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JOURNAL OF Algebra
Journal of Algebra 273 (2004) 33–59
www.elsevier.com/locate/jalgebra

Toward a classification of compact complex homogeneous spaces

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Received 19 October 2001
Communicated by Robert Kottwitz

Abstract

In this note, we prove some results on the classification of compact complex homogeneous spaces. We first consider the case of a parallelizable space $M = G/\Gamma$, where G is a complex connected Lie group and Γ is a discrete cocompact subgroup of G. Using a generalization of results in [M. Otte, J. Potters, Manuscripta Math. 10 (1973) 117–127; D. Guan, Trans. Amer. Math. Soc. 354 (2002) 4493–4504, see also Electron. Res. Announc. Amer. Math. Soc. 3 (1997) 90], it will be shown that, up to a finite covering, G/Γ is a torus bundle over the product of two such quotients, one where G is semisimple, the other where the simple factors of the Levi subgroups of G are all of type A_l. In the general case of compact complex homogeneous spaces, there is a similar decomposition into three types of building blocks.

Keywords: Homogeneous spaces; Product; Fiber bundles; Complex manifolds; Parallelizable manifolds; Discrete subgroups; Classifications

1. Introduction

In this paper, M is a compact complex homogeneous manifold and G a connected complex Lie group acting almost effectively, holomorphically, and transitively on M. We refer to the literature [5,6,8–12,14,21,22,31,32] quoted at the end of this paper for the classification of complex homogeneous spaces which are pseudo-kählerian, symplectic or admit invariant volume forms.

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[1] Supported by NSF Grant DMS-9627434 and DMS-0103282.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2003.11.007

We separate our introduction into 5 subsections. The readers who are only interested in a general picture might go directly to Sections 1.1 and 1.5. We build a general compact complex homogeneous space, up to a finite covering, as a torus bundle over the product of two special spaces in Section 1.1. One of them, which we call a *semisimple space*, was already exhibited in both [12,15]. We devote our Section 1.5 to the other, which we call a *reduced space*, which is, up to a finite covering, a fiber bundle over a torus with a typical fiber a product of *primary spaces* and a *parallelizable manifold*.

Section 1.4 is dedicated to the case of the *spaces of* 1*-step*, which is the major case we considered here. The general case (respectively the primary spaces) can be built up from this case (respectively the primitive spaces). We consider the primitive spaces in Section 1.3. In the Lie algebra level, the primitive space is decomposed into the data related to those B *factors* in Section 1.2, and we will classify these data there.

Section 1.2 is the core of this paper. An expert might mainly be interested in Theorem D, in which one also finds many examples.

In this paper, we regard the Lie algebra $\mathcal{G}$ of a Lie group G as a part of that Lie group. Therefore, if there is no confusion, *we also use G to represent the Lie algebra $\mathcal{G}$*.

1.1. Our starting point is the following proposition of [23].

Proposition 1. *Let $M = G/\Gamma$, where Γ is discrete and cocompact. If a Levi subgroup of G is simple and acts non-trivially on the radical R of G, then it is a Lie group of type A_l or E_8.*

The possibility of factors of type E_8 was left open because at that time it was not known whether groups of type E_8 would satisfy the Hasse principle. Subsequently Chernousov [7] showed that only factors of type A_l can occur.

We shall use following generalization (see also [33, Proposition 2.8.2][2]).

Theorem A. *Let G, Γ, and R as in Proposition* 1 *and S a Levi subgroup of G. Then any simple factor of S which acts non-trivially on R is of type A_l.*

The foundation of the proof for Theorem A comes from the theory of Galois cohomology (see [24]). The cornerstone of our solution to the classification of compact complex homogeneous spaces is our Theorem D which naturally leads to our theorems. Although we use Theorem A to prove Theorem D, we could just as well have done the proof with only the results of Galois cohomology of classical Lie groups which can be found in [20]. One of difficulties in constructing all our proofs is that for a connected complex Lie group G with an abelian nilradical N, the reductive group G/N does not always act reductively on N. We shall deal with this difficulty in different ways (in particular, in the proof of Theorem A and those of Theorems D–H).

[2] We were told about [33] by a referee in 1998 after we had finished this paper. We shall give a complete proof here. Winkelmann's proof works for a special case in which G/N is semisimple, where N is the nilradical of G. See our Section 4 for further discussion.

Theorem B. *Let G, Γ, S, and R be as in Theorem A. S_1 (respectively S_2) be the normal subgroup of S such that each simple factor of S_1 (respectively of S_2) acts trivially (respectively non-trivially) on R. Then, up to a finite covering, G/Γ is a torus fiber bundle over a product $S_1/\Gamma_1 \times S_2R/\Gamma_2C^0$, where C^0 is the identity component of the center C of G and Γ_1 (respectively Γ_2) is a discrete subgroup of S_1 (respectively of S_2R/C^0). In particular, S_2 has only factors of type A_l.*

This structure theorem was suggested in [12]. We use a generalization of [12, Lemma 5] for its proof.

An indirect application of Theorem B is the following *Main Theorem I* to the classification of compact complex homogeneous spaces.

Theorem C. *Let M be a compact complex homogeneous space. Then M, up to a finite covering, is a torus bundle over $S_1/H_1 \times S_2R/H_2$ with S_1, S_2 semisimple and R being the radical of S_2R such that each factors of S_2 acts non-trivially on R. If $J_1 = N_{S_1}(H_1^0)$, $J_2 = N_{S_2R}(H_2^0)$, then J_1/H_1^0 is semisimple and $H_2^0 \cap S_2$ is unipotent, J_2 has only simple factors of type A_l, which are not in H_2. Moreover, each simple factor of S_2 is a classical Lie group and each simple factor of J_2 acts non-trivially on $R/R \cap H$.*

The first factor S_1/H_1 is a reductive compact complex homogeneous space and has a description in [12, Announcement, Main Theorem C] (see also [15]), which is a fiber bundle over a compact rational homogeneous space S_1/P_1 with reductive parallelizable manifold as a fiber. And H_1 contains the nilradical of the parabolic subgroup P_1 of S_1.

We call S_1/H_1 a *semisimple space* and S_2R/H_2 a *reduced space*.

Theorem C cannot be derived directly from Theorem B. Its proof requires a case by case checking. This is where Theorem A is applied.

We also make use of several observations. Theorem 2 in Section 3 is one of them. We use Morita theory in the proof of Theorem 2 *which basically gives the existence of the compact complex homogeneous spaces in the following theorems*. And *any cocompact discrete subgroup of $N_{S_2R}(H_2^0)/H_2^0$ induces a compact complex homogeneous space with the universal covering S_2R/H_2^0. Our major work here is to find all the possible pairs* (S_2R, H^0). The computation of the multiplicities of the k-representations for an algebraic number field k is a very powerful tool in our proof and was fortunately done in [30]. At the beginning, we cannot expect that the reductive part

$$F = \left(J_2/H_2^0\right) \big/ \operatorname{nil}\left(J_2/H_2^0\right)$$

of J_2/H_2^0 acts reductively on its nilradical quotient

$$\operatorname{nil}\left(J_2/H_2^0\right) \big/ \left[\operatorname{nil}\left(J_2/H_2^0\right), \operatorname{nil}\left(J_2/H_2^0\right)\right].$$

But we can consider the case in which F acts reductively, and prove later that F indeed acts reductively on the submodule of the Lie algebra of $\operatorname{nil}(J_2/H_2^0)$ consisting of all the non-trivial representations of the simple factors of J_2 which are not simple factors of S_2. This is one of the reasons Theorems F–H are true.

This reduces the classification of compact complex homogeneous spaces to that of S_2R/H_2 in Theorem C, which is much more tractable than the original problem.

1.2. From now on, in this introduction, we shall assume that $M = G/H$ is a reduced space and $G = SR$ is a Levy decomposition $J = N_G(H^0)$.

This condition makes our description simpler, while the corresponding results below still hold without this condition.

We let S^1 (respectively S^2) be the normal subgroup of S containing the simple factors of S which are not contained in J (respectively are contained in J). We call any one of the simple factors of J which is not in S a *factor A* or an *A factor*, and any simple factor of S^1 (respectively of S^2) a *factor B* (respectively a *factor C*) or a *B factor* (respectively a *C factor*). In Theorem C, we show that the only possible B factors are classical Lie groups.

For a homogeneous space G/H we try to find all the possible pairs of data (B, A), where B is one of the B factor and A are those A factors in B. We call a non-trivial representation of a factor B, as a submodule of the Lie algebra of R, a *primitive* representation if the nilradical of J is *abelian* and the rational quotient $F_{\mathbf{Q}}$ of $J_{\mathbf{Q}} = (J/H^0)_{\mathbf{Q}}$, which comes from H/H^0, acts *irreducibly* on the nilradical of $J_{\mathbf{Q}}$. We call a primitive representation a *representation E* and the homogeneous space with a primitive representation a *primitive space*. Without loss of generality, we try to find all possible pairs (B, A) with a non-trivial primitive space. Fixing a Cartan subalgebra in $\mathcal{B} \cap \mathcal{J}$, we notice that A is generated by some simple root vectors of B and denote the indices of these simple roots by $I = I(B, A)$. Hence, we shall actually try to find all the possible data (B, A, I, E) (some time we use the rank n instead of B if we already know the type of B). If A, and hence I, is empty, we denote it by (B, E).

One observation in Corollary 2 which we found was that F contains the complete given Cartan subalgebra of B in $\mathcal{J}$. We state another observation as Lemma 4 that the Lie algebra of H^0 contains no negative simple root vector. The inequality (1) in our Corollary 3 is very powerful in narrowing down the possibilities. These are some of the reasons why we can classify all these data (B, A, I, E).

When we describe the A factors in a factor B, we should use the notation G_0 to describe the simple factors of the realization $G_{\mathbf{R}}$, in the factor B, of the arithmetic group G. But *by abuse of notation, we use G instead of G_0 in this paper*. We also note here that *all the division algebra of degree* 2 *are quaternions*. We let H_i be the fundamental weights of the factor B, we have the following theorem.

Theorem D. *For each type of classical Lie algebra, we have a classification of the data in the lists below*:

- *In the case $B = A_n$, we have*
  (a) *if $E = E_0 = H_1 + H_n$, then (n, A, I) is one of*
    (1) $(2l+1, SU_1(D, f) \times SU_1(D, g), \langle 1, \dots, l\rangle \cup \langle l+2, \dots, 2l+1\rangle)$ *with* $l > 1$,
    (2) $(4l+3, SU_2(D, f) \times SU_2(D, g), \langle 1, \dots, 2l+1\rangle \cup \langle 2l+3, \dots, 4l+3\rangle)$,
    (3) $(3, SL_1(D)^{\otimes 2}, \langle 1\rangle \cup \langle 3\rangle)$,
    (4) $(3l+2, SU_2(D, f) \times SU_1(D, g), \langle 1, \dots, 2l+1\rangle \cup \langle 2l+3, \dots, 3l+2\rangle)$,
    (5) $(2, SL_1(D), \langle 1\rangle)$,

(6) $(3l+1, SL_1(D)^{\otimes(l+1)}, \langle 1\rangle \cup \langle 4\rangle \cup \cdots \cup \langle 3l+1\rangle)$,
(7) (n);

(b) *if $E \neq E_0$, then (n, A, I, E) is one of*
(1) $(3, SL_1(D), \langle 1, 2\rangle, H_2)$ *with D a division field of degree* 3,
(2) $(3, SL_1(D), \langle 2\rangle, H_2 \oplus H_2)$,
(3) $(4, SU_2(D, f), \langle 1, 2, 3\rangle, H_1 \oplus H_1 \oplus H_3)$,
(4) $(n, H_1 \oplus H_n)$,
where D's are division fields and are of degree 2 (*i.e., quaternions*) *except the ones in* (a1), (a2), (a4), *which are of degree l, and the one in* (b1), *which is of degree* 3.

- *In the case $B = B_n$, we have one of the following situations for* (n, A, I, E):
(1) (n, H_1),
(2) $(2, SL_1(D), \langle 2\rangle, 2H_2)$,
(3) $(2, SL_1(D), \langle 1\rangle, H_1)$,
where the D's are quaternions.
- *In the case $B = C_n$, we have $n = 4$ and $(A, I, E) = (SU_2(D, f), \langle 1, 2, 3\rangle, 2H_1)$ with D a quaternion.*
- *In the case $B = D_n$, we have one of the following*:
(1) $(6, SU_2(D, f), \langle 2, 3, 4\rangle, H_2)$,
(2) $(4, SU_2(D, f), \langle 1, 2, 3\rangle, H_2)$,
(3) $(5, SU_2(D, f), \langle 1, 2, 3\rangle, H_1 \oplus H_1)$,
(4) $(4, SL_1(D), \langle 2\rangle, H_2)$,
where D's are quaternions.

Altogether we have 7 series and 12 exceptional ones for the possible data (B, A, I, E) for a primitive representation including E.

To have some examples of reduced space, one might take anyone in above lists and let $G = BE$. H^0 consists of the group generated those one parameter subgroups corresponding to the negative root vectors which are not in A and the A irreducible representations in E which do not contain any highest weight vector. F is the product of the given Cartan subgroup of B and A. H/H^0 comes from the integer part of the arithematic group of A and the integer part of the A irreducible representations in $N = \mathrm{nil}(N_G(H^0))$ the nilradical of $N_G(H^0)$ as well as the integer part[3] of the center of $F = N_G(H^0)/N$ (the rational extension of this center is just $F_{\mathbf{Q}}/A_{\mathbf{Q}}$). The A irreducible representations in N are copies of the A irreducible representations in E containing the highest weight vectors.

One of the simplest examples comes from (a7) of the case $B = A_n$ with $n = 1$. In this case $E = sl(2, \mathbf{C})$ and B acts on E as the adjoint representation. Let α be the simple root of B, then $E_{-\alpha}, \alpha, E_\alpha$ generate B. Let $F_{-\alpha}, \alpha_E, F_\alpha$ be the corresponding triple basis in E. Then $H^0 = \mathbf{C}\alpha_E + \mathbf{C}F_{-\alpha}$, $F = \mathbf{C}\alpha$, $J = \mathbf{C}\alpha + \mathbf{C}E_{-\alpha} + E$, $N = \mathbf{C}E_{-\alpha} + \mathbf{C}F_\alpha$. We can assume that the integer part of N is $(\mathbf{Z} + \mathbf{Z}i)e_1 + (\mathbf{Z} + \mathbf{Z}i)e_2$ where

$$e_1 = 2E_{-\alpha} - 2F_\alpha, \qquad e_2 = \big(\sqrt{5} + 3\big)E_{-\alpha} + \big(\sqrt{5} + 1\big)F_\alpha,$$

[3] For the detail of this part, see our proofs.

and the integer part of F can be generated by the matrix $M = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. We have defined e_1 and e_2 such that $E_{-\alpha}$ and F_α are the eigenvectors of M. $F_{\mathbf{Z}} = \{M^k|_{k\in\mathbf{Z}}\}$.

Another example comes from (a3) of the case $B = A_n$ with $n = 3$. In this case $E = sl(4, \mathbf{C})$. B acts on E as the adjoint representation. Let α_i be the simple roots of B with $i = 1, 2, 3$, then $E_{-\alpha_i}, E_{\alpha_i}$ generate B. $E_{-\alpha_i}, E_{\alpha_i}, \alpha_i, E_{-\alpha_j-\alpha_{j+1}}, E_{\alpha_j+\alpha_{j+1}}$, $j = 1, 2$, $E_{\alpha_1+\alpha_2+\alpha_3}, E_{-\alpha_1-\alpha_2-\alpha_3}$ is a basis of B. Let F_α for all roots α and α_i^E be a corresponding basis in E. Then $H^0 = \sum_i \mathbf{C}\alpha_i^E + \sum_{\alpha\notin\Delta_1} \mathbf{C}F_\alpha$ where $\Delta_1 = \{\alpha_2, \alpha_j + \alpha_{j+1}, \sum_i \alpha_i\}$. A_j, $j = 1, 2$ is generated by $E_{\alpha_{2j-1}}, E_{-\alpha_{2j-1}}$. $F = A_1 + A_2 + \mathbf{C}\alpha_2$, $N = N \cap B + N \cap E$, $N \cap B = \sum_{\alpha\in\Delta_1} \mathbf{C}E_{-\alpha}$, $N \cap E = \sum_{\alpha\in\Delta_1} \mathbf{C}F_\alpha$. $J = F + N + H^0$. Let D be a quaternion field over $k = \mathbf{Q}(i) = \mathbf{Q} + i\mathbf{Q}$, e.g., D can have a k basis $1, e_1, e_2, e_3$ with $e_1^2 = 3$, $e_2^2 = 1 + i$, $e_3 = e_1e_2 = -e_2e_1$. Then $SL_1(D)$ are the quaternions with norm 1. In our example above, they are those $a + be_1 + ce_2 + de_3$ with $a^2 - 3b^2 - (1+i)c^2 + 3(1+i)d^2 = 1$. Then we can consider $N \cap B$, $N \cap E$ as the $\mathbf{C}$ extensions of two copies of D with A_1 acting on the left and A_2 on the right. To get a k torus action which corresponds to $\mathbf{C}(\alpha_1 + 2\alpha_2 + \alpha_3)$, we apply the matrix M in the last example. Let $e_\alpha^1 = 2E_{-\alpha} - 2F_\alpha$, $e_\alpha^2 = (\sqrt{5}+3)E_{-\alpha} + (\sqrt{5}+1)F_\alpha$ with $\alpha \in \Delta_1$, then M^k can be regarded as transformations on $\mathbf{C}e_\alpha^1 + \mathbf{C}e_\alpha^2$ for each α. We can choose the integer part of N to be the $\mathbf{Z}(i)$ module generated by e_α^1, e_α^2 with $\alpha \in \Delta_1$. The integer part of $SL_1(D)$ can be chosen as the set of $\mathbf{Z}(i)$ integers in D with norm 1.

In these two examples every primary space is primitive.

Since a primitive space might have several different data with similar A factors and similar A irreducible representations which have the same highest weights, not all of the primitive spaces can be obtained in this way. In addition there is also a possible torus factor T which is neither in B nor in E, and $G = (\prod B_i \otimes T)\prod E_i$. We shall discuss this in the next subsection.

1.3. For convenience we denote the data in the above lists by

$$A_l^{a,1},\ A_l^{a,2},\ A^{a,3},\ A_l^{a,4},\ A^{a,5},\ A_l^{a,6},\ A_n^{a,7};\quad A^{b,1},\ A^{b,2},\ A^{b,3},\ A_n^{b,4};$$
$$B_n^1,\ B^2,\ B^3,\ C,\ D^1,\ D^2,\ D^3,\ D^4.$$

We also use the small letter and the lower case of these notations to denote the corresponding types of J/H, i.e., we have

$$a_{1,l}^a,\ a_{2,l}^a,\ a_3^a,\ a_{4,l}^a,\ a_5^a,\ a_{6,l}^a,\ a_{7,n}^a;\quad a_1^b,\ a_2^b,\ b = b_2,\ c,\ d_1,\ d_2.$$

We observe that only $A_n^{a,7}$, $A_k^{b,4}$, and B_l^1; $A^{a,5}$, $A_l^{a,6}$, and B^3; $A^{b,2}$ and D^4; $A^{b,3}$, D^1, and D^3 have similar type of J/H, which are denoted by a_7^a, a_5^a, a_2^b, and d_1, respectively. Now we consider primitive compact complex homogeneous spaces with non-trivial B factors. Without loss of generality we assume that the adjoint action of the reductive quotient of J/H^0 on its nilradical is locally faithful. We call this kind of space a *reduced primitive space* (we notice here that this condition is stronger than a space being reduced and primitive). Otherwise, we just consider the quotient by the kernel of the adjoint action of J/H^0.

Before we state our further results we need some terminology. We say that two complex homogeneous manifolds are *isogenous* if they are isomorphic up to a finite covering, i.e., they have a common finite covering which is comparable with the group actions. We say that a space G^1/H^1 is, *up to an action of a torus, isogenous to* G^2/H^2 if G^1 is isogenous to a normal subgroup of G^3 of G^2 such that G^1/H^1 is isogenous to $G^3/(H^2 \cap G^3)$ with a finite $H^2/(H^2 \cap G^3)$ and $G^2 = G^3T$ with a torus T which is semisimple in G^2. In this situation we denote $G^1/H^1 \leqslant G^2/H^2$. We say that a space G^1/H^1 is, *up to an action of a torus, isogenous in the Lie algebra level to* G^2/H^2 if the Lie algebra $\mathcal{G}^1$ of G^1 is isomorphic to a normal Lie subalgebra $\mathcal{G}^3$ of the Lie algebra $\mathcal{G}^2$ of G^2 such that the image of the Lie algebra $\mathcal{H}^1$ of H^1 is the Lie algebra $\mathcal{H}^2$ of H^2 and if G^3 is the normal Lie subgroup of G^2 with Lie algebra $\mathcal{G}^3$, then $G^2 = G^3T$ with a torus T which is semisimple in G^2. We say that two spaces G^1/H^1 and G^2/H^2 are, *up to actions of torus, isogenous* (respectively *isogenous in the Lie algebra level*) if they are both, up to actions of torus, isogenous (respectively isogenous in the Lie algebra level) to a same space. We also *abuse* the notation $\prod^*$ to express a *twist concept of product* $\prod^*(B_i, E_i = \bigoplus H_{i,j}) = (\prod B_i, E^*)$ with $E^* = \bigoplus \bigotimes H_{i,j}$ for some combination of $H_{i,j}$ such that each $H_{i,j}$ appears once and only once, and H_{i,j_1}, H_{i,j_2}, $j_1 \neq j_2$ cannot appear in a same summand (examples can be found in the last part of the fifth section). We have the following theorem.

Theorem E. *The reduced primitive compact complex homogeneous spaces, up to actions of some torus, are isogenous in the Lie algebra level to one of following* 12 *cases*:

(1) $k_1 A_l^{a,1} \times k_2 a_{1,l}^a$,
(2) $k_1 A_l^{a,2} \times k_2 a_{2,l}^a$,
(3) $k_1 A^{a,3} \times k_2 a_3^a$,
(4) $k_1 A_l^{a,4} \times k_2 a_{4,l}^a$,
(5) $k_1 A^{a,5} \times k_2 B^3 \times k_3 a_5^a \times (\prod_i A_{l_i}^{a,6})$,
(6) $(\prod_i^* A_{n_i}^{a,7}) \times (\prod_j^* B_{n_j}^1) \times (\prod_k^* A_{n_k}^{b,4}) \times (\prod_l^* a_{7,n_l}^a)$,
(7) $k_1 A^{b,1} \times k_2 a_1^b$,
(8) $k_1 A^{b,2} \times k_2 D^4 \times k_3 a_2^b$,
(9) $k_1 A^{b,3} \times k_2 D^1 \times k_3 D^3 \times k_4 d_1$,
(10) $k_1 B^2 \times k_2 b$,
(11) $k_1 C \times k_2 c$,
(12) $k_1 D^2 \times k_2 d_2$,

where k_i, $i = 1, 2, 3, 4$ *are nonnegative integers, and* kB *means* k *copies of* B.

Once we have non-trivial A factors, the structure of the reduced primitive spaces can be described by the theory of division algebras. Therefore, the classification is basically finished. But the case (6) is more difficult, since we need more on the **Q**-structure of the torus action including the Cartan subgroup.

1.4. For the case that the nilradical of J/H^0 is abelian, we call the manifold a compact homogeneous space of 1-*step* (the nilradical of J/H^0 has one step). We observe that all the examples in [1,23,28] fall in this class.[4] We have the following theorem.

Theorem F. *Any reduced space of* 1*-step is isogenous to* $T \times L$ *with* T *a torus and* L *is, up to an action of a torus, isogenous* (*i.e.*, $\leqslant$) *to* (*up to a finite covering, this induces an embedding*) *a product of a parallelizable manifold and several reduced primitive compact complex homogeneous spaces such that each projection of the image is onto.*

To have another description, we need the following definition: A complex homogeneous space is called a *complete reduced primitive space* if it is a primitive space and is in a minimal isogeny class respect to the partial order $\leqslant$. Then we have:

Theorem G. *Any reduced space* M *of* 1*-step is in an isogeny class* $\geqslant$ *a product* M_0 *of a parallelizable manifold and some complete reduced primitive homogeneous spaces. In particular,* M *is a homogeneous* M_0 *bundle over a torus.*

One can also see the last part of Section 6 for a more detail construction.

1.5. We will consider the general spaces in detail in [13]. To complete the picture here, we call a compact complex homogeneous space a *primary space* if the 1-step space obtained by modulo the right action of the commutator of the nilpotent radical of $N_G(H^0)/H^0$ on its universal covering (see Section 4, for example) is a complete reduced primitive space and prove the following *Main Theorem II*:

Theorem H. *Any reduced space* M *is in an isogeny class* $\geqslant$ *a product* M_0 *of a parallelizable manifold and some primary spaces. In particular,* M *is a homogeneous* M_0 *bundle over a torus.*

These theorems give a classification of compact complex homogeneous spaces up to building blocks.

In Section 2 of this paper, we give some basic background for compact complex homogeneous spaces and general representation theory of a semisimple Lie algebra. In Section 3, we give some results on the representation theory of a k-linear algebraic group over an algebraic number field k. In Section 4 we give a complete proof of Theorem A, which is the Lie algebra foundation for Theorem B and part of Theorem C. In Section 5 we shall deal with the representational part of the Lie algebra aspect of Theorem C and prove Theorem D, which is the core of this paper. Finally, in the last section, we finish the global picture of Theorem B and that of Theorem C and those of the others.

[4] It turns out that their results are similar to the last three paragraphs in our Section 5.

2. Preliminaries

2.1. A *rational homogeneous manifold* Q is a compact complex manifold which can be realized as a closed orbit of a linear algebraic group in some projective space. Equivalently, $Q = S/P$ where S is a complex semisimple Lie group and P a parabolic subgroup, i.e., a subgroup of S which contains a maximal connected solvable subgroup (Borel subgroup). Every homogeneous rational manifold is simply-connected and is therefore an orbit of a compact group. In general, a quotient K/L with K compact and semisimple carries a K-invariant complex structure which is projective algebraic if and only if L is the centralizer $C(T)$ of a torus $T \subset K$.

A *parallelizable complex manifold* is the quotient of a complex Lie group by a discrete subgroup [31]. It is a *solv-manifold* or *nil-manifold* according as the complex Lie group is solvable or nilpotent. In the same way, we can define *reductive parallelizable manifolds* and *semisimple parallelizable manifolds*.

2.2. We recall Tits' result [29] on the fibration of compact homogeneous spaces.

Proposition 2. *Let G be a connected complex Lie group and H a closed complex subgroup such that G/H is compact. Then $G/\operatorname{Norm}_G(H^0)$ is a rational homogeneous space and $\operatorname{Norm}_G(H^0)/H$ is connected and parallelizable. Moreover, if $G/H \to G/R$ is a holomorphic fibration with parallelizable fiber R/H, then $R \subset \operatorname{Norm}_G(H^0)$; if in addition the base G/R is rational homogeneous, then $R = \operatorname{Norm}_G(H^0)$.*

2.3. Here we collect some results we need from the representation theory of the semisimple Lie algebras (cf. [17, pp. 67–69, 113]). Let s be a semisimple Lie algebra, t a Cartan subalgebra, Δ an ordered root system, Δ^+ the positive roots. We let $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ and $\{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots. We also let $\{H_1, \ldots, H_l\} \subset t$ be a set of elements dual to the simple roots such that $2(H_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$. We have the following proposition.

Proposition 3. *Let s be a semisimple Lie algebra. Then*:

(a) *An element in t is a highest weight for an irreducible representation if and only if it can be expressed as $\sum a_i H_i$ with a_i nonnegative integers.*
(b) $\delta = \sum H_i$.
(c) $H_i = \sum_j a_{ij}\alpha_i$ *with positive a_{ij}.*
(d) *Let π_i be the representation corresponding to H_i. Then the unique irreducible representation with highest weight as in* (a) *is a submodule of $\bigotimes(\pi_i)^{a_i}$ generated by the highest weight vector which is the tensor product of the highest weight vectors of π_i.*

The statements (a), (b), and (d) come from the standard representation theory, while (c) will be explicitly described in Appendix A, which is very useful in this paper.

2.4. Here we give some algebraic concepts which we used throughout this paper.

Let k be an algebraic number field. Therefore, $\mathbf{Q} \subset k$. Then, there are only finite many embeddings of k to $\mathbf{C}$, which are identity maps over $\mathbf{Q}$. If there is no embedding τ of k such that $\tau(k) \subset \mathbf{R}$, we call k a *total imaginary* algebraic number field.

For any ring A with identity, let a, b be two elements of A, $Q = A + Ae_1 + Ae_2 + Ae_3$ be a A algebra with the condition that $e_1^2 = a$, $e_2^2 = b$, $e_3 = e_1e_2 = -e_2e_1$. We call Q a *quaternion algebra* (a, b) over A. If Q is a division algebra, we call Q a *quaternion* (a, b). For the details of the quaternions, we refer the readers to [27, 2.11, 6.4].

For any algebraic number field k, let v be a valuation (or an *absolute value*). Then the valuation v gives a metric on k. One denote k_v to be the completion. For more details of the field k_v, we refer the readers to [24, Chapter 1] and [27, 5.6, 6.4].

3. Representation of k-linear algebraic groups over an algebraic number field

Here we collect some results on the representation theory of the k-algebraic reductive groups with k an algebraic number field. First we have (see [4, p. 87, Theorem 2]):

Theorem 1. *Every finite-dimensional representation of an k-linear algebraic reductive group* (*i.e., k points of an* $\mathbf{C}$*-algebraic reductive group*) *is completely reducible.*

Proof. Let M be a module of G_k, then $M \otimes_k \mathbf{C}$ is a $G_{\mathbf{C}}$ module. If N is a submodule of M, $N \otimes_k \mathbf{C}$ is a submodule of $M \otimes_k \mathbf{C}$. Since $M \otimes_k \mathbf{C}$ is completely reducible, there is a projection e from $M \otimes_k \mathbf{C}$ to $N \otimes_k \mathbf{C}$. Let h be a k-linear map from $\mathbf{C}$ to k such that $h|_k$ is the identity. Then we get a projection π from M to N by $m \to m \otimes 1 \to (1 \otimes h)e(m \otimes 1)$ and $M = N \oplus \ker(\pi)$. This implies that M is completely reducible. $\square$

Theorem 2. *If an absolutely simple k-algebraic group G of type A_l^2 over a total imaginary number field is anisotropic, then G is $SU_m(D, f)$, $m = 1, 2$ with a central division field D of dimension $((l + 1)/m)^2$ over a quadratic extension field of k.*

Proof. We see from [24, p. 88, Proposition 2.18] that $G = SU_m(D, f)$, where D is a central division field over a quadratic extension field of k and $f : D^m \times D^m \to D$ is a hermitian form with an involution of second kind over D. Now by [24, p. 86, Proposition 2.15], we have that G is isotropic if and only if f is isotropic.

From [27, p. 373, Theorem 6.2] we have that f is isotropic if and only if it is locally isotropic. And f is locally isotropic for an imaginary valuation by the discussion of the decomposable case in [27, p. 374] ($k_{\mathcal{P}} = \mathbf{C}$). Therefore, we only need to consider the p-adic case. For the p-adic case we have by [27, p. 353, Theorem 2.2(ii)] that D actually splits and is $M(n, K)$ with involution $\tau(a) =^t \bar{a}$ where $^-$ is the automorphism of K induced by a nonzero element of the Galois group.

In the p-adic case we apply the Morita theory as in [27, p. 362, Lemma 3.5]. Let $e_i = E_{ii} = \mathrm{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$ and $\tilde{f} = f(e_1, e_1)$ which is a hermitian form on Ve_1 over K/k. It is not difficult to see that $x = \sum xe_i = \sum xE_{1i}e_1E_{1i}$. The argument there implies that f is isotropic if and only if $\tilde{f}$ has Witt index $\geqslant n$.

Now if $m > 2$, then $\tilde{f}$ is a hermitian form of dimension $mn \geqslant 3n$ over K and hence of dimension $\geqslant 6n > 4$ over k. We observe that it is isotropic over k and hence over K. Let x_1 be an isotropic vector in Ve_1 and $y_1 \in Ve_1$ such that $\tilde{f}(x_1, y_1) \neq 0$, $V_1 = Ve_1/(Kx_1 + Ky_1) = \{x_1, y_1\}^{\perp}$. Then $\tilde{f}$ induced a hermitian form on V_1 and V_1 has dimension $\geqslant 6n - 4$ over k. In this way, we can get x_i, y_i, V_i for $i < n$ and V_{n-1} has dimension $\geqslant 6n - 4(n-1) = 2n + 4 > 5$ over k. Therefore, the Witt index $\geqslant n$, and G is isotropic. □

Remark.

(1) Another argument for the last part of this proof is: for a p-adic local field $D_p = M(n, K_p)$, every hermitian form is isometric to $h = \langle a_1, \ldots, a_m \rangle$ with a_i diagonal matrics with coefficiences in k_p. It represents 0 if each component of h represents 0 as a hermitian form over K_p. But all these n equations are independent, that is, there is a solution for $m > 2$ which is nonzero for each component and h is isotropic.

(2) But the original argument also shows that for $m = 2$, $\tilde{f}$ can be anisotropic. We might choose a quadratic extension K such that $K = k(\sqrt{d})$ with (d, π) is the unique k_p nonsplit quaternion and hence $\langle 1, -d, -\pi, d\pi \rangle$ is anisotropic. We just let $\tilde{f} = \langle 1, -\pi, 1, -1, \ldots, 1, -1 \rangle$.

Corollary 1. *In the case that D is a quaternion, G is A_3^2.*

We also need the following theorem for the representation of the reductive k-linear algebraic groups.

Theorem 3. *Any k-irreducible representation of a reductive k-linear algebraic group G which is non-trivial to its semisimple part is a sum of several copies of similar* (*i.e., the highest weight are same up to the Galois group*) *k-irreducible representations of its semisimple part.*

Proof. By Tits' result on the representation theory of k-semisimple groups (see [30, Theorem 7.2]), we observe that the irreducible representation of a semisimple group is determined by the highest weight up to the action of the Galois group. Now any irreducible representation of G which is non-trivial to its semisimple part is a sum of irreducible representations over $\mathbf{C}$ which is non-trivial to its semisimple part. The G irreducible representation contains just the representations over $\mathbf{C}$ which have the same highest weight up to the Galois group action and the G action (the action on G may not be the same as that on the semisimple part of G). □

For further results on the representation theory of a k-reductive linear algebraic groups, we refer the readers to [30].

4. Determination of the simple factors of a compact parallelizable manifold

In this section, we consider M to be a compact complex parallelizable manifold as in Theorem A. Here we need the following theorem of Wang (see [25, p. 150, Corollaries 8.27, 8.28]).

Proposition 4. *Let N be the maximal closed connected normal nilpotent subgroup of G and R be the radical, then $R/\Gamma \cap R$ and $N/\Gamma \cap N$ are compact parallelizable manifolds.*

This theorem give us a tower of two fibrations $G/\Gamma \to G/N\Gamma \to G/R\Gamma$. By [25, p. 31, Corollary 1] we have that $[N,N]/\Gamma \cap [N,N]$ is a compact complex parallelizable manifold. This gives us another tower of two fibrations $G/\Gamma \to G/[N,N]\Gamma \to G/N\Gamma$. Since each factor of S_1 (respectively S_2) acts on N trivially (respectively non-trivially) we have that each factor of S_1 (respectively S_2) acts trivially (respectively non-trivially) on $N/[N,N]$. To prove Theorem A we can assume that N is abelian.

Since $[G,R] \subset N$, S_1S_2R/N is reductive. By Lemma 14 in Section 6 the kernel G_1 of the action of this group on N has a discrete cocompact subgroup Γ_{G_1} and there is a fibration $(G/N)/(\Gamma N/N) \to ((G/N)/G_1)/\Gamma^1$, where $\Gamma^1 = ((\Gamma N/N)G_1)/G_1$. Since N is abelian, by using $(G/N)/G_1$ instead of G/N and $\Gamma^1(\Gamma \cap N)$ instead of Γ, we can assume that G/N acts almost faithfully[5] on N.

Now we observe that the lattice $\Gamma^2 = \Gamma \cap N$ is isomorphic to $\mathbf{Z}^{2\dim N}$. By Γ^1 acting on[6] Γ^2, we observe that S_2 is isogenous to the real form of a $\mathbf{Q}$-anisotropic algebraic group as the semisimple part of a reductive $\mathbf{Q}$-linear algebraic group (see [24, p. 58, Theorems 2.3, 2.4]. By taking the algebraic closure of G/N if it is necessary, we shall have an algebraic $\mathbf{Q}$-group). By the classification of $\mathbf{Q}$-algebraic groups, we have $S_2 = (\prod_i \mathbf{R}_{k_i/\mathbf{Q}} G_i)_\mathbf{R}$ with G_i a (absolutely) simple k_i-algebraic group, i.e., $(G_i)_\mathbf{C}$ simple. But $(\mathbf{R}_{k_i/\mathbf{Q}} G_i)_\mathbf{R} = (G_i)^s_\mathbf{R}(G_i)^t_\mathbf{C}$ if k_i has s real embeddings and $2t$ complex embeddings into $\mathbf{C}$. Since S_2 is a complex group, we have that $s = 0$ and k_i is a totally imaginary algebraic number field. By the following theorem (see [24, p. 352, Theorem 6.25]), we have our Theorem A.

Proposition 5. *Let G be a simple anisotropic group over a totally imaginary number field. Then G is of type A_l.*

5. Determination of the triple B, A, and I

In this section, we shall deal with the Lie algebra aspects of our theorems. We notice that arguing as above we can reduce Theorem D to the situation where the nilpotent radical

[5] This paragraph is unnecessary if we pass to the $\mathbf{Q}$-group structure as in the following paragraph. However, the construction here will be used later.

[6] After we finished this paper, we were told about [33] and Propositions 2.8.1 and 2.8.2 there by a referee in 1998. Proposition 2.8.2 is similar to our Theorem A. In the proof of Proposition 2.8.2, the construction of the semiproduct of S_2 and N, which corresponds to our paragraph, does not work here since the discrete subgroup of S_2 alone might not keep $N \cap \Gamma$ invariant.

N_1 of $G_1 = J/H^0$ is abelian. We can also reduce our Theorem D to the situation where the nilradical is a $\mathbf{Q}$-irreducible representation of G_1/N_1, which is either a non-trivial representation or a trivial representation for the semisimple part and with non-trivial B factors. To achieve these reductions, we have to do two things.

First, we want to reduce our situation to the case in which G_1/N_1 acts reductively on N_1. The torus part of G_1/N_1 as the center of G_1/N_1 is a $\mathbf{Q}$-torus and may not act reductively on N_1. But for each element $t \in \Gamma N_1/N_1$ we have $(\mathrm{ad}\, t)|_{N_1} = s + n$ with s semisimple and n unipotent rational actions (see [3, Chapter 7, Section 5, no. 9, Theorem 1]). Let $t_1, \ldots, t_k \in \Gamma N_1/N_1$ generate $\Gamma N_1/N_1$, then by noticing that the n_i, the nilpotent part of t_i, commute with each other, we observe that the subgroup Γ_0 of $\Gamma \cap N_1$ which is invariant under all n_i is not the identity. The action of G_1/N_1 keeps all the information of the action of the semisimple part of G_1/N_1 ($\Gamma \cap N_1$ can be regarded as a direct sum of subgroups of the copies of the subgroups in Γ_0 which can be regarded as irreducible representations of $\Gamma N_1/N_1$ appeared in Γ_0) and all the information of (B, A) is in Γ_0. Therefore, we can use $\Gamma_{0,\mathbf{R}}$ instead of N_1 without losing the information we need. Since all n_i act trivially on $\Gamma_{0,\mathbf{R}}$, we have that G_1/N_1 acts reductively on $\Gamma_{0,\mathbf{R}}$.

Second, we want to reduce our situation to the case in which the G_1/N_1 representation is irreducible. If we suppose $\Gamma_{\mathbf{Q}} = \Gamma_{N_1} \otimes \mathbf{Q} = C_1 \oplus C_2$ as a $\mathbf{Q}$ representation and p_1, p_2 are the projections, then $p_1(\Gamma_{N_1}) \oplus p_2(\Gamma_{N_1})$ is a lattice of dimension $\dim_{\mathbf{Q}} \Gamma_{\mathbf{Q}}$, i.e., the dimension of Γ_{N_1}. $\Gamma = H/H^0$ acts on both $\Gamma_1 = p_1(\Gamma_{N_1})$ and $\Gamma_2 = p_2(\Gamma_{N_1})$. So we can use either Γ_1 or Γ_2 instead of Γ_{N_1} by Borel's density theorem.

Lemma 1. *Let M be a compact complex homogeneous space. Let G be a connected complex Lie group of holomorphic automorphisms acting transitively and effectively on M, H be the isotropy subgroup, and $J = N_G(H^0)$ be the normalizer of H^0 in G. Let $G = SR$ be a Levi decomposition of G. Then with respect to a Cartan subalgebra in $\mathcal{S} \cap \mathcal{J}$, $\mathcal{H}$ decomposes into eigenvector spaces.*

If $h \in \mathcal{H}$ is an eigenvector with nonzero eigenvalue, then $h = h_s + h_r$ such that $h_s \in \mathcal{S} \cap \mathcal{H}$ and $h_r \in \mathcal{R} \cap \mathcal{H}$.

Proof. Since $J \cap S$ is parabolic, its Lie algebra contains a Cartan subalgebra in $\mathcal{S}$. Since $\mathcal{H}$ is an ideal of $\mathcal{J}$, it must be decomposed into its eigenvector spaces.

If h is an eigenvector with nonzero eigenvalue such that h is not in $\mathcal{R}$, then there is an $s = sl(2, \mathbf{C})$ generated by root vectors in $\mathcal{S}$ such that $h = h_s + h_r$ and $h_s \in s$, $h_r \in \mathcal{R}$ with weight α.

If $h_r \neq 0$, then there is an $h_r^- \in \mathcal{R}$ which is eigenvector with weight $-\alpha$ such that $[h_s, [h_s, h_r^-]] = -h_r$. We have $h_r, h_r^- \in \mathrm{nil}(\mathcal{G})$ and

$$\begin{aligned} h + \left[h, \left[h, h_r^-\right]\right] &= h_s + h_r + \left[h_s, \left[h_s, h_r^-\right]\right] + \left[h_r, \left[h_s, h_r^-\right]\right] + \left[h_s, \left[h_r, h_r^-\right]\right] \\ &\quad + \left[h_r, \left[h_r, h_r^-\right]\right] \\ &= h_s + [h_r, h_1] + \left[h_r^-, h_2\right] = h_s + h_r^2 \in \mathcal{H}, \end{aligned}$$

where $h_1, h_2 \in \mathrm{nil}(\mathcal{G}) := n$, hence $h_r^2 \in [n,n] := n_2$. In this way, we can find $h_r^k \in n_k := [n_{k-1}, n_{k-1}]$ such that $h_s + h_r^k \in \mathcal{H}$. By n being nilpotent, we have that $h_s \in \mathcal{H}$. And hence $h_r \in \mathcal{H}$ also. □

Lemma 2 (cf. [1]). *Let M be as in Lemma* 1 *and $\mathcal{S} = s_1 + s_2$ such that s_1 contains all the simple factors acting non-trivially on G/J. Then $\mathcal{G} = W_1 + \cdots + W_l + W_0$, where W_i are non-trivial s_1 irreducible representations for $1 \leqslant i \leqslant l$ and W_0 is a vector space containing all the s_1 fixed vectors. If $w_1, \ldots, w_l$ are the highest weight vectors, then they are linearly independent modulo $\mathcal{H}$. Moreover,* $\dim W_0 \leqslant \dim J/H$.

Proof. The direct sum comes from the representation theory of semisimple Lie groups. If $w = \sum a_i w_i \in \mathcal{H}$ and $p = \mathcal{J} \cap s_1$, then $[p, w] \subset \mathcal{H}$ and $[s_2 + \mathcal{R}, w] \subset \mathcal{H}$ since $\mathcal{H}$ is an ideal of $\mathcal{J}$. But $[B, w] = 0$ where B is the Borel subalgebra which contains all the positive root vectors, and we have that $[s_1, w] \subset \mathcal{H}$. Therefore, $m_1 = [\mathcal{G}, w] \subset \mathcal{H}$. And $[B, m_1] = [[B, \mathcal{G}], w] \subset m_1$. If we let $m_k = [\mathcal{G}, m_{k-1}]$ and assume that $m_k \subset \mathcal{H}$, $[B, m_k] \subset m_k$, then $m_{k+1} = [B + \mathcal{J}, m_k] \subset [[B, \mathcal{G}], m_{k-1}] + [\mathcal{G}, m_{k-1}] + \mathcal{H} \subset m_k + \mathcal{H} \subset \mathcal{H}$. Therefore, w generates a $\mathcal{G}$-ideal in $\mathcal{H}$. This implies that $w = 0$. We have that all the weight vectors w_i are linearly independent modulo $\mathcal{H}$.

All the vectors in W_0 correspond to the fiberwise actions of the bundle $G/H \to G/J$ being without any fixed point and invariant under the action of subgroup S_1 of G, which corresponds to s_1 and acts transitively on G/J. These vector fields are determined by their values at any fixed fiber of $G/H \to G/J$. We have $\dim W_0 \leqslant \dim J/H$. □

Lemma 3. *Let A be a simple factor of J which acts trivially on the radical $\mathcal{R}_J$ of $\mathcal{J}/\mathcal{H}$. Then the simple factor B of $\mathcal{G}$ which contains A acts trivially on $\mathcal{R}$.*

Proof. There is a Cartan subalgebra contained in $B \cap \mathcal{J}$ such that A is generated by the $\mathcal{G}_\alpha$ for a set of the simple roots $(\alpha_1, \ldots, \alpha_k)$. By our assumption, all the negative root vectors $e_{-\alpha}$ such that the coefficient of some α_i, $i \in (1, \ldots, k)$ is nonzero are in $\mathcal{H}$. Hence the actions of any fundamental weight H_i corresponding to these simple roots are trivial, but the actions of H_i on $\mathcal{J} \cap \mathcal{R}/\mathcal{H} \cap \mathcal{R}$ have the same positive eigenvalue as for the highest weight vector for each non-trivial irreducible representation of B in $\mathcal{R}$. Therefore, B acts trivially on $\mathcal{R}$. □

Corollary 2. *If a simple factor B of $\mathcal{G}$ has an element in $\mathcal{J}$ which acts non-trivially on $\mathcal{R}_J$, then every element in the Cartan subalgebra of B acts non-trivially on the nilradical of $\mathcal{R}_J$.*

Proof. We may assume the element b is in the Cartan subalgebra. Otherwise we may assume that $b = \sum b_i \in \mathcal{B}, a = \sum a_j \in \mathcal{R}_J$ such that $[b, a] \neq 0$ with b_i, a_j being the eigenvectors. Then, there is an i and a j such that $[b_i, a_j] \neq 0$. If b_i is in the Cartan subalgebra, we are done. If b_i is not in the Cartan subalgebra, then either $e_1 = [b_i, a_j]$ is an eigenvector with a nonzero eigenvalue or $e_2 = a_j$ is an eigenvector with a nonzero eigenvalue. In both cases, we can easily obtain a b in the Cartan subalgebra which acts non-trivially on e_1 or e_2.

Now we want to prove that every element c in the Cartan subalgebra of B acts non-trivially on $\mathcal{R}_J$. If c is in the Cartan subalgebra of some factor A in Lemma 3, then we are done. If c is not in the Cartan subalgebra of any factor A, then $[c, e_{-\alpha}] \neq 0$ for a simple root α such that $e_{-\alpha} \in \mathcal{R}_J$. If corollary does not hold for this c, we have that $e_{-\alpha} \in \mathcal{H}$. Applying the argument in the proof of Lemma 3 to the fundamental weight H corresponding to α we have a contradiction. □

By the result of the last section, we know that the factors A in Lemma 3 must have the type of A_l. Now we want to discuss the possible representations of these factors on $\mathcal{R}_J$.

Lemma 4. *Every negative simple root eigenvector which is not in any A factor is a highest weight vector of a representation of A factors. Conversely, every highest weight vector of a representation of A factors as a subspace in some factor B of G comes from a negative simple root.*

Proof. If $e_{-\alpha_{i_0}}$ is H^0 for a simple root α_{i_0}, then all $e_\alpha \in H^0$ for any α such that α_{i_0} is a component of α. But the action of H_{i_0} on J/H^0 cannot be unimodular by Proposition 3(c).

The second statement follows from the first statement, otherwise the nilradical of J/H^0 is not abelian. □

The application of the unimodular property in the proof of this lemma is the basic method we used in both this paper and in [12].

Now we come to the point of clarifying the possibility of the representations. The representations of A_l are classified by the fundamental weights. If the highest weight is H_1 (respectively $H_1 \oplus H_l$), the representation comes from the standard one of SL_{l+1} (respectively SU_{l+1}). And the $2H_1$ (respectively $2H_1 \oplus 2H_l$) comes from the symmetric quadratic form representations. The H_2 (respectively $H_2 \oplus H_{l-2}$) comes from the antisymmetric quadratic form representation.

Lemma 5. *The only possible representation of A on $\mathcal{R}_J$ are*:

(1) H_1;
(2) $2H_1 \oplus 2H_3$, $B = C_4$ $A = SU_2(D, f)$;
(3) $2H_1$, $B = B_2$ $A = SL_1(D)$;
(4) $H_1 \oplus H_l$;
(5) H_2, $B = D_4$ $A = SU_2(D, f)$,

where D's are quaternions.

Proof. Suppose that B is of type A_n, then the highest weight vector in B must be $e_{-\alpha}$ for a simple root α. It must be type of H_1.

Suppose that B is of type B_n. If A is generated by α_k, $k < n$, the highest weight vector is $e_{-\alpha_k}$, $k \leqslant n$. It is a type of H_1.

If A is generated by $e_{-\alpha_n}$, then the representation of A is of type $2H_1$. This situation occurs only if $n = 2$. If $n > 2$, then the other simple factor must be generated by

$e_{-\alpha_i}, i < n-1$. And this cannot happen, since the only possible representation of B is either $H_1 + 2H_n$ (if A is of type A^2) or $H_{n-2} + 2H_n$ (if A is of type A^1). But the action of H_{n-1} on the representation of A in B has an eigenvalue $\langle H_{n-1}, -\alpha_{n-1}\rangle = -1$, while the action of H_{n-1} on both possible representations of B have eigenvalues $\langle H_{n-1}, H_1+2H_n\rangle = 1+n-1 > 1$ and $\langle H_{n-1}, H_{n-2}+2H_n\rangle = n-2+n-1 = 2n-3 > 1$ (for the calculation of H_i we refer to the Appendix A).

Suppose that B is of type C_n, then the highest weights are either $e_{-\alpha_k}, k < n$, which are of type H_1, or $e_{-\alpha_n}$, which is a type of $2H_1$.

The second case occurs only when A is of type A^2_{n-1}, which must be $SU_m(D, f)$. By counting the multiplicity of the representation (see [30, 8.2]), we observe that D must be a quaternion, therefore $n = 4$ or 2. In the case $n = 2$, $B = B_2$. Therefore, $n = 4$.

Suppose that B is of type D_n, the representation is either H_1 or H_2.

The later case occurs only if the highest weight is $-\alpha_n$ (respectively $-\alpha_{n-1}$), A is generated by α_i, $i < n$ (respectively $i < n-1$ and $i = n$). By the multiplicity of the representation (see [30] again), we obtain that A is of type $A^2_3 = SU_2(D, f)$ for a quaternion D. In this case $n = 4$.

Suppose that B is of type E_k, $k = 6, 7, 8$, then the representation are of type H_1 except the case that the highest weight comes from α_{k-2} which is the root at the end of the shortest branch of the graph and A is generated by other simple roots, which is a type of H_3. The H_3 case cannot occur by calculating both $\langle H_{k-2}, H_3\rangle = 9/(9-k) > 1$ and $\langle H_{k-2}, H_{k-3}\rangle = (3(k-3))/(9-k) > 1$.

Suppose that B is of type F_4, the representations are of type H_1 except the case that the highest weight comes from α_2 with A generated by α_i, $i = 3, 4$ (refer to the Appendix A), which is a type of H_2. This cannot happen since both $\langle H_2, H_1 + 2H_3\rangle = 22 > 2$ and $\langle H_2, H_1 + 2H_4\rangle = 14 > 2$.

Suppose that B is of type G_2, then the representation are of type H_1 except the case the highest weight is α_2 and A is generated by α_1, which is a type of $3H_1$. This cannot happen since $\langle H_2, 3H_1\rangle = 9 > 3$. □

As in the proof of the last lemma, we observe:

Corollary 3. *If*

$$\langle H_i, \alpha_i\rangle < \langle H_i, H_j\rangle \tag{1}$$

for all j then α_i must be in some factor A.

We then have the following lemma.

Lemma 6. *Any factor B must be a classical Lie group.*

Proof. If B is of type G_2, the only possible situation in the last lemma is that A is generated by α_2 and α_1 is not in A. We check that in this case the inequality (1) holds, therefore it cannot occur.

If B is of type F_4, the only i for which the inequality (1) does not hold is $i=1$ with $j=4$. But this cannot happen, otherwise there must be another i for which the inequality does not hold since A must be type of A_l.

We observe that the inequality always holds for E_8.

If B is of type E_7, the possible i's for which the inequality does not hold are $i=1$ with $j=7$, or $i=7$ with $j=1$. In both cases $i=5$ should not be in any factor A, since A must be type of A_l. But the inequality holds for $i=5$, therefore both cases cannot occur.

If B is of type E_6, the only H_i such that certain coefficients of α_j are $\leqslant 1$ are $H_1=\frac{1}{3}(4\alpha_1+5\alpha_2+6\alpha_3+3\alpha_4+4\alpha_5+2\alpha_6)$, $H_4=\frac{1}{3}(3\alpha_1+6\alpha_2+9\alpha_3+6\alpha_4+6\alpha_5+3\alpha_6)$, and $H_6=\frac{1}{3}(2\alpha_1+4\alpha_2+6\alpha_3+3\alpha_4+5\alpha_5+4\alpha_6)$. The only possible i's are $1, 4, 6$. When $i=1$, $j=4$ ($j=6$ cannot occur, otherwise we need another k such that $\langle H_1,\alpha_1\rangle \geqslant \langle H_1, H_6+H_k\rangle$. But $\langle H_1, H_6+H_k\rangle > \langle H_1,\alpha_1\rangle$ always) and A is generated by α_k, $k=4,3,2$, this cannot occur. In the same way, we observe that $i=6$ does not occur. We also observe that if $i=4$, A is generated by α_k, $k=1,2,3,5,6$, this does not occur by the last lemma.

We also observe that for the non-classical groups, all linear sum of H_i's with nonnegative integer coefficients cannot be the sum of the simple roots since at least one of the coefficients of some α_i is >1. Therefore, the case of abelian nilradical does not occur. $\square$

Lemma 7. *If B is a classical group of a type other than A_n, then one of the situations in the lists after Theorem D holds.*

Proof. In the case of D_n, we have two situations: (1) α_1 is not in any A. (2) α_1 is in some A.

If (1) occurs, then there is an α_i, $i>2$ which is not in any A. We observe that i must be $n-1$ and n. In this case the highest weight of the representation of B in $\mathcal{R}$ must be H_2. A can be $SL_1(D)$ or $SU_2(D,f)$ for a quaternion D.

If $A=SL_1(D)$ then $n=4$. The torus commuting with $SL_1(D)$ and acting not as a multiplication of a constant on each $\mathbf{C}$ component of D^2 corresponding to each B must be a torus in $SL_2(D)$ and has at most dimension 3.

If $A=SU_2(D,f)$ for a quaternion D over a quadric extension K/k, then $n=6$. Let a be the non-trivial element of the Galois group of K/k. Then by H_1 acting trivially on the representation generated by α_k, $k=5,6$, the torus generated by H_1 is defined over K. The torus generated by H_1, H_5-H_6 is invariant under a and is a k-torus, which comes from a maximal torus in $SL_1(D)$ multiplying from the other side. And the torus generated by $H_1-H_5-H_6$ is a k-torus as the multiplication of the elements in K whose k-determinants are 1.

If (2) occurs, then A is either generated by α_i, $i<n-1$ or by α_i, $i<n$. By $n>3$, we observe that in the first case we have $n=5$ by counting multiplicities (see [30] again) and $E=H_1\oplus H_1$. This can only occur if there is another 1-dimensional $\mathbf{C}$ torus which acts on one copy of H_1 as a and on the other as a^{-1} since the two copies of H_1 have the same eigenvalues for any element in the torus and this is not true for the other two representations coming from α_i, $i=4,5$.

And in the second case, we have $n = 4$ by counting multiplicities (see [30]) and $E = H_2$. Thus the torus comes from the K multiplication of elements whose determinants are 1.

In the case of C_n, either (1) α_1 is not in any A, we have $n = 2$, i.e., $B = B_2$; or (2) α_1 is in some A, this only happen in the situation of Lemma 5(2).

In both the D_n and C_n cases, there is no linear combination of H_i's with nonnegative integer coefficients such that it is the sum of all simple roots. Therefore, the abelian case cannot occur.

But for B_n, H_1 is the sum of all simple roots. Therefore, we have (1) in the second list after Theorem D.

In the case of B_n, either α_1 is not in any A or α_1 is in some A. In the first situation, we have $n = 2$, which is just the case (3) of Lemma 5. Otherwise, there is another α_i which is not in any A, then $i > 2$ and this can not happen.

In the second situation, α_1 is in an A. Then $n = 2$ and $A = SL_1(D, f)$ which is generated by α_1. □

Now we are coming to the most difficult case of A_n.

Lemma 8. *In the case of A_n, if there is an α_i which is not in any factor A and $e_{-\alpha_i}$ is the highest weight vector of the product of two factors $A_1 = SL_{l_1+1}$ and $A_2 = SL_{l_2+1}$, then $E = H_1 + H_n$ and we have the following cases*:

(1) $B = A_{2l+1}$, $A_1 = SU_1(D, f_1)$, $A_2 = SU_1(D, f_2)$, $i = l + 1$, $l > 1$.
(2) $B = A_{4l+3}$, $A_1 = SU_2(D, f_1)$, $A_2 = SU_2(D, f_2)$, $i = 2l + 2$, $l > 0$.
(3) $B = A_3$, $A_1 = SL_1(D)$, $A_2 = SL_1(D)$, $i = 2$.
(4) $B = A_{3l+2}$, $A_1 = SU_1(D, f_1)$, $A_2 = SU_2(D, f_2)$, $i = 2l + 2$.

And in all these cases, the representation generated by $e_{-\alpha_i}$ as a representation of A is of type $H^1_{l_1} \otimes H^2_1$.

Proof. In this case, we must have

$$\langle H_i, \alpha_i\rangle \geqslant \langle H_i, H_j\rangle + \langle H_i, H_k\rangle \tag{2}$$

for some $k < i < j$ such that $H_j + H_k$ provides the highest weight for another representation in R of the factors A_1 and A_2 with A_2 next to A_1. But we have that the coefficient of α_i in $H_k + H_j$ is

$$\begin{aligned}&\frac{1}{n+1}\big((n+1-i)k + (n+1-j)i\big)\\&\quad = \frac{1}{n+1}\big(n+1+(k-1)(n+1-i) - i + (n+1-j)i\big) \geqslant 1,\end{aligned}$$

and is equal to 1 if and only if $k = 1$, $j = n$. That is, $A_1 = SL_{l_1+1}$ and $A_2 = SL_{l_2+1}$ are both of type A^2, and there is no other simple root outside $A_m, m = 1, 2$ which is not α_i.

We observe that A_m are $SU_{k_m}(D_m, f_m)$, $m = 1, 2$. By counting the multiplicities of the representations (apply [30] again), we also observe that D_1 is the opposite of D_2. □

Lemma 9. *If there is no $e_{-\alpha_i}$ which is a highest weight vector of a product representation of factors A_i, and $B \neq A_3$, then in the case* (1) *of Lemma* 5, $A = SL_1(D)$, $l = 1$ *and in the case* (4) *of Lemma* 5, $A = SU_2(D, f)$, $l = 3$, *where D is a quaternion. In the case $B = A_3$, the only possible case is $A = SL_1(D)$ with D a division field of degree* 3.

Proof. We only need to consider $B = A_n$ and only need to check the situation near α_1.

If α_1 is not in any A, we have two cases: (1) A is $SL_1(D)$, or (2) A is of type A^2. In the first case, the other representations of the A factor next to α_1 come from the representation of B in R involving the representation of B with highest weight H_2. We want to see that there is only one of them. The coefficient of α_1 in H_2 is

$$\frac{1}{n+1}(n+1-2) = 1 - \frac{2}{n+1},$$

that is, the twice of this coefficient is > 1 since $n > 3$. By counting the multiplicity, we observe that D must be a quaternion.

In the second case, the same consideration shows that D is a quaternion.

If α_1 is in some A, we let α_i be the first simple root which is not in any A. Then the coefficient of α_i in the H_{i-1} will be

$$\begin{aligned}\frac{1}{n+1}(n+1-i)(i-1) &= 1 + \frac{1}{n+1}\big((n+1-i)(i-2) - i\big)\\ &= 1 + \frac{1}{n+1}\big((n-i)(i-2) - 2\big).\end{aligned}$$

Twice of this coefficient is > 1 if $n > 3$. As before, we observe that lemma holds.

In the case $B = A_3$, A can be chosen to be generated by α_i, $i = 2, 3$. If $A = SL_1(D)$ then D is a division field of degree 3. If A is of type A^2, the coefficient of α_1 in $H_2 + 2H_3$ is $\frac{4}{3} > 1$, therefore this cannot happen. We want to see that the case $A = SL_1(D)$ can occur only if there is a 1-dimensional **C** torus acting on one copy of the representation of B with highest weight H_2 as $a : x \to ax$ and the other as $a : x \to a^{-1}x$. Otherwise, any element of the torus acts on the two copies of the representation which comes from H_2's of B with the same eigenvalues which is different from that of the representation coming from $e_{-\alpha_1}$, that is, the representation of A cannot be irreducible. □

Lemma 10. *If $B = A_n$ and the A factors are copies of $SL_1(D)$ which do not have any joint representation, then we have one of the following*:

(1) $n = 3$, *A is generated by α_2, the representation of B is $H_2 \oplus H_2$.*
(2) $n = 2$, *A is generated by α_2, the representation of B is either $H_2 \oplus H_2 \oplus H_2$ or $H_1 + H_2$.*
(3) $n = 3l + 1$, *the A factors are generated by α_{3i+1}, $0 \leqslant i \leqslant l$, the B representation is $H_1 \oplus H_n$.*

Proof. If α_1 is not in any A, the coefficient of α_1 in H_2 is $(n+1-2)/(n+1)$. $2(n+1-2)/(n+1) \geqslant 1$ if $n > 2$, and it is equal to 1 if and only if $n = 3$. This proves (1). If $n = 2$, the coefficient is $\frac{1}{3}$ we have (2).

If α_1 is in some A, then the coefficient of α_2 in H_1 is $(n-1)/(n+1)$. Twice of this coefficient is $\geqslant 1$ since $n > 3$. Therefore, we can have only one representation of B with highest weight H_1. As we observed in the proof of Lemma 8, the other representation of B must be the representation with highest weight H_n. Therefore, α_n is in some A and $n = 3l + 1$. This is (3). □

Lemma 11. *If $B = A_n$ and the A factors are copies of $SU_2(D, f)$ and do not have any joint representation in B, then we have one of the following*:

(1) *$n = 4$, A is generated by α_i, $i = 1, 2, 3$, the representation of B is $H_1 \oplus H_1 \oplus H_3$.*
(2) *$n = 5l + 3$, A is generated by α_i, $i = 5k + 1, 5k + 2, 5k + 3$. the representation of B is $H_1 \oplus H_n$, but this does not occur.*

Proof. If α_1 is in some A, the coefficient of α_4 in $2H_1 + H_3$ is

$$\frac{1}{n+1}\big(2(n-3) + 3(n-3)\big) = 1 + \frac{4}{n+1}(n-4) \geqslant 1$$

since $n > 3$ and $= 1$ if and only if $n = 4$. We have (1).

If $n > 4$, we have a representation of B with highest weight H_1 and the other representation can only be the representation with highest weight H_n as before. We have (2). But this does not occur as we see by counting the multiplicities. □

Lemma 12. *If $B = A_n$ without any factor A, then the nilradical of $\mathcal{R}$ is generated by all the simple root vectors and*

(1) *the highest weight vector of $H_1 + H_n$, or*
(2) *those of H_1 and H_n.*

Proof. In this case $\mathcal{R}_J$ is generated by $e_{-\alpha_i}$ for all the simple roots and some highest weight vectors. We observe that the only possible representations of B such that the sum of the highest weights is the sum of the simple roots are those in the lemma. □

In considering the situation that the representation of the semisimple part of J/H^0 is trivial on the abelian N, we need to look more closely at the structure of the Lie group $F = J/H^0$. $F = S \times T_1T_2N$ with T_1 a maximal reductive subgroup in the radical and T_2 a T_1 invariant complement of N in the Lie group T_2N. Without loss of generality, we consider the case in which S is the identity. The lattice of F induces a lattice in T_1T_2 as the quotient group F/N and induces a $\mathbf{Q}$ structure $T_{\mathbf{Q}}$ of F/N regarded as a subset of T_1T_2. With the $\mathbf{Q}$ structure $N_{\mathbf{Q}}$ of N, we obtain a $\mathbf{Q}$ structure $F_{\mathbf{Q}} = T_{\mathbf{Q}}N_{\mathbf{Q}}$ of F. Now we consider the subgroup F_1 of F which is generated by $T_{\mathbf{Q}}$. Then $N^1_{\mathbf{Q}} = F_1 \cap N_{\mathbf{Q}}$ is invariant under the action of $F_{\mathbf{Q}}$, i.e., invariant under the adjoint action of $T_{\mathbf{Q}}$. But T_1 acts trivially on $(N^1_{\mathbf{Q}})_{\mathbf{R}}$, this cannot happen in the situation in Lemma 12. We observe that $N^1_{\mathbf{Q}}$ is trivial, i.e., T_1T_2 is actually an abelian group.

The second case in Lemma 12 can actually occur, e.g., if there is an extra 1-dimensional torus $\mathbf{C}^*$ which acts on H_1 the same as a but on H_n as a^{-1}. In this situation the group F is exactly as those in $B = B_{n+1}$ and the first case of the Lemma 12 with $B = A_{n+1}$. To prove this, we can modify the construction in [2, pp. 95–96] as follows: first $F = T\mathbf{C}^{n+2}$ as a semidirect product with action of T on $\mathbf{C}^{n+2}$ defined by $(t_1, \ldots, t_n) \otimes a \in T_n \times \mathbf{C}^* = T \subset SL(n+1, \mathbf{C}) \times \mathbf{C}^*$:

$$\begin{aligned}\mathbf{C}^{n+2} &= \mathbf{C}^{n+1} \times \mathbf{C} \ni (z_1, \ldots, z_{n+1}) \oplus (z_{n+2}) \\ &\to \big(\hat{\alpha}_1(t)z_1, \ldots, \hat{\alpha}_n(t)z_n, \hat{h}_1(t)az_{n+1}, \hat{h}_n(t)a^{-1}z_{n+2}\big) \in \mathbf{C}^{n+2}\end{aligned}$$

and define a homomorphism

$$\begin{gathered}\varphi : T\mathbf{C}^{n+2} \to D\mathbf{C}^{n+2}, \\ \varphi(t, z) = \operatorname{diag}\big(\hat{\alpha}_1(t), \ldots, \hat{\alpha}_n(t), \hat{h}_1(t)a, \hat{h}_n(t)a^{-1}, z\big).\end{gathered}$$

It can also happen that several pairs of $(A_{n_i}, H_1 + H_{n_i})$ with different $\{n_i\}$ and several pairs of $(A_{n_j}, H_1 \oplus H_{n_j})$ with different $\{n_j\}$ as well as several pairs of (B_{n_k}, H_1) with different $\{n_k\}$ occur together. For example, if we have a torus D_n acting on $\mathbf{C}^{n+1}$, we choose a group of $\{n_i\}$, $\{n_j\}$, $\{n_k\}$ such that $\sum_i (n_i + 1) + \sum_j (n_j + 1) + \sum_k (n_k + 2) = n + 1$, and we regard the $\mathbf{C}^{n+1}$ as $(\bigoplus_i \mathbf{C}^{n_i+1}) \oplus (\bigoplus_j \mathbf{C}^{n_j+1}) \oplus (\bigoplus_k \mathbf{C}^{n_k+2})$ and apply above construction individually and regard $(\bigotimes_i T_{n_i}) \otimes (\bigotimes_j T_{n_j}) \otimes (\bigotimes_k T_{n_k+1})$ as the subgroup of T_n with each T_{n_i} (respectively T_{n_j}, T_{n_k+1}) acting only non-trivially on $\mathbf{C}^{n_i+1}$ (respectively $\mathbf{C}^{n_j+1}, \mathbf{C}^{n_k+2}$).

The above construction also works on the non-trivial twist product of two of the factors in the case (6) of Theorem E. For examples,

$$\begin{aligned}&(B_n, H_1) \times^* (A_m, H_1 + H_m) = \big(B_n \times A_m, H_1 \otimes (H_1 + H_m)\big) \quad \text{and} \\ &(B_n, H_1) \times^* (A_m, H_1 \oplus H_m) = (B_n \times A_m, H_1 \otimes H_m \oplus H_1).\end{aligned}$$

6. Global structure theorems

Now we are able to place our manifolds in a global structure. First, we prove some lemmas.

Lemma 13 (cf. [12]). *If G is a connected complex Lie group, H is a cocompact discrete subgroup. Then H is finitely generated.*

Proof. We consider the universal covering $\widetilde{G}$ of G and the preimage Γ of H. Then $\widetilde{G}$ can be regarded as a complex linear group by [16, p. 225, Theorem 4.6]. If Ω is a fundamental domain of Γ in $\widetilde{G}$, and if there are finite number of elements $\{\gamma_i|_{1\leqslant i\leqslant s}\}$ such that $\{\gamma_i\Omega|_{1\leqslant i\leqslant s}\}$ are all the fundamental domains next to Ω, then since $\widetilde{G}/\Gamma$ is compact $\{\gamma_i|_{1\leqslant i\leqslant s}\}$ generates Γ, that is, Γ is finitely generated. So is H. $\square$

Lemma 14 (cf. [12,26]). *Let G be a connected complex Lie group acting on another complex Lie group M as automorphisms and G/H be a compact complex parallelizable manifold with H being discrete such that H fixes a discrete cocompact subgroup N of M. If $C = C_G(M)$ is the centralizer of N in G, then $C/C \cap H$ is compact.*

Proof. Let $B = \{n_i|_{1\leqslant i\leqslant l}\}$ be a finite set of elements in N such that B generates N. Then $C = C_G(M) = C_G(N) = \{g \in G|_{\rho(g)n_i=n_i \text{ for all } i}\}$ by the main theorem in [18] (see also [19, p. 5, Proposition 2.4], here we regard G and M as subgroups of the semiproduct GM, i.e., $\rho(g)n_i = n_i$ is the same as $n_i g n_i^{-1} = g(\rho(g)n_i)n_i^{-1} = g$). Let Ω be a compact fundamental domain in G as in the proof of the last lemma. For any element $c \in C$, there is an element $h_c \in H$ such that $h_c c \in \Omega$. Therefore, $\{\rho(h_c)n_i\} = \{\rho(h_c c)n_i\}$ lies in a compact region $\bigcup \rho(\Omega n_i)$ of M. This means that there is a finite set $\{c_j|_{1\leqslant j\leqslant k}\}$ such that for any $c \in C$ there is a j with $h_{c_j}^{-1}h_c \in H \cap C_{n_i}$ for all i, where $C_{n_i} = \{g \in G|_{\rho(g)n_i=n_i}\}$. And hence $h_{c_j}^{-1}h_c \in H \cap C$. From this we easily observe that there exists a compact region $A = \bigcup h_{c_j}^{-1}\Omega \cap C$ of C such that for any $c \in C$, there is an element $c_c \in H \cap C$ such that $c_c = h_{c_j}^{-1}h_c$, hence $c_c c = h_{c_j}^{-1}h_c c \in h_{c_j}^{-1}\Omega \cap C \subset A$. We finally have that $C/C \cap H$ is compact. $\square$

Theorem 4. *Let G, S_1, S_2, R, H be as in Theorem C, and $G_1 = S_2R$. Then, up to a finite covering, $M = G/H$ is a holomorphic principal torus bundle on a product $S_1/H_1 \times G_1/H_2$, and if $J_1 = N_{S_1}(H_1^0)$ and $J_2 = N_{G_1}(H_2^0)$, then J_1/H_1^0 is semisimple and H_2^0 is unipotent. J_2 has only simple factors of type A_l. Each simple factor of S_2 is a classical group and each simple factor of J_2 acts non-trivially on $R/R \cap H$. The torus action comes from the center of J/H.*

Proof. We first prove the theorem in the case when M is parallelizable manifold. By Lemma 14, we observe that the centralizer of R is S_1C, where C is the center of G, and $S_1C \cap H$ is a discrete cocompact subgroup of S_1C. Similarly, $C/C \cap H$ and $S_2R/S_2R \cap H$ (since $S_2R = C_G(S_1C)$) are compact parallelizable manifolds. By $S_1C \cap S_2R = C$ we obtain the torus bundle over the product of $S_1/(HC^0/C^0)$ and S_2R/HC^0 up to a finite covering.

We apply the result in the parallelizable case to J/H in the general case and by the result of the last section we observe that S_1 is exactly the product of B factors whose Cartan subalgebra in J acts trivially on the radical part of J/H and S_2 is exactly the product of B factors whose Cartan subalgebra in J acts non-trivially on the radical part of J/H. $\square$

Proof of Theorems E, F. We observe from the list of Theorem D that the semisimple part of J/H^0 in S^1, i.e., the product of the A factors, consists of either copies of the same simple arithmetic group, or copies of the same product of two simple arithmetic group acting on joint irreducible representation with non-trivial actions, which occurs only in the cases (a1, 2, 3, 4) of the first list after Theorem D. Therefore, the situations in the list of Theorem E are the only possible combinations (see also the paragraph after the proof of Lemma 12). We have Theorem E.

If we have a 1-step compact complex homogeneous space, we assume that the nilradical of $F = J/H^0$ is a product of almost irreducible representations (i.e., these representations cannot be decomposed into a sum of non-trivial submodules) $\prod_{i=1}^{s} V_i$ of the adjoint action of the reductive quotient of F with $V_1, \ldots, V_k$ in the center of F, $V_{k+1}, \ldots, V_{k+l}$ not coming from any reduced primitive complex homogeneous space and $V_{k+l+1}, \ldots, V_s$ coming from primitive complex homogeneous spaces (i.e., their reduction to $(\Gamma_0)_{\mathbf{R}}$ as we described before Lemma 1 is a sum of irreducible G_1/N_1 representations with some non-trivial B factors). From the classification of primitive spaces, we observe that all the n_i in the construction before Lemma 1 *act actually trivially* on V_i for $i \in (k + l + 1, \ldots, s)$. Hence, V_i are *actually irreducible* for the G_1/N_1 action with $i \in (k + l + 1, \ldots, s)$. Then, we let $T = V_1 \ldots V_k$, P be the parallelizable manifold corresponding to $F_0 = F/V_1 \ldots V_k V_{k+l+1} \ldots V_s$ and Q be the parallelizable manifold corresponding to $F_0/C_{F_0}(F_0)$. We also let P_i be the primitive complex homogeneous space corresponding to V_i, $i > k + l$, and Q_i be its reduced primitive complex homogeneous space. Then, $S_2R/H_2 \to TP\prod P_i \to TQ\prod Q_i$ is isogenous to a homogeneous submanifold. We have Theorem F. $\square$

Notice that the isogeny in the proof of Theorem F may not be an onto homomorphism. For example, if we have two reduced primitive spaces $M^1 = G^1/H^1$ and $M^2 = G^2/H^2$ with Tits fibrations $P^i = J^i/H^i \subset M^i \to Q^i = G^i/J^i$ such that P^i is defined over the Gauss numbers $\mathbf{Q}(\sqrt{-1})$. Let T be a complex anisotropic torus with two $\mathbf{Q}(\sqrt{-1})$-representation V^i, $i = 1, 2$. Then

$$T_{\mathbf{C}} \times \left(G^1/H^{1,0}\right)^{\otimes \dim_{\mathbf{C}} V^1} \times \left(G^2/H^{2,0}\right)^{\otimes \dim_{\mathbf{C}} V^2}$$

will have a structure of a complex homogeneous space and it is a covering space of a compact complex homogeneous space which can be isogenous to a homogeneous subspace of a product of reduced primitive homogeneous spaces but the homomorphism is not onto.

To construct this example, we can assume that the nilradical of G_1 is $V = V^1 \otimes N^1 \oplus V^2 \otimes N^2$ where each N^i is the nilradical of M^i. The T action on V is the natural one. While we do not require that the $T_{\mathbf{C}}$ action be the complexification of T, we assume that the $T_{\mathbf{C}}$ acts trivially on the product of $\dim_{\mathbf{C}} V^i$ copies of semisimple part of G^i and properly on the nilradical of G^i such that if $e_1, \ldots, e_k, e_{k+1}, \ldots, e_l$ is a basis of eigenvectors of the T action on V with eigenvalue α_i and only $e_1, \ldots, e_k$ are in the nilradical of $(J^1/H^{1,0})^{\dim_{\mathbf{C}} V^1} \times (J^2/H^{2,0})^{\dim_{\mathbf{C}} V^2}$, and we let $T_{\mathbf{C}}$ act on e_i with eigenvalue $\alpha_i + (\alpha_{k+1} + \cdots + \alpha_l)/k$ for each $i < k + 1$. The action above can be extended to the whole nilradical of $(G^1)^{\dim_{\mathbf{C}} V^1} \times (G_2)^{\dim_{\mathbf{C}} V^2}$ such that for each element of the same irreducible representation of a B factor $T_{\mathbf{C}}$ has a common eigenvalue (this is well defined since the complex homogeneous spaces are 1-step).

We shall see that the above construction can be regarded as a general structure for compact complex homogeneous spaces.

Proof of Theorem G. Following the proof of Theorem F, we define further

$$C = \left\{g \in G_1/N_1|_{\operatorname{ad} g|_{\bigoplus_{i\in(k+l+1,\dots,s)} V_i}=0}\right\} \quad \text{and}$$
$$C_i = \left\{g \in G_1/N_1|_{\operatorname{ad} g|_{\bigoplus_{j\neq i} V_j}=0}\right\}, \quad i \in (k+l+1,\dots,s).$$

Then, we have onto morphism $S_2R/H_2 \to A = (G_1/N_1)/C\prod C_i$ and the fiber of this morphism is a product of a parallelizable manifold and some complete reduced primitive homogeneous spaces. □

Proof of Theorem H. For any compact complex homogeneous space $M = G/H$, we have the Tits fibration $G/H \to G/N_G(H^0)$ with fiber $F = G_1/\Gamma = N_G(H^0)/H$. Then as in Section 4, we have a fibration $F \to G_1/\Gamma \cap [N,N]$ with N being the nilradical of G_1. This fibration of F introduce a fibration of M to a 1-step space M_1. We apply this construction to SR/H and obtain the 1-step space M_1 in Theorem H. To prove Theorem H, we assume that $[N,[N,N]] = 0$, and the same proof works otherwise.

Apply Theorem G to M_1. Then we obtain a $(M_1)_0$ which is a product of a parallelizable manifold P and some complete reduced primitive spaces $N_1,\dots,N_m$. What we need to prove is that each **Q**-irreducible representation A of the reductive part of Γ comes from only those in one of P and $N_1,\dots,N_m$, i.e., cannot be a product representation from distinct elements among P and $N_1,\dots N_m$. Otherwise, we can construct a compact complex homogeneous space N_A with the same M_1 and $(\Gamma \cap [N,N])_{\mathbf{Q}} = A$. We may also assume that there are only two of P and $N_1,\dots,N_m$. Then the reductive part L_1 of the group of N_1 acts non-trivially on A. This is a contradiction to the unimodular property of G_1. To see this, we notice that $(N \cap \Gamma)_{\mathbf{Q}} = B_1 \oplus B \oplus A$ with B_1 corresponding to the nilradical of N_1 and B corresponding to the nilradical of the other while L_1 acts trivially on B and unimodularly on B_1 but non-unimodularly on A. L_1 acts non-unimodularly on A because basically A only comes from some other representations of the B factor of N_1 and we have Lemma 2 of Section 5. □

Acknowledgments

I thank Professor Sarnak for showing me [24] in 1996, which turns out to be very helpful for this research. I thank the Department of Mathematics, Princeton University, and University of California at Riverside, Professors W.C. Hsiang and B. Wong, as well as NSF for their supports, which make this work possible to be done and to be published.

Appendix A. Fundamental weights of the simple Lie algebras

This appendix is devoted to the proving of the following theorem (see also a similar list in [17, p. 69]).

Proposition 6. *For all the simple Lie algebras, we have H_i in Proposition 5 as follows*:

(1) A_l: $H_i = \frac{1}{l+1}\big((l-i+1)\alpha_1 + \cdots + (l-i+1)i\alpha_i + \cdots + i\alpha_l\big)$,

where $(\alpha_k, \alpha_k) = 2$, $(\alpha_k, \alpha_{k+1}) = -1$.

(2) B_l: $H_i = \alpha_1 + 2\alpha_2 + \cdots + i\alpha_i + \cdots + i\alpha_l$ *for* $i < l$,

$$H_l = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + l\alpha_l),$$

where $(\alpha_i, \alpha_{i+1}) = -1$, $(\alpha_i, \alpha_i) = 2$ *for* $i < l$, *and* $(\alpha_l, \alpha_l) = 1$.

(3) C_l: $H_i = \alpha_1 + 2\alpha_2 + \cdots + i\alpha_i + \cdots + i\alpha_{l-1} + \frac{i}{2}\alpha_l$ *for* $i < l$,

$$H_l = \alpha_1 + 2\alpha_2 + \cdots + (l-1)\alpha_{l-1} + \frac{l}{2}\alpha_l,$$

where $(\alpha_{i-1}, \alpha_i) = -1$, $(\alpha_i, \alpha_i) = 2$ *for* $i < l$, *and* $(\alpha_l, \alpha_l) = 4$, $(\alpha_{l-1}, \alpha_l) = -2$.

(4) D_l: $H_i = \alpha_1 + \cdots + i\alpha_i + \cdots + i\alpha_{l-2} + \frac{i}{2}(\alpha_{l-1} + \alpha_l)$ *for* $i \leqslant l-2$,

$$H_i = \frac{1}{2}\big(\alpha_1 + 2\alpha_2 + \cdots + (l-2)\alpha_{l-2}\big) + \frac{1}{4}(a_i\alpha_{l-1} + b_i\alpha_l)$$

with $i = l-1$ *or* l, $a_{l-1} = b_l = l$, *and* $a_l = b_{l-1} = l-2$, *where* $(\alpha_i, \alpha_i) = 2$, $(\alpha_k, \alpha_{k+1}) = -1$ *for* $k \leqslant l-2$, *and* $(\alpha_{l-2}, \alpha_l) = -1$.

(5) E_k, $k = 6, 7, 8$:

$$H_k = \frac{1}{9-k}\big(2\alpha_1 + \cdots + 2(k-3)\alpha_{k-3} + (k-3)\alpha_{k-2} + (k-1)\alpha_{k-1} + 4\alpha_k\big),$$

$$H_{k-1} = \frac{1}{9-k}(4\alpha_1 + \cdots + 4(k-3)\alpha_{k-3} + 2(k-3)\alpha_{k-2} + 2(k-1)\alpha_{k-1} + (k-1)\alpha_k),$$

$$H_{k-2} = \frac{1}{9-k}\big(3\alpha_1 + \cdots + 3(k-3)\alpha_{k-3} + k\alpha_{k-2} + 2(k-3)\alpha_{k-1} + (k-3)\alpha_k\big),$$

$$H_i = \frac{1}{9-k}\big((9+i-k)\alpha_1 + \cdots + i(9+i-k)\alpha_i + i(9+i+1-k)\alpha_{i+1} + \cdots + 6i\alpha_{k-3} + 3i\alpha_{k-2} + 4i\alpha_{k-1} + 2i\alpha_k\big) \quad \text{for } i \leqslant k-3,$$

where $(\alpha_i, \alpha_i) = 2$, $(\alpha_j, \alpha_{j+1}) = -1$ *for* $j \leqslant k-3$ *or* $j = k-1$, *and* $(\alpha_{k-3}, \alpha_{k-1}) = -1$.

(6) F_4: $H_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, $H_2 = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4$,

$H_3 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4$, $H_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$,

where $(\alpha_i, \alpha_i) = 4$ *for* $i \leqslant 2$ *or* $= 2$ *for* $i > 2$, *and* $(\alpha_j, \alpha_{j+1}) = -2$ *for* $j \leqslant 2$ *or* $= -1$ *for* $i = 3$.

(7) G_2: $H_1 = 2\alpha_1 + \alpha_2$, $H_2 = 3\alpha_1 + 2\alpha_2$,

where $(\alpha_i, \alpha_i) = 2$ *for* $i = 1$ *or* $= 6$ *for* $i = 2$, *and* $(\alpha_1, \alpha_2) = -3$.

And hence $a_{ij} > 0$ *for all the cases.*

Proof. By direct checking, we observe that all these H_i satisfy the condition $(2(H_i, \alpha_j))/(\alpha_j, \alpha_j) = \delta_{ij}$. □

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