|_{0>l>a_i-B}\}$.

To calculate the total volume, we notice that

$$T \wedge JT \wedge F \wedge JF \bigwedge_{\alpha=\alpha_2}^{\alpha_l} (F_\alpha \wedge JF_\alpha) = M \frac{T \wedge H \wedge F \wedge G \bigwedge_{\alpha=\alpha_2}^{\alpha_l} (F_\alpha \wedge G_\alpha)}{t}$$

with a possitive number M.

$$\mathcal{U}(0) = B^2,\ \mathcal{U}(+\infty) = (l+B)^2.$$

Therefore, the total volume is

$$V_T = \int_{B^2}^{(l+B)^2} Q(\mathcal{U}) d\mathcal{U}.$$

We also see that

$$f_\rho = h' - \frac{2(k-1)}{t} = \frac{\mathcal{U}''}{\mathcal{U}'} + \frac{Q'(\mathcal{U})\mathcal{U}'}{Q(\mathcal{U})} - \frac{2k-1}{t}.$$

One can easily check that

$$\left(\frac{\mathcal{U}''}{\mathcal{U}'} - \frac{1}{t}\right)(0) = 0,$$

$$\left(\frac{\mathcal{U}'}{\mathcal{U} - B^2} - \frac{2}{t}\right)(0) = \mathcal{U}'(0) = 0$$

by f being an odd function and therefore $f_\rho(0) = 0$.

Now, from

$$\begin{aligned} \mathcal{U} &= (tf+B)^2 + df^2 \\ &= (t^2+d)f^2 + 2Btf + B^2 \\ &= (t^2+d)\left(f + \frac{Bt}{t^2+d}\right)^2 + \frac{dB^2}{d+t^2} \end{aligned}$$

we have that

$$\left(f + \frac{Bt}{t^2+d}\right)^2 = \frac{1}{(d+t^2)^2}((d+t^2)U - dB^2).$$

We have that

$$-f - \frac{Bt}{d+t^2} = \frac{\sqrt{(d+t^2)\mathcal{U} - dB^2}}{d+t^2}.$$

That is,

$$f = -\frac{\sqrt{(d+t^2)U - dB^2} + Bt}{d+t^2}.$$

To make the things clearer, we replace t by $\theta = \frac{t^2}{d+t^2}$. We have that

$$\begin{aligned} tf_\rho &= M[[\log[\mathcal{U}_\theta Q(\mathcal{U})(1-\theta)^2]]_\theta 2\theta(1-\theta) - 2(k-1)] \\ &= M[2\theta(1-\theta)\left[\frac{\mathcal{U}_{\theta\theta}}{\mathcal{U}_\theta} + \frac{Q'(\mathcal{U})\mathcal{U}_\theta}{Q(\mathcal{U})}\right] - 4\theta - 2(k-1)], \end{aligned}$$

which has a limit $-2M(k+1)$ at $\theta = 1$.

Therefore, we obtain:

Proposition 9. *For the Ricci class, we have*

$$l_\rho = -2M(k+1),$$

where $M = \frac{1}{2}$ for $S = B_n$ and $M = 1$ for $S = C_n$. Therefore, the Ricci class is a class of a type $(B_\rho, -2M(k+1))$ using the notation of Theorem 6.

We also have that

$$\mathcal{U}_\rho(1) = (B_\rho + l_\rho)(B + l) = (B_\rho - 2M(k+1))(B+l).$$

Now, we have the Kähler Einstein equation

$$\begin{aligned} M[2\theta(1-\theta)\left[\frac{\mathcal{U}_{\theta\theta}}{\mathcal{U}_\theta} + \frac{Q'(\mathcal{U})\mathcal{U}_\theta}{Q(\mathcal{U})}\right] - 4\theta - 2(k-1)] &= tf \\ = \quad -\frac{t\sqrt{(d+t^2)\mathcal{U} - dB_\rho^2} + B_\rho t^2}{d+t^2} \\ = \quad -\sqrt{\theta[\mathcal{U} - B_\rho^2(1-\theta)]} - B_\rho\theta. \end{aligned}$$

Let

$$u = \mathcal{U} - B_\rho^2$$

we have that:

$$\begin{aligned} & M[\theta(1-\theta)\left(\frac{u''}{u'} + \frac{Q'(u)u'}{Q(u)}\right) - 2\theta - k + 1] \\ = \quad & -\frac{1}{2}(B_\rho\theta + \sqrt{\theta(u + B_\rho^2\theta)}) \qquad (2) \\ = \quad & -\frac{\theta^{\frac{1}{2}}u}{2(-B_\rho\theta^{\frac{1}{2}} + \sqrt{u + B_\rho^2\theta})}, \end{aligned}$$

where the derivatives are the derivatives with respect to θ.

The total scalar curvature is

$$R_T = \int_0^1 [p(\mathcal{U})\mathcal{U}' + 2(\mathcal{U}_\rho Q(\mathcal{U}))']d\theta.$$

And from this, we have the average scalar curvature

$$\begin{aligned} R_0 &= \frac{R_T}{V_T} \\ &= \frac{\int_{B^2}^{(B+l)^2} p(\mathcal{U})d\mathcal{U} + 2(\mathcal{U}_\rho Q(\mathcal{U}))|_{B^2}^{(B+l)^2}}{\int_{B^2}^{(B+l)^2} Q(\mathcal{U})d\mathcal{U}} \\ &= \frac{\int_{B^2}^{(B+l)^2} p(\mathcal{U})d\mathcal{U} + 2(B_\rho + l_\rho)(B+l)Q((B+l)^2)}{\int_{B^2}^{(B+l)^2} Q(\mathcal{U})d\mathcal{U}}. \end{aligned}$$

If $G = S$ and $S \neq G_2$ (we shall see later on that this is the same as the assumption that the manifold being homogeneous), then $Q = (\mathcal{U} - B^2)^{k-1}$ and $p = -2BB_\rho(k-1)(\mathcal{U} - B^2)^{k-2}$. Therefore,

$$R_0 = \frac{-2BB_\rho + 2(B_\rho + l_\rho)(B+l)}{k^{-1}((B+l)^2 - B^2)} = 2k\frac{B_\rho l + Bl_\rho + ll_\rho}{2Bl + l^2}.$$

The equation of constant scalar curvature is $\frac{R}{V} = R_0$. Therefore, we have that

$$\begin{aligned} & 2\mathcal{U}_\rho Q(\mathcal{U}) + \int_{B^2}^{\mathcal{U}} p(\mathcal{U})d\mathcal{U} \\ = \; & R_0 \int_{B^2}^{\mathcal{U}} Q(\mathcal{U})d\mathcal{U} + A_0 \end{aligned} \tag{3}$$

with A_0 a constant.

Let $\theta = 0$, we have that

$$2BB_\rho Q(B^2) = A_0.$$

Therefore,

$$A_0 = 0$$

since $k > 1$. If we put $\theta = 1$ in, we get the same A_0.

We have that

$$\mathcal{U}_\rho = \frac{R_0 \int_{B^2}^{\mathcal{U}} Qd\mathcal{U} - \int_{B^2}^{\mathcal{U}} pd\mathcal{U}}{2Q(\mathcal{U})}$$

where $Q(\mathcal{U}) = (\mathcal{U} - B^2)^{k-1} Q_1(\mathcal{U})$.

Applying Theorem 3 and integration by parts, we have that

$$\begin{aligned}
\mathcal{U}_\rho &= \frac{R_0 \int_{B^2}^{\mathcal{U}} Q d\mathcal{U} + 2(k-1)BB_\rho \int_{B^2}^{\mathcal{U}} (\mathcal{U} - B^2)^{k-2} Q_1 d\mathcal{U} - \int_{B^2}^{\mathcal{U}} (\mathcal{U} - B^2)^{k-1} P_1 d\mathcal{U}}{2Q} \\
&= \frac{\int_{B^2}^{\mathcal{U}} (R_0 Q - (\mathcal{U} - B^2)^{k-1} (P_1 + 2BB_\rho Q_1')) d\mathcal{U} + 2BB_\rho (\mathcal{U} - B^2)^{k-1} Q_1}{2Q} \\
&= \frac{R(\mathcal{U})}{2Q_1(\mathcal{U})},
\end{aligned}$$

where $R(\mathcal{U})$ is a polynomial of $\mathcal{U}$. Therefore,

$$f_\rho((t^2 + d)f + Bt) = -B_\rho t f + \frac{um(u)}{Q_1(u)}$$

where we let $R(u) = 2um(u) + 2BB_\rho Q_1(\mathcal{U})$.

If $G = S$ and $S \neq G_2$, we have that

$$\mathcal{U}_\rho = \frac{R_0}{2k}(\mathcal{U} - B^2) + BB_\rho.$$

And $R(\mathcal{U}) = \frac{R_0}{k}(\mathcal{U} - B^2) + 2BB_\rho$, $m(u) = \frac{R_0}{2k}$.

Now, by

$$tf = -B\theta - \sqrt{\theta(u + B^2\theta)}$$

we have that

$$(d + t^2)tf + Bt^2 = -\frac{d\sqrt{\theta(u + B^2\theta)}}{1 - \theta},$$

and therefore

$$\begin{aligned}
& M[\theta(1-\theta)\left[\frac{u''}{u'} + \frac{Q'(u)u'}{Q(u)}\right] - 2\theta - k + 1] \\
&= 2^{-1}\sqrt{\frac{\theta}{u + B^2\theta}}\, u \left[\frac{B_\rho \theta}{B\theta - \sqrt{\theta(u + B^2\theta)}} - \frac{m(u)}{Q_1(u)}\right]. \qquad (4)
\end{aligned}$$

Comparing with (2), we see that

$$m(u) = Q_1(u)$$

if the Kähler metric is in the Ricci class.

If $G = S$ and $S \neq G_2$, then we have that $\frac{m(u)}{Q_1}$ is a constant. There is a solution with $u = c\theta$. Actually, if we use $f = \frac{lt}{d+t^2}$ in the proof of the Theorem 6 we obtain that $u = (2B + l)l\theta$ which solves our equation.

From (4), we have that

$$[\log[u'Q(u)]]' = \frac{P}{\theta(1-\theta)}.$$

We also have that

$$2\theta + k - 1 - A_{B,l}\theta^{\frac{1}{2}} \leq P \leq 2\theta + k - 1 + C_{B,l}\theta^{\frac{1}{2}}.$$

for some positive constant $A_{B,l}, C_{B,l}$ which only depend on B and l. Since $P(1) = k + 1 + 2^{-1}l_\rho = 0$, we have that $A_{B,l} \geq k + 1$.

By integration, we have that

$$\begin{aligned} \frac{a^{k-1}(1-a^{\frac{1}{2}})^{A_{B,l}-k-1}(1+\theta^{\frac{1}{2}})^{A_{B,l}+k+1}}{\theta^{k-1}(1-\theta^{\frac{1}{2}})^{A_{B,l}-k-1}(1+a^{\frac{1}{2}})^{A_{B,l}+k+1}} &\leq \frac{u'(a)u^{k-1}(a)Q_1(u(a))}{u'(\theta)u^{k-1}(\theta)Q_1(u(\theta))} \\ &\leq \frac{a^{k-1}(1-\theta^{\frac{1}{2}})^{k+1+C_{B,l}}(1+\theta^{\frac{1}{2}})^{k+1-C_{B,l}}}{\theta^{k-1}(1-a^{\frac{1}{2}})^{k+1+C_{B,l}}(1+a^{\frac{1}{2}})^{k+1-C_{B,l}}} \end{aligned} \tag{5}$$

for $0 < \theta \leq a < 1$. We let $V = u^k$ and $x = \theta^k$, and obtain the following Harnack inequality:

$$\begin{aligned} \frac{(1-a^{\frac{1}{2}})^{A_{B,l}-k-1}(1+\theta^{\frac{1}{2}})^{A_{B,l}+k+1}}{(1-\theta^{\frac{1}{2}})^{A_{B,l}-k-1}(1+a^{\frac{1}{2}})^{A_{B,l}+k+1}} &\leq \frac{V_x(a)Q_1(u(a))}{V_x(\theta)Q_1(u(\theta))} \\ &\leq \frac{(1-\theta^{\frac{1}{2}})^{k+1+C_{B,l}}(1+\theta^{\frac{1}{2}})^{k+1-C_{B,l}}}{(1-a^{\frac{1}{2}})^{k+1+C_{B,l}}(1+a^{\frac{1}{2}})^{k+1-C_{B,l}}}. \end{aligned} \tag{6}$$

Arguing as in [Gu4], we have that

Theorem 7. *If there is a solution $0 \leq u \leq l(l+2B)$ of above equation with $u(0) = 0$ and $u(1) = l(l+2B)$. Then there is a Kähler metric with constant scalar curvature in the considered Kähler class.*

Theorem 8. *For any small positive number f, we have a solution $u(0) = 0, u(1-f) = l(l+2B)$. This corresponds to a Kähler metric with constant scalar curvature on the manifold with boundary $\theta \leq 1 - f$.*

8 Global solutions

In this section, we shall extend our solutions to the hypersurface orbit. We shall let $f \to 0$. As we did in [Gu4], we let $\tau = -\log(1-\theta)$ and have that

$$[\log[u_\tau Q(u)]]_\tau = \frac{P-\theta}{\theta}.$$

Therefore, we have that

$$\begin{aligned}
&\left[\log\left[\frac{ku^{k-1}u_\tau}{\theta^{k-1}}Q_1(u)\right]\right]_\tau = \frac{P-\theta}{\theta} - \frac{(k-1)\theta_\tau}{\theta} \\
= &\quad \frac{P-\theta}{\theta} - (k-1)\left(\frac{1}{\theta}-1\right) \\
= &\quad \frac{P-k+1+(k-2)\theta}{\theta} \qquad (7)\\
= &\quad k + \frac{2^{-1}u}{M\sqrt{\theta(u+B^2\theta)}}\left[\frac{B_\rho\theta}{B\theta-\sqrt{\theta(u+B^2\theta)}} - \frac{m(u)}{Q_1(u)}\right] \\
= &\quad T(u,\theta) \\
\to &\quad k - 2^{-1}\frac{B_\rho}{M}\left[1+\frac{B}{\sqrt{u+B^2}}\right] - \frac{um(u)}{2MQ_1(u)\sqrt{u+B^2}} \\
= &\quad k-\alpha,
\end{aligned}$$

when θ tends to 1 and it converges unformly for $u \geq u_0$ with any $u_0 > 0$.

If ω is in the Ricci class, then $m(u) = Q_1(u)$ and

$$\alpha = (2M)^{-1}[B_\rho + \sqrt{u+B_\rho^2}].$$

Let u_i be a series of solutions corresponding to $f_i \to 0$. By $P(1) = 0$, for any $e_0 \in (k, k+1)$ there are two numbers $A(e_0) < l(l+2B)$ and $B(e_0) > 0$ such that if $u > A(e_0)$ and $\tau > B(e_0)$ then $\alpha > e_0 > k$ and $T(u,\theta(\tau)) < k - e_0$. Let τ_i be a point of τ such that $u_i(\tau_i) = A(e_0)$, and if we also have $\tau_i > B(e_0)$ then

$$\left[\log\left[\frac{ku_i^{k-1}u_{i,\tau}}{\theta^{k-1}}Q_1(u_i)\right]\right]_\tau = \frac{P-k+1+(k-2)\theta}{\theta} = T(u,\theta) < k - e_0$$

for $\tau \geq \tau_i$.

Let $w = \frac{ku^{k-1}u'}{\theta^{k-1}}Q_1(u)$, then

$$w_i \leq e^{(k-e_0)(\tau-\tau_i)}w_i(\tau_i).$$

If there is no subsequence of τ_i which tends to $+\infty$, then there is a subsequence of τ_i which tends to a finite number τ_0. By the left side of the Harnack inequality (6), we see that $V_{i,x}(\theta(\tau_0))$ must be bounded from above, otherwise $V_{i,x}$ will be bounded from below by a very large number such that V_i will be bigger than $l(l+2B)$ before x reaching the point 1. That is, there is a subsequence of u_i converging to a solution u of our equation with $u(1) > A(e_0)$.

We shall observe that there is no subsequence of τ_i which tends to $+\infty$ under *certain* condition below.

If there is a subsequence of τ_i which tends to $+\infty$, we might assume that

$$\lim_{i\to+\infty} \tau_i = +\infty,$$

and $\tau_i > B(e_0)$. To make the things simpler, we should avoid the cases in which $G = S$. In those cases, the second Betti numbers are 2 and the manifolds are homogeneous. By Calabi's result, all the extremal metrics are homogeneous and therefore they are unique since there is only one invariant metric in the given Kähler class. As we see before in the last section in the paragraph after (4), $u = c\theta$ will solve the equations.

Thus, we can assume that $G \neq S$, and therefore there is at least one a_i. From the equation (4), we observe that if

$$u_{i,\tau}(\tau_i)u_i^{k-1}(\tau_i) > 2(l(2B+l))^{k-1}(a_1^2 - B^2)A_{B,l} > 2u^{k-1}(a_1^2 - u - B^2)A_{B,l},$$

then

$$\frac{u_{i,\tau}(\tau_i)}{a_1^2 - u(\tau_i) - B^2} > 2A_{B,l}$$

and we have that $v_\tau = u_i^{k-1}u_{i,\tau}$ is increasing for $\tau \geq \tau_i$. This can not happen. Therefore, $u_{i,\tau}(\tau_i)$ is bounded from above.

We shall see that in this circumstance there is a subsequence of

$$\tilde{u}_i(\tau) = u_i(\tau + \tau_i)$$

which converges in C^1 norm to a nonconstant function $\tilde{u}$. We see that for each $\tau \geq 0$, w_i is decreasing and $\tilde{u}_{i,\tau}$ are uniformly bounded. For each $\tau < 0$, $-A_{B,l} < [\log w_i]_\tau < k + C_{B,l}$ when i big enough, that is, $\tilde{V}_{i,\tau}$ are also bounded uniformly on i over any closed intervals. Therefore, a subsequence of $\tilde{V}_i$ converges in the C^1 norm to a function $\tilde{u}$. Thus, the same thing happens for a subsequence of $\tilde{u}_i$.

To observe that $\tilde{u}$ is not a constant, we notice that

$$\frac{ku_i^{k-1}u_{i,\tau}}{\theta^{k-1}} \leq C_i \frac{ku_i^{k-1}(\tau_i)u_{i,\tau}(\tau_i)}{\theta^{k-1}(\tau_i)} e^{(k-e_0)(\tau-\tau_i)}$$

for $\tau \geq \tau_i$, where C_i does not depend on u_i. That is,

$$ku_i^{k-1}u_{i,\tau} \leq Cu_{i,\tau}(\tau_i)e^{(k-e_0)(\tau-\tau_i)}.$$

By integrating both side we have that

$$(l(l+2B))^k - A(e_0)^k \leq -\frac{C}{k-e_0}u_{i,\tau}(\tau_i),$$

i.e., $u_{i,\tau}(\tau_i)$ is bounded from below. Therefore, $\tilde{u}_{i,\tau}(0)$ are bounded from below. We have that $\tilde{u}_\tau(0) > 0$. This implies that $\tilde{u}$ is not a constant.

Then, $\tilde{u}$ satisfies the equation

$$[\log[x^{k-1}x'Q_1(x)]]' = -\alpha + k$$

on $(-\infty, +\infty)$. Therefore,

$$[x^{k-1}x'Q_1(x)]' = (-\alpha + k)x^{k-1}Q_1(x)x'.$$

Integrating as in [Gu4], we have that

$$\int_{x(-\infty)}^{x(+\infty)} f_l dx = 0,$$

where

$$f_l = (-\alpha + k)x^{k-1}Q_1(x).$$

As in [Gu4], we see that $x(+\infty) = l(l+2B)$.

As in [Gu4], we shall prove:

Lemma 5. $k - \alpha$ has only one zero.

Proof: As in [Gu4], we may expect that x is related to a Kähler metric of constant scalar curvature on the normal line bundle over the hypersurface orbit. Hence, we may apply the method of counting zeros in [Gu2,4] to this circumstance. $x^{k-1}x'Q_1(x)$ is proportional to "φQ" in [Gu2]. Therefore, the counting of zeros of $k - \alpha$ should be the same as counting the zeros of the derivative of "φQ" to "U" there.

Let $v = \sqrt{u + B^2}$, then $u = v^2 - B^2$ and $a_i^2 - u = (-a_i + v)(-a_i - v)$. We observe that $g_l = 2vf_l$ is actually a polynomial of v and should be proportional to the derivative of "φQ" in [Gu2]. Therefore, we may expect that

$$y = \frac{2}{l}(-B - v) - 1$$

corresponds to the "U" in [Gu2]. We let

$$q = 2vQ(v),$$

and observe that q is proportional to the "Q" in [Gu2].

We see that

$$\begin{aligned} g_l = & \frac{1}{2}(2k - \frac{B_\rho}{M})q - \frac{BB_\rho Q}{M} - \frac{um(u)}{M}u^{k-1} \qquad (8) \\ = & \frac{1}{2}(2k - \frac{B_\rho}{M})q - \frac{R(\mathcal{U})}{2M}u^{k-1} \\ = & \frac{1}{2}(2k - \frac{B_\rho}{M})q - \frac{R_0}{2M}\int Q d\mathcal{U} + \frac{1}{2M}\int p d\mathcal{U}. \end{aligned}$$

Let g_l' be the derivative of g_l to v, we have that

$$\begin{aligned} g_l' = & \frac{1}{2}(2k - \frac{B_\rho}{M})q' - v\frac{R_0}{M}Q + \frac{vp}{M} \qquad (9) \\ = & \frac{1}{2}(2k - \frac{B_\rho}{M})q' + vP_2 - v\frac{R_0}{M}Q + vP_3 \\ = & \Delta - mq, \end{aligned}$$

where $P_3 = 2m_1 Q$ is the Q term in $\frac{p}{M}$ and $P_2 = \frac{p}{M} - P_3$ is the positive linear combination of $\frac{Q}{q_i}$,

$$\Delta = \frac{1}{2}(2k - \frac{B_\rho}{M})q' + vP_2,$$

$m = \frac{R_0}{2} - m_1$. Therefore,

$$g_l = \int_0^v (\Delta - mq)dv.$$

Lemma 6. *The coefficients of Δ are always positive.*

Proof of Lemma 6: From Theorem 3, we see that the 1-strings do not have any contribution to Δ.

The contibution to P_2 of each 2-strings and 3-strings, 4-strings of the $\mathcal{U} - B^2$ factor is in the first term of the $p(\mathcal{U})$ in the Theorem 3.

The contribution to P_2 of each 2-string and 3-string, 4-string related to the Q_1 factors is $\frac{a_{\rho,i}a_i}{Mq_i}q$.

For the first term of Δ, we have $(2k-\frac{B_\rho}{M})Q$ (one might call it the term of v factor since $Q=\frac{q}{2v}$) with $2k-\frac{B_\rho}{M}>0$.

Then, we have the $\mathcal{U}-B^2$ term (or the term of $\frac{q}{\mathcal{U}-B^2}$)

$$\begin{aligned}&2(k-1)v[v(2k-\frac{B_\rho}{M})-\frac{BB_\rho}{M}](\mathcal{U}-B^2)^{k-2}Q_1\\ =\ &(k-1)v[2k(v-B)+2(k-\frac{B_\rho}{M})(v+B)](\mathcal{U}-B^2)^{k-2}Q_1\end{aligned}$$

with both k and $k-\frac{B_\rho}{M}$ positive.

Similarly, we have q_s factor of Q_1 term (or the term of $\frac{q}{q_s}$)

$$\begin{aligned}&2v[-v(2k-\frac{B_\rho}{M})+\frac{a_s a_{\rho,s}}{M}]\frac{Q}{q_s}\\ =\ &v[(2k-\frac{B_\rho+a_{\rho,s}}{M})(a_s-v)-(2k-\frac{B_\rho-a_{\rho,s}}{M})(a_s+v)]\frac{Q}{q_s}\end{aligned}$$

with coefficients $2k-\frac{B_\rho+a_{\rho,s}}{M}>0$ and $-2k+\frac{B_\rho-a_{\rho,s}}{M}$.

So we need to check that the last coefficient is also positive. There are two ways to prove this. First we notice that this actually is the same to check that the coefficients

$$2Mk-B_\rho, 2Mk=2Mk-B_\rho+B_\rho, 2(Mk-B_\rho)=2Mk-B_\rho-B_\rho$$

and

$$2Mk-B_\rho-a_{\rho,s}, -2Mk+B_\rho-a_{\rho,s}$$

are all positive. We claim that these are the components of the Ricci curvature of the exceptional divisor, then the positivity comes from the positivity of the Ricci curvature of the projective rational homogeneous spaces. The point is that v is corresponding to an H in the calculation of the metric and the volume form, and we should prove that the contribution of H to the Ricci curvature is exactly $2Mk-B_\rho$, i.e.,

$$(q_{G/P_\infty}, H)_0=(q_{S/(S\cap P_\infty)}, H)_0=2Mk-B_\rho,$$

where P_∞ is the isotropic group of the exceptional divisor at p_∞. Notice that P_∞ is parabolic.

If $S = B_n$, then the semisimple part of $P_{\infty,1} = S \cap P_\infty$ is generated by $\alpha_2, \cdots, \alpha_n$ with the same orientation. Therefore,

$$(q_{S/P_{\infty,1}}, H)_0 = 1 + 2(n-1) = 2n - 1.$$

But, we also have that

$$2Mk - B_\rho = k + (n-1) = n + n - 1 = 2n - 1.$$

If $S = C_n$, then the semisimple part of $P_{\infty,1}$ is generated by $\alpha_3, \cdots, \alpha_n$ with an orientation in which $e'_i = e_i \ i \neq 2, e'_2 = -e_2$. Therefore,

$$(q_{S/P_{\infty,1}}, H)_0 = 2 + 2(n-1) + 2 + 2(n-2) = 2(2n-1).$$

But, we also have that

$$2Mk - B_\rho = 2k + 2 = 2(k+1) = 2(2(n-1)+1) = 2(2n-1).$$

Secondly, we could check the positivity of the last coefficient with a case by case checking. That will also give all the $a_{\rho,s}$ in concrete calculations. This is extremally useful when we check the Fano property of the manifolds and classify the manifolds with higher codimensional end (see [Gu9]). For example, from Theorem 6 and Proposition 9 we can check that:

Proposition 10. *If $S = B_n$ or C_n the manifold is Fano if and only if*

$$-2M(k+1) + B_\rho - a_{\rho,s} > 0.$$

If $S = B_n$, $M = \frac{1}{2}$ and we shall check that the last coefficient is $2[-(2n-1) - a_{\rho,i}] > 0$. If $G = B_{m+n}$, then

$$-a_{\rho,i} = -B_{\rho,i} = 2(m+n) + 1 - l_1 - l_2 \geq 2n + 1 = k - B_\rho + 2.$$

The corresponding affine manifolds are Fano.

If $G = C_n \ n > 2$ and $S = B_2$, then

$$-a_{\rho,i} = \mp B_\rho - 2B_{\rho,i} = \mp 1 + 2n + 2 - l_1 - l_2 \geq 4 + 1 = -B_\rho + 4 = -B_\rho + k + 2.$$

The corresponding affine manifolds are Fano.

If $G = F_4$ and $S = B_2$, we take our notations as in the third section, then $\alpha = e_3$ and

$$a_1 = B_1, a_2 = B_2, a_3 = B_1 + B_2 + B,$$

$$a_4 = B_1 - B_2 + B, a_5 = B_1 + B_2 - B, a_6 = B_1 - B_2 - B.$$

We have that $B_{\rho,1} = -11$ and $B_{\rho,2} = -5$, therefore,

$$-a_{\rho,3} = 11 + 5 + 1 = 17, -a_{\rho,4} = 7, -a_{\rho,5} = 15, -a_{\rho,6} = 5.$$

We have all of $-a_{\rho,s} \geq 5 = 2n + 1 = -B_\rho + k + 2$. The corresponding affine manifold is Fano.

If $G = F_4$ and $S = B_3$, $\alpha = e_2$ and

$$a_1 = B_1, a_2 = B_1 + 2B, a_3 = B_1 = a_4, a_5 = B_1 - 2B.$$

We have that $B_{\rho,1} = -11$ and

$$-a_{\rho,2} = 15, -a_{\rho,5} = 7.$$

We have all of $-a_{\rho,s} \geq 9 - 2n + 1 = -B_\rho + k + 2$. The corresponding affine manifold is Fano.

Altogether, we see that the last coefficient is positive for the case $S = B_n$ and the corresponding affine manifolds are Fano.

Otherwise, $S \neq B_n$ and $M = 1$, we shall prove that the last coefficient is $-2k + B_\rho - a_{\rho,s} > 0$ also. If $S = C_n, G = C_{m+n}$, we have that

$$k = 2(n-1),\ B_\rho = -2, a_i = B_i \pm B, \alpha = e_{m+1} - e_{m+2}.$$

We have that

$$B_{\rho,i} = -2(2(m+n+1) - l_1 - l_2) \leq -4(n+1) = -4(n-1) - 6 = -2(k+2) + B_\rho.$$

The corresponding affine manifolds are Fano.

If $S = C_3, G = F_4$, we have that

$$k = 4, B_\rho = -2, a_1 = \frac{B_1}{2}, a_2 = B_1 + B, a_3 = B_1 - B.$$

We have that $B_{\rho,1} = -28$, and therefore

$$a_{\rho,1} = -14, a_{\rho,2} = -30, a_{\rho,3} = -26 \leq 14 = -2(k+1) + B_\rho.$$

The corresponding manifold is Fano.

Q. E. D.

Therefore, as we argued in [Gu4 p.73], if $k - \alpha$ has two zeros, then $\Delta - mq$ has $\deg q - 3 + 4 = \deg q + 1$ zeros. That will be a contradiction to the degree of this polynomial which is $2 \deg Q + 1$. Thus, we obtain our Lemma 5.

Q. E. D.

A corollary of our proof of our Lemma 6 is that:

Corollary 1. *The manifolds are Fano if* $S = B_n$ *or* C_n.

That is, all the manifolds we considered in this paper are Fano. Combinning with [Gu8], we have that:

Corollary 2. *The type II manifolds are Fano if* $S \neq A_n$.

Now, we have that f_l has a unique zero. Therefore, if

$$\int_0^{l(l+2B)} f_l dx < 0, \tag{10}$$

we can not have that

$$0 = \int_{x(-\infty)}^{l(l+2B)} f_l dx \leq \int_0^{l(l+2B)} f_l dx.$$

Otherwise, we have a contradiction.

By choosing $A(e_0)$ close to $l(l+2B)$ we have that $u(1) = l(l+2B)$. Arguing as in [Gu4], we have that $u'(1)$ exists and is finite. Similarly, $u''(0)$ and $u''(1)$ exist and are finite.

Also, we already see that if $G = S$ and $S \neq G_2$, the manifold is homogeneous and admits unique extremal metric in any given Kähler class. Therefore, we have that:

Theorem 9. *There is a Kähler metric of constant scalar curvature in a given Kähler class if the condition* (10) *is satisfied.*

We shall prove the converse in [Gu6].

We could easily argue as in [Gu5 p.273–274] and [Gu4] that the right side of (10) is the Ding-Tian generalized Futaki invariant for a (possibly singular) completion of the normal line bundle of the exceptional divisor, although we do not really know that there is an actually analytic degeneration with this completion as the central fiber. Our condition here is stronger than the Ross-Thomas version of Donaldson's version of K-stability (Cf. [Gu9]).

9 Kähler-Einstein metrics

If the Kähler class is the Ricci class, we have that $B = B_\rho$, $l = l_\rho$,

$$m(u) = Q_1(u),$$

$$\alpha = (2M)^{-1}[B_\rho + \sqrt{u + B_\rho^2}].$$

Therefore,

$$f_l = [k - (2M)^{-1}[B_\rho + \sqrt{u + B_\rho^2}]]u^{k-1}Q_1(u).$$

In this section, we show how we can check the Kähler-Einstein property case by case on the pairs of groups (S, G).

First, if $S = B_n$ $n \geq 2$, we have that $B_\rho = -(n-1)$, $k = n$, Q_1 is a constant, $l_\rho = -(n+1)$.

$$f_\rho = (2n - 1 - \sqrt{u + (n-1)^2})u^{n-1}Q_1(u).$$

Therefore, the integral is

$$\int_0^{(n+1)(3n-1)} (2n - 1 - \sqrt{u + (n-1)^2}Q_1(u)du. \tag{11}$$

If $G = S = B_n$, then

$$\begin{aligned}
c_n &= \int_0^{l_\rho(l_\rho + 2B_\rho)} f_{l_\rho} du \\
&= \int_0^{(n+1)(3n-1)} (2n - 1 - \sqrt{u + (n-1)^2})u^{n-1}du \\
&= \int_{n-1}^{2n} (2n - 1 - v)2v(v^2 - (n-1)^2)^{n-1}dv \\
&= \frac{1}{n}[(2n - 1 - v)(v^2 - (n-1)^2)^n|_{(n-1)}^{2n} + \int_{n-1}^{2n}(v^2 - (n-1)^2)^n dv] \\
&= \frac{1}{n}[-((2n)^2 - (n-1)^2)^n + \int_{n-1}^{2n}(v^2 - (n-1)^2)^n dv] \\
&= \frac{1}{n}\int_{n-1}^{2n}[(v^2 - (n-1)^2)^n - ((v^2 - (n-1)^2)^n)']dv \\
&= \frac{1}{n}\int_{n-1}^{2n}(v^2 - 2nv - (n-1)^2)(v^2 - (n-1)^2)^{n-1}dv \\
&< 0
\end{aligned}$$

since $v^2 - 2nv = v(v - 2n) \leq 0$ and $= 0$ only if $v = 2n$. Therefore, the conditon of the Theorem 9 holds for this case, and it is known that there is an Einstein metric since the manifolds are homogeneous.

Now, we consider the circumstance in which $G = B_{n+1}$ and $S = B_n$. Then the corresponding integral is

$$\begin{aligned}
I_n = \int_{n-1}^{2n} & 2v(2n-1-v)(v^2-(n-1)^2)^{n-1}((2n+1)^2-v^2)dv \\
= & \int_0^{(3n-1)(n+1)} (2n-1-v)u^{n-1}(3n(n+2)-u)du \\
= & -3(n+2)u^n(2n) + \frac{u^{n+1}(2n)}{n+1} + 3(n+2)L(n) - \frac{L(n+1)}{n+1} \\
= & \int_{n-1}^{2n} [(3(n+2) - \frac{u}{n+1})u - 2v((2n+1)^2-v^2)]u^{n-1}dv.
\end{aligned}$$

We have that

$$\begin{aligned}
(n+1)I_n = & \int_{n-1}^{2n} 2v(2n-1-v)((2n+1)^2-v^2)u^{n-1}dv \\
+ & \ n\int_{n-1}^{2n}[(3(n+2) - \frac{u}{n+1})u - 2v((2n+1)^2-v^2)]u^{n-1}dv \\
= & \int_{n-1}^{2n}[(3(n+2) - \frac{u}{n+1})nu + 2v((2n+1)^2-v^2)(n-1-v)]u^{n-1}dv \\
= & \ (n+1)^{-1}\int_{n-1}^{2n}[-(3(n+1)(n+2)-u)n(n-1+v) \\
+ & \ 2(n+1)v((2n+1)^2-v^2)](n-1-v)u^{n-1}dv \\
= & \ (n+1)^{-1}\int_{n-1}^{2n}[-((n+1)(2n+1)^2-(n-1)^2-nv^2)(n-1+v) \\
+ & \ 2(n+1)v(2n+1)^2-v^2)](n-1-v)u^{n-1}dv \\
= & \ (n+1)^{-1}\int_{n-1}^{2n}[(n+2)v^2+2v(n-1) \\
- & \ (n+1)(2n+1)^2+(n-1)^2](n-1-v)^2u^{n-1}dv \\
= & \ (n+1)^{-1}\int_{n-1}^{2n} p(v)(n-1-v)^2u^{n-1}dv,
\end{aligned}$$

where $p(v) = (n+2)v^2 + 2(n-1)v - (n+1)(2n+1)^2 + (n-1)^2$.

We have that

$$\begin{aligned}
p(n-1) = & \ (n+5)(n-1)^2 - (n+1)(2n+1)^2 \\
< & \ (n-1)(n^2+4n-5-4n^2-4n+1) \\
= & \ -(n-1)(3n^2+4) < 0
\end{aligned}$$

and

$$p(2n) = 4(n+2)n^2 + 4n(n-1) - (n+1)(4n^2+4n+1) + (n-1)^2$$
$$= \quad (5n-1)(n-1) - 5n - 1 = 5n^2 - 11n = n(5n-11).$$

Therefore, when $n = 2$, $p(2n) = -n = -2 < 0$ also. But $p(v)$ is positive for $|v|$ big enough, we have that $p(v) < 0$ for $v \in [n-1, 2n] = [2-1, 4]$. In particular $I_2 < 0$ and there is a Kähler-Einstein metric when $S = B_2, G = B_3$.

For the case $n > 2$, we have that

$$p(2n) > 0$$

and

$$p(2n-1) = (n+2)(2n-1)^2 + (n-1)(5n-3) - (n+1)(2n+1)^2$$
$$= \quad -8(n+1)n + 4n^2 - 4n + 1 + 5n^2 - 8n + 3 = n^2 - 20n + 2 > 0$$

if and only if $n \geq 20$.

We want to see that $(5n-11)c_n + 2I_n > 0$ for n big enough. If this is true, then we have that $2I_n > (11-5n)c_n > 0$ since $c_n < 0$.

Let

$$g(v) = (5n-11)(2n-1-v)v$$
$$+ \quad \frac{2(v-n+1)^2}{(n+1)^2}[(n+2)v^2 + 2(n-1)v(n-1)^2 - (n+1)(2n+1)^2].$$

Then $g(n-1) = (5n-11)n(n-1) > 0$ for $n > 3$. $g(2n-1) > 0$ for $n \geq 20$. $g(2n) = -2n(5n-11) + 2n(5n-11) = 0$. It is not difficult to check that $g'''(v) > 0$ on $[n-1, 2n]$. We also have that

$$g'(v) = (5n-11)(2n-1-2v)$$
$$+ \quad \frac{4(v-n+1)}{(n+1)^2}[(n+2)v^2 + 2(n-1)v + (n-1)^2 - (n+1)(2n+1)^2]$$
$$+ \quad \frac{2(v-n+1)^2}{(n+1)^2}]2(n+2)v + 2(n-1)].$$

Therefore,

$$g(2n) = -(5n-11)(2n+1) + \frac{4n(5n-11)}{n+1} + 4(2n^2+5n-1)$$
$$< \quad -10n^2 + 17n + 11 + 4(5n-11) + 4(2n^2+5n-1)$$
$$= \quad -2n^2 + 57n - 37 < 0$$

if $n \geq 28$. Now, we have that:

$$(5n-11)c_n + 2I_n = \int_{n-1}^{2n} g(v)u^{n-1}dv.$$

Lemma 7. *$g(v) > 0$ on $[n-1, 2n)$ if $n \geq 28$.*

Proof: If $n \geq 28$, we have that $g'(2n) < 0$. Then $g(v) > 0$ in $[n-1, 2n)$. Otherwise, there is a root of g in $[n-1, 2n)$. Now, by $g(n-1) > 0, g(2n) = 0, g'(2n) < 0$ we observe that in $[n-1, 2n)$ there is a minimal point a and a locally maximal point $b \in (a, 2n)$. $g''(v)$ is positive near a and negative near b. But $g'''(v)$ is positive in $(n-1, 2n)$, a contradiction.

Q. E. D.

Now, by $g(v) > 0$ we have that $(5n-11)c_n + 2I_n > 0$ for $n \geq 28$, i.e., $I_n > 0, n \geq 28$.

Now, there are only 25 integers between 2 and 28. We could actually check these I_n by using Mathematica with:

```
Integrate[2v(2n-1-v)(v^2 -(n-1)^2 )^(n-1)
        ((2n+1)^2 -v^2 ), {v, n-1, 2n}]
```

we obtain that:

Lemma 8. *$I_n < 0$ if $n = 2, 3, 4, 5, 6$ and $I_n > 0$ if $7 \leq n \leq 27$.*

Now, we consider the general circumstance in which $S = B_n$, $G = B_{n+m}$ and P be the smallest parabolic subgroup of G containning S as a semisimple factor. In this case, $Q_1(v) = \prod_{k=0}^{m-1}(2n+2k+1)^2 - v^2)$. Since each $((2n+2k+1)^2 - v^2)$ decreases, when v increases we have that if

$$I_{n,m} = \int_{n-1}^{2n} 2v(2n-1-v)u^{n-1}\prod_{k=0}^{m-1}((2n+2k+1)^2 - v^2)dv$$

then $I_{n,m+1} > ((2n+2m+1)^2 - (2n-1)^2)I_{n,m}$. Therefore, $I_{n,m} > 0$ if $n \geq 7$.

Using Mathematica with:

```
Integrate[2v(2n-1-v)(v^2 -(n-1)^2 )^(n-1)
((2n+1)^2 -v^2 )((2n+3)^2 -v^2 ), {v, n-1, 2n}]
```

we obtain that:

Lemma 9. $I_{n,2} > 0$ *if* $n = 3, 4, 5, 6$.

Similarly, we can use Mathematica to calculate $I_{2,2}, I_{2,3}$ etc. and obtain that:

Lemma 10. $I_{2,m} < 0$ *if* $m = 2, 3, 4$ *and* $I_{2,5} > 0$.

Therefore, if we denote the corresponding Fano manifolds by $M_{n,m}$ with $n \geq 2, m \geq 0$, then we have that:

Theorem 10. $M_{n,0}$ *are homogeneous with Kähler-Einstein metrics.*

$$M_{2,1}, M_{3,1}, M_{4,1}, M_{5,1}, M_{6,1}, M_{2,2}, M_{2,3}, M_{2,4}$$

are nonhomogeneous Kähler-Einstein manifolds. Other $M_{n,m}$ *do not admit any Kähler-Einstein metric.*

We delay our proof of the nonexistence to [Gu6]. See Theorem 12 in the next section for the nonhomogeneity of $M_{n,k}$ with $k > 0$.

Next, we consider the case in which $S = B_n, G = B_{n+m}$ and S_1 in the section 2 is maximal. In this case, we have that $Q_1(v) = ((2n+m)^2 - v^2)^m$. The integral is

$$J_{n,m} = \int_{n-1}^{2n} 2v(2n-1-v)(v^2-(n-1)^2)^{n-1}((2n+m)^2-v^2)^m dv$$

and

$$m^{-2m} J_{n,m} \to e^{4n} C_n < 0.$$

Therefore, $J_{n.m} < 0$ when m is big enough.

Again, we can compare the change rate of the factor $h(v) = ((2n+m)^2 - v^2)^m$. We let

$$\begin{aligned} t(m) &= (\log h)' \\ &= m\left(\frac{1}{2n+m+v} - \frac{1}{2n+m-v}\right) \\ &= -\frac{2n+v}{2n+m+v} + \frac{2n-v}{2n+m-v}. \end{aligned}$$

Then,

$$t(m+1) - t(m) = \frac{-2v[4n^2 - m(m+1) - v^2]}{((2n+m)^{-}v^2)((2n+m+1)^2 - v^2)} > 0$$

if $m \geq 2n$. Therefore, if $J_{n,m} \leq 0$ with $m \geq 2n$, then $J_{n,m+1} < 0$.

Now, we can use Mathematica to check $J_{n,2n}$ with

```
Integrate[2v(2n-1-v)(v^2 -(n-1)^2 )^(n-1)
          (16n^2 -v^2 )^(2n) , {v, n-1, 2n}]
```

we get that $J_{n,2n} < 0$ when $n = 2, 3$ but $J_{4,8} > 0$.

We then use Mathematica to check $J_{4,12}$ with

```
Integrate[2v(8-1-v)(v^2 -9)^3 (400 -v^2 )^(12) , {v, 3, 8}]
```

and have that $J_{4,12} < 0$. Therefore, $J_{4,m} < 0$ for $m \geq 12$.

Similarly, by using Mathematica we have that $J_{4,m} > 0$ if $2 \leq m \leq 10$ and $J_{4,11} < 0$. Therefore, when $m = 1$ or ≥ 11, we have that $J_{4,m} < 0$, otherwise, $J_{4,m} > 0$.

Similarly, we use Mathematica to check $J_{2,m}$ for $m = 2, 3$ and $J_{3,m}$ for $m = 2, 3, 4, 5$. We find that all of them < 0.

Therefore, we obtain that if we denote the corresponding manifolds by $N_{n.m}$, then:

Theorem 11. *$N_{2,m}, N_{3,m}$ admit Kähler-Einstein metric for all m. $N_{4,m}$ admit Kähler-Einstein metric if and only if $m = 1$ or $m > 10$. In general, $N_{n,m}$ admit Kähler-Einstein metric when m big enough, i.e., there is an integer $N(n)$ such that if $m > N(n)$ then $N_{n,m}$ admit Kähler-Einstein metric. Moreover, if $m \geq 2n$ and $N_{n,m}$ admit a Kähler-Einstein metric, then $N_{n,m+1}$ also admit a Kähler-Einstein metric.*

We now leave other examples to the readers, since Theorem 11 (see also Theorem 13 in the next section) gives us enough new Kähler-Einstein manifolds and Theorem 10 gives us a large class of Fano manifolds which do not admit any Kähler-Einstein metric.

However, for the readers' benefit, we should give the integral for the Case in which $S = C_n$. In that case, $M = 1$, $l_\rho = -2(k+1) = -2(2n-1)$ and $B_\rho = -2$. We have

$$\alpha = \frac{1}{2}(-2 + \sqrt{u+4}).$$

$$f_\rho = \frac{1}{2}(4n - 2 + \sqrt{u+4})u^{2n-3}Q_1(u).$$

Therefore, the integral is proportional to

$$\int_0^{4(4n^2-1)} (4n - 2 + \sqrt{u+4})u^{2n-3}Q_1(u)du. \tag{12}$$

In the next section, we shall discuss some properties of these manifolds.

10 Further comments

In [Gu8], we observe that the third example in [Ak p.68] is not homogeneous and the identity component of the automorphism group is G_2. However, it was mentioned in [Ak p.69] that the first and the second examples in [Ak p.68] are homogeneous. Moreover, in the case 1) and 5) in [Ak p.73] which correspond to the first and the second with $n = 3$ in [Ak p.67], the manifolds are also homogeneous. For the nonaffine type II case of the third with $n = 3$ in [Ak p.67], the manifold is also homogeneous. That is, when $G = S$ the manifolds in this paper and [Gu12] are homogeneous and the automorphism groups are some simple complex Lie groups which are strictly larger. What will happen if G is strictly larger than S? Applying the Theorem in [St], we have that:

Theorem 12. *Let M be a compact complex almost-homogeneous manifold with one hypersurface end and a complex semisimple Lie group G action. If G is strickly larger than S, then the identity component of the automorphism group is G and M is not homogeneous. Consequently, all these complex manifolds are biholomorphically different from each other.*

Proof: First, we consider the case in which M is affine. We know that if G is strictly larger than S, by [Ak], M is a fiber bundle over a rational homogeneous manifold Q with a transitive G action. Actually, $Q = G/P$ and $\dim Q > 0$. The fiber F is just our manifold in the case $G = S$. Therefore, F is a fiber bundle over a rational homogeneous manifold Q_1 with a transitive S action. The fiber is $\mathbf{C}P^k$. $\dim Q_1 > 0$. Therefore, M is a $\mathbf{C}P^k$ bundle over a rational homogeneous manifold Q_2 with a transitive G action. By our construction, Q_2 is a Q_1 bundle over Q. If there is another connected complex Lie group G_1 acting on M and containning G, then by [Ak] and [GC] (see also [Gu6]), G_1 is semisimple. G_1 also acts transitively on Q_2. Comparing Q_2 to the possible manifolds in the Theorem in [St], we have that $G_1 = G$.

If M is not affine but of type II, then $S = A_1$. If M is homogeneous, then according to [St p.427 Theorem] A_1 should be one of the semisimple part of the isotropic group of the smaller group actions there. There are only 3 possibilities: C_n in 1) with $n \geq 1$ there, A_1 in 2), A_{n-1} in 3) with $n \geq 3$. The only possibility are $n = 1$ in 1) and the case 2). In the case 1), $F = \mathbf{C}P^2$. The semisimple part of the isotropic group of the larger group A_{2n+1} action is A_{2n} in 1). When $n = 1$, we have A_2. It happens that A_2 does actually act on F. But, then M is the flag manifold parametrizing the planes π and the line $l \subset \pi$ in $\mathbf{C}^4$. Let the Cartan subalgebra of the larger group A_3 be:

$$\left\{ \left[\begin{array}{cccc} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{array} \right] \Big|_{a_1+a_2+a_3+a_4=0} \right\}.$$

The Cartan subalgebra of the smaller group C_2 could be:

$$\left\{ \left[\begin{array}{cccc} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & -a_2 \end{array} \right] \Big|_{a_1,a_2\in\mathbf{C}} \right\}.$$

Let $l = (a,0,0,0)^T$, $\pi = \{(a,0,b,0)^T|_{a,b\in\mathbf{C}}\}$. We obtain that M is a manifold with $G = S = B_2$ (compare also the description for the affine case of $G = S = C_2$ in [Ak p.69]), that is different from the case in which $S = A_1$ since they have quite different A_1 actions—A_1 actually has three orbits on F. Therefore, 1) does not occur. In the case of 2), the isotropic group of the larger group B_3 in [St] is B_2, which does not act on $\mathbf{C}P^2$, a contradiction. Therefore, M can not be homogeneous.

If M is of type I and is homogeneous. By $G \neq S$ we have that S must be one of the semisimple part of the isotropic groups of the smaller groups G in [St p.427 Theorem] and the semisimple part $\tilde{S}$ of the isotropic group of the larger groups $\tilde{G}$ must act on F transitively. We have that $(S, G, \tilde{S}, \tilde{G}) =$ 1) $(C_n, C_{n+1}, A_{2n}, A_{2n+1})$; 2) (A_1, G_2, B_2, B_3); 3) $(A_{n-1}, B_n, A_n, D_{n+1})$. Now, we go through the possible list in [Ak p.67].

If $S = A_k$, we have the first case, and the second, the third cases with $n = 6$ in which $k = 3$. In the first case, $F = \mathbf{C}P^k \times \mathbf{C}P^k$ and when $k = 1$, none of A_2, B_2, A_2 as the possible $\tilde{S}$ above can act on F nontrivially; when $k > 1, A_{k+1}$, as the $\tilde{S}$ above, can not act on F nontrivially. In the second and the third case, $F = Q(6)$ being the 5 complex dimensional hyperquadric or $\mathbf{C}P^5$, but A_4 does not act on F transitively. That is, $S \neq A_k$. So $S = C_k$ with $n > 1$. The only possible cases are the second and third cases in [Ak p.67] with $n = 5$ in which $k = 2$, and the fourth case. $\tilde{S} = A_{2k}$. The second case can not occur, since A_4 does not act on $Q(5)$ nontrivially. The fourth case can not occur, since A_{2k} does not act on $Gr(2k, 2)$ nontrivially. Therefore, M is a $\mathbf{C}P^4$ bundle over $\mathbf{C}P^5$ and it parametrizes the planes

π and the lines $l \subset \pi$ in $\mathbf{C}^6$, see the description of the our affine case of $G = S = C_3$ in [Ak p.69]. But, then we have $G = S$, a contradiction.

Q. E. D.

Next thing we like to point out is that it is not difficult to check that all the homogeneous ones have the condition (10) for the Ricci class by checking our integrals. We already checked the cases with $G = S = B_n$ and we shall check a similar condition for $G = S = A_n$ in [Gu12]. One can also check the case with $G = S = C_n$.

If $G = C_{m+n}$ and $S = C_n$, we have that $B_\rho = -2$,

$$l_\rho = -2(k+1) = -2(2(n-1)+1) = -2(2n-1).$$

$$\begin{aligned} a_{\rho,i} &= B_{\rho,i} + B_\rho \\ &= -2(2(m+n+1) - l_1 - l_2) - 2 \\ &= -2(2m + 2n + 3 - l_1 - l_2) \\ &\leq -2(2n+3) \\ &< -4n \\ &= l_\rho + B_\rho. \end{aligned}$$

The manifolds are always Fano. The integrals are

$$\int_2^{4n} v(2(n-1) - 2^{-1}(-2+v))(v^2-4)^{2n-3} \prod_{i=1}^{m} (a_{\rho,i}^2 - v^2) dv.$$

Now, if $m = 0$ we let $v = 2x$, then the integral is the same as

$$\begin{aligned} C\int_1^{2n} x(2n-1-x)(x^2-1)^{2n-3}dx &= C_1[(2n-1-x)(x^2-1)^{2(n-1)}|_0^{2n} \\ &+ \int_1^{2n}(x^2-1)^{2(n-1)}dx] \\ &= C_1[-(x^2-1)^{2(n-1)}|_0^{2n} + \int_0^{2n}(x^2-1)^{2(n-1))}dx] \\ &= C_1\int_0^{2n}(x^2-1-4(n-1)x)(x^2-1)^{2n-3}dx \end{aligned}$$

with $C, C_1 > 0$. But

$$x^2 - 1 - 4(n-1)x = -1 - (4(n-1) - x)x \leq -1,$$

since $4(n-1)-2n = 2(n-2) \geq 0$ if $n \geq 2$. Therefore, the integrals are also negative with $G = S = C_n$.

One further observation: If we compare those two Ricci curvtures in the proof of the Lemma 6 for the hypersurface divisor D and in Proposition 10 for the whole manifold, we see that the canonical line bundle of the fiber F is related $K_F = l_\rho$ and the divisor itself as a line bundle is related to the difference $D_F = 2M$. If we let $x = v - B_\rho$ and $q(x) = vQ(v)$, then by (11) and (12) we have:

Theorem 13. *The manifolds we considered has a Kähler-Einstein metric if and only if*

$$\int_0^{-K_F} (K_F + D_F + x)q(x)dx > 0, \tag{13}$$

where $(K_F, D_F) = (-n-1, 1)$ *if* $S = B_n$ *and* $(K_F, D_F) = (-2(2n-1), 2)$ *if* $S = C_n$. *Moreover,* $(K_F, D_F) = (-(k+1)D_F, D_F)$ *with* k *being the dimension of corresponding affine spaces* $\mathbf{C}^k$ *as it is in Theorem 1.*

This is related to a stronger version of the Ross-Thomas slope stability. Therefore, our result in [Gu6] is stronger than the result of Ross-Thomas for the necessary direction even for the Kähler-Einstein cases of our manifolds here.

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