Modification and the Cohomology Groups of Compact Solvmanifolds

Daniel Guan
e-mail: zguan@math.ucr.edu

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In this note we give a modification theorem for a compact homogeneous solvmanifold such that a certain Mostow type condition will be satisfied. An application of this result is a simpler way to calculate the cohomology groups of compact quotients of real solvable Lie group over a cocompact discrete subgroup. Furthermore, We apply the second result to obtain a splitting theorem of compact complex homogeneous manifolds with symplectic structures. In particular, we are able to classify compact complex homogeneous spaces with pseudo-kählerian structures.

1 Introduction

A compact real homogeneous manifold $M = G/H$ is called a solvmanifold if G is solvable and H is discrete. Here we always assume that G is connected and *simply connected.* If $\mathrm{Ad}(G)$ has the same real algebraic closure as that

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of $\mathrm{Ad}(H)$, we say that M has the Mostow condition with respect to G and H. The Mostow condition can be also defined for any given compact homogeneous space. In general case, the compact factors in the semisimple part has to be the identity. To make the things simpler, we only consider the case in which G is a real solvable Lie group and H is a cocompact discrete subgroup, i.e., M is a solvmanifold. When M is a solvmanifold and satisfies the Mostow condition, the cohomology of M can be calculated by the cohomology of the Lie algebra (see [Rg Corollary 7.29]). But in general, it is very difficult to calculate the cohomology for a general compact solvmanifold. In this paper we prove the following:

Main Theorem 1. *If $M = G/H$ is a compact real homogeneous solvmanifold, there is a finite covering space $M' = G/H'$, i.e., H/H' is a finite group, such that there is another simply connected solvable real Lie group G' which contains H' and is diffeomorphic to G such that (1) $M' = G'/H'$, (2) M' satisfies the Mostow condition with respect to G' and H'.*

In particular, we have:

Main Theorem 2.: *If G is solvable and H is discrete, $M = G/H$ is compact, then we have $H^*(M) = H^*(\mathcal{G}')$, where $\mathcal{G}'$ is the Lie algebra of the Lie group G' in the Main Theorem 1.*

A smooth $2n$-dimensional manifold M equiped with a smooth transitive action of a Lie group is what we call a homogeneous space. If in additional M is a symplectic manifold, we refer to it as a *homogeneous space with a symplectic structure* and, if the structure is invariant, a *homogeneous space with an invariant symplectic structure*.

Recently there has been much progress in the area of symplectic manifolds and group actions. I was interested in the *classical* problem of classifying compact homogeneous spaces with symplectic structure. The difficulty is that we do not know anything about the transitive group and the isotropy group (Cf. [DG], [Hk], [Gu3]). In the Kähler case we know that the isometric group is compact. In an earlier paper [Gu3] we prove following theorem:

Proposition 1.: *Every finite dimension Lie subgroup of the automorphism group of a compact symplectic manifold is locally a product of a compact semisimple group and a 2-step solvable group R. Moreover, the adjoint representation of R on R' is a subgroup of a compact torus.*

I am also interested in the structure of compact homogeneous manifold with symplectic structure (which might not be invariant under the group action)

We also posted following conjecture therein [Gu3]:

CONJECTURE. *If G/H is a compact homogeneous space with a symplectic structure, then G/H is diffeomorphic to a product of a rational homogeneous space and another homogeneous space N, where N up to a finite quotent is a compact quotient of a compact locally flat parallelizable manifold with a symplectic structure.*

Our Main Theorem 1 will be a major step toward a proof of this conjecture. Here we call a manifold N *locally flat parallelizable* if $N = G/H$ for a simply connected Lie group G which is diffeomorphic to $\mathbf{R}^k$ for some integer k and H is a uniform (i.e., cocompact) discrete subgroup.

In this paper, we should finish the complex case. We should prove:

Main Theorem 3.: *If $M = G/H$ is a compact complex homogeneous space with a symplectic structure and G is complex (we can always assume this), then M is a product of a rational projective homogeneous space and a complex solvmanifold $N = N_G(H^0)/H$. Moreover, if we let*

$G_1 = N_G(H^0)/H^0$ and $\Gamma = H/H^0$, then there is a subgroup Γ' of Γ such that Γ/Γ' is finite and N up to a finite quotent is a solvmanifold G_2/Γ' with a right invariant symplectic structure of a real Lie group G_2 which contains the nilradical of G_1 and is possible different from G_1. In particular, if M has a pseudo-kählerian structure, so is N and if N has a right invariant pseudo-kählerian structure N must be a complex torus. And, if M has a holomorphic symplectic structure, then $M = N$.

We notice that for the splitting theorem of [Gu1,2[to hold we only need a real symplectic structure which is invariant under the maximal compact subgroup and this is provided by the existence of G' in our Main Theorem 1 and 2. Also, the complex structure is right invariant under the action of G_2, see Corollary 1 in the next section.

The case when $M = N$ with a right invariant pseudo-kählerian metric was proven in [Gu1,2] (see [Gu2 proof of Lemma 1] or [DG 3.3] in 1989). The theorem in [DG] follows also from our main theorem 3. Applying the same method in [DG] to G_2 one can easily prove that if N is pseudo-kählerian, then the nilradical N_{G_1} of G_1 has at most two steps. An application of the Leray spectral sequence to the fiber bundle $G_2/\Gamma' \to G_2/N_{G_2}\Gamma'$ shows that the derivator $[N_{G_2}, N_{G_2}] = 0$, see Corollary 2 in the next section.

Moreover, the Lie algebra of G_2 is a direct orthogonal sum of two vector spaces A and N_{G_2} (regarded as its Lie algebra) respect to a pseudo-kählerian structure which ω is right invariant on the universal covering. Moveover, both $\omega_1 = \omega|_A$ and $\omega_2 = \omega|_{N_{G_2}}$ are closed, and nonzero cohomology classes. A is an abelian Lie algebra and acts on N_{G_2} semisimply with real eigenvalues. ω_2 has zero index. This is done with a new algebra called *complex-parallelizabe-right-invariant-pseudo-kählerian algebra*, which is similar to what we used in [DG, Gu3] (see our proof of the Corollary 1 in the next section). We should deal with the pseudo-kählerian case in another paper [Gu4]. One might also notice that when ω is Kähler, then $\omega_2 = 0$ and we have the Borel-Remmert Thorem.

When the pseudo-kählerian structure is Kähler, the original average process in [Gu1,2] works and gives an averaged Kähler structure. Then the splitting theorem holds there and an application of our argument there produces another proof of the Borel-Remmert Theorem.

The main purpose of [Gu1] is finding nonkählerian holomorphic symplectic manifolds. That mission was completed in [Gu1]. Actually, we should also see that *Theorem B there is still true*[1]. The reason is that assuming our main theorem 3, for the complex parallelizable manifold $M = N$ the holomorphic tangent bundle is trivial and therefore $H^{2,0}(M) = \wedge^2 \mathcal{G}^*$, which all come from the right invariant forms. Therefore, the averaging process goes through trivially.

2 The Proof

1. Here we collect some results we need from the splitting theory of the Lie group (see [Gb1]). Let $G = SR$ be a Levi decomposition of a semisimple Lie group. In this section, we assume that G is simply connected. We call G a *splittable Lie group* if $R = TU$ with $T \cap U = \{e\}$ such that T acts semisimplely and U acts unipotently on the Lie algebra $\mathcal{G}$. We call a Lie

[1] I did not realized that there was a mistake in [Gu2] until I saw [Ym] in August 2006, he (they) did not tell me. This is unfortunately an odd situation. However, this eventually turns out to be a positive turn. Thanks to the Gorbatservich's construction that we wrote down our Main Theorem 1 in a few days and are able to go around Iwamoto's result which we misused there to prove a weaker version, i.e., our Main Theorem 3. The examples in [Ym] gave strong and beautiful evidences for our Main Theorem 3. Finally, I received a letter from Yamada with several of his reprints on April 24, 2007.

group embedding $\alpha: \ G \to M(G)$ from G to a splittable simply connected Lie group $M(G) = T \cdot S \cdot U$ a *Mal'cev splitting* or *M-splitting* if $\alpha(G)$ is a normal subgroup of $M(G)$ and $M(G)$ is a semidirect product of T and $\alpha(G)$, and $\alpha(G) \cdot U = M(G)$.

Proposition 2. *For any simply connected Lie group G there is a unique Mal'cev splitting.*

The Mal'cev splitting can be constructed as following:

Let $G = S \cdot R$ be the Levi decomposition of a connected simply connected Lie group G. Consider the adjoint representation $\mathrm{Ad}_G: \ G \to GL(\mathcal{G})$; put $G^* = \mathrm{Ad}_G(G)$, and let $< G >$ be the algebraic closure of G^* in $GL(\mathcal{G})$. Since $< G >$ is algebraic, it has a Chevalley decomposition

$$< G >= T^* S^* U^*,$$

where U^* is the unipotent radical, S^* is semisimple, and T^* is abelian and consists of semisimple (i.e., completely reducible) elements. Put $W^* = S^*U^*$; then $< G >= T^*W^*$, with $T^* \cap W^*$ finite. Let $t^*: \ T^*W^* \to T^*/T^* \cap W^*$ be the natural epimorphism, with kernel W^*. Writing $\hat{T} = T^*/T^* \cap W^*$, we have clearly $t^*(\mathrm{Ad}G) \subset (\hat{T})^0$, since G is connected. If for the connected abelian Lie group $(T^*)^0$ we consider the universal covering for $\pi_T: \ \tilde{T} \to (T^*)^0$, it is obvious that $t^* \cdot \pi_T: \ \tilde{T} \to (\hat{T})^0$ is the universal covering for $(\hat{T})^0$. Since G is connected and simply connected, there exists a unique homomorphism $\tilde{t}: \ G \to \tilde{T}$ such that $t^* \cdot \pi_T \cdot \tilde{t} = t^* \cdot \mathrm{Ad}_G$. Put $T = \tilde{t}(G)$, $T^*_G = \pi_T \cdot \tilde{t}(G)$; then T is a connected simply connected abelian Lie group covering of T^*, while $T^*_G \subset < G >$. We see that T^*_G can be regarded as a subgroup of $\mathrm{Aut}G$. The imbedding $T^*_G \to \mathrm{Aut}G$ and the homomorphism π_T induce a homomorphism $\phi: \ T \to \mathrm{Aut}G$, with $\ker\phi = \ker\pi_T \cap T$ discrete.

Then we can get the Mal'cev splitting

$$M(G) = T \times_\phi G$$

and

$$M(G) = TSU$$

for a unipotent group U such that

$$\dim U = \dim R,\ \dim U/N_R = \dim T,$$

where N_R is the nilpotent radical of G.

Now we let

$$W_G = SU,\ W_{G,l} = S/l(S) \cdot U,$$

where $l(S)$ is the minimal discrete subgroup of the center of S such that $S_l = S/l(S)$ is linear. Then, both $\mathrm{Aut}W_{G,l}$ and the semidirect product $\mathrm{Aut}W_{G,l} \propto W_{G,l}$ are prealgebraic groups. As W_G is a normal subgroup containning the commutator of $M(G)$, we can regard T^*_G as a subgroup of $\mathrm{Aut}W_{G,l}$. Let $a(T^*_G)$ be the prealgebraic hull of T^*_G in $\mathrm{Aut}W_{G,l}$, and

$$\mathcal{A}_l(G) = a(T^*_G) \propto W_{G,l}.$$

We see that $\mathcal{A}_l(G)$ is prealgebraic. Let $M_l(G) = T^*_G S_l U$ as a quotient of $M(G)$, then:

Proposition 3. *The group $\mathcal{A}_l(G)$ is prealgebraic, and there exists an imbedding $\beta :\ M_l(G) \to \mathcal{A}_l(G)$ such that the following properties hold:*

1) *$\mathcal{A}_l(G)$ is splittable, and if $\mathcal{A}_l(G) = T'S'U'$, where U' is unipotent, S' semisimple and T' a prealgebraic torus, then $\beta(M_l(G)) \supset S'U'$ and $S' = S_l$, where S is the semisimple part of G and $U' = U$.*

2) *The prealgebraic closure of each of the subgroup* $\beta(G_l)$ *and* $\beta(M_l(G))$ *in* $\mathcal{A}_l(G)$ *is* $\mathcal{A}_l(G)$ *itself.*

Here we like to give a very simple example: Let $G = G_1 \times G_2,\ G_1 = TN$ with T, N, G_2 abelian and T acts on N almost faithfully and as a compact torus without any eigenvector. Then $< G >= \mathrm{Ad}_G(T)N,\ W^* = N$,

$$t^* :\ \mathrm{Ad}_G(T)N \to \mathrm{Ad}_G(T) = \hat{T}$$

$$\pi_T :\ T \to \mathrm{Ad}_G(T)$$

$$\tilde{t} :\ TN \times G_2 \to T$$

$$T_G^* = \mathrm{Ad}_G(T),\ \phi :\ T \to \mathrm{Ad}_G(T)$$

$$M(G) = T \times_\phi G = TU,\ U = \{(t, t^{-1}, n, g)|_{t\in T\ n\in N\ g\in G_2}\}$$

$$W_G = W_{G,l} = U,\ \mathcal{A}_l(G) = M_l(G) = \mathrm{Ad}_G(T)U.$$

2. Here we prove the *modification* for a compact homogeneous solvmanifolds. This method was first used in [Gb2]. Similar construction can be found in the study of homogeneous Kähler manifolds, e.g., [Dm], [DN].

In this subsection we only deal with the case when G is solvable, i.e., S is the identity and H is discrete, in particular H^0 is also the identity.

Let $M = G/H$ be a compact homogeneous solvmanifold of a simply connected solvable real Lie group G. We go through the proof with the similar notations as in [Gu3] 3.1.7., that might help us understand the both constructions here and therein. In our case, we set $G_* = G_l = G/l(S) = G$ be the image of $G(H_* = H/H \cap l(S) = H$ be the image of $H)$ in $\mathcal{A}_l(G) = \mathcal{A}(G)$. We also set P_* be the algebraic closure of $\beta(H)$ in $\mathcal{A}(G)$. Then the nilradical $N_R \subset P_*$ by a theorem from Mostow [Rg Theorem 3.3] , see also [Rg Corollary 8.27, 8.28] (in [Gu3] 3.1.7. this condition was true by the fact that the nilradical is in the normalizer of H^0, which of course is trivial in our case here since H^0 is the identity and we see that our modification is quite different from the one we mentioned in [Gu3]). Since the subgroup P_* is an algebraic group, the group $\pi_0(P_*)$ is finite. Passing from H to the subgroup $H_1 = H \cap \pi^{-1}(P_*^0 \cap H_*)$ of finite index, where $\pi : M(G) \to M_l(G) = M(G)$ is the natural epimorphism, we might assume that $H_* \subset P_*^0$ by considering a finite covering M' of M. This inclusion will be assumed to hold in what follows.

We consider the natural epimorphism $\gamma : \mathcal{A}_l(G) \to \mathcal{A}_l(G)/W_l$. We have $\mathcal{A}_l(G)/W_l = T_* \times \pi(W_G)/W_l$ with $W_G = SU = U,\ W_l = S_l N_R = N_R$ (our W_l is the same as in [Gb2] but different from the one in [Gb1], in [Gb1] $W_l = S/l(S) \cdot U = U$) and T_* is a prealgebraic torus; $\pi(W_G)/W_l = U/N_R$. So $\mathrm{Im}\gamma = T_* \times U/N_R$, we denote it by A. A is connected and Abelian. There is a natural embedding of the group $G_*/W_l = R/N_R$ in $M_l(G)/W_l$ which is contained in A.

We denote the image of R/N_R by B. We have U/N_R is the projection of B to the second factor U/N_R. By our construction we have $\dim U/N_R = \dim R/N_R$. The composition of γ and the projection restricted on R/N_R is an onto linear map between two Eucliden spaces with same dimensions, and therefore is an isomorphism. That implies $B \cap T_* = \{e\}$. We see that the projection $\mu : T_* \times U/N_R \to U/N_R$ to the second factor is an isomorphism on B, i.e., B is closed in A. Now we consider the subgroup $H_*/H_* \cap W_l$ of A and its closure $\overline{H_*/H_* \cap W_l}$ (in the Euclidean topology) which we denote by A_1. Since $H_*/H_* \cap W_l \subset B$ we have $A_1 \subset B$. Since the group B is simply connected and Abelian, A_1 is a closed subgroup of it, A_1 is torsion free and isomorphic to $\mathbf{R}^p \times \mathbf{Z}^q$ for some $p, q \geq 0$.

Finally we consider the subgroup $\gamma(P_*) \subset A$. The subgroup $\mathrm{Ker}\gamma = W_l = N_R$ is closed in the "Zariski topology" on $\mathcal{A}_l(G)$, so does the Lie group P_*, therefore $\gamma(P_*)$ is a closed subgroup of A. But $H_* \subset P_*$, so $H_*/H_* \cap W_l \subset \gamma(P_*)$ and hence $A_1 \subset \gamma(P_*)$, i.e., $A_1 \subset \gamma(P_*^0)$ by our convention. The group $\gamma(P_*^0)$ is connected and Abelian and hence $\gamma(P_*^0) = K \times V$, where K is a maximal compact subgroup of $\gamma(P_*^0)$ (which is a torus), and V is simply connected. Since A_1 is closed in A and torsion free, $A_1 \cap K = \{e\}$. Hence the projection $K \times V \to V$ onto the second direct factor on A_1 is a monomorphism. Now it follows from this that there exists a closed simply connected subgroup $C \subset \gamma(P_*^0)$, such that $A_1 \subset C$ and A_1 is uniform in C (we notice that C is not always in B). We set $\Phi_l = \gamma^{-1}(C)$. Then $\Phi_l \subset P_*^0$ and therefore has the same algebraic closure as H_l. Then Φ_l is a closed connected subgroup of $\mathcal{A}_l(G)$. To it corresponds a closed connected subgroup Φ of $\mathcal{A}(G)$.

With this construction at hand, we have the following theorem:

Theorem. *Let $M = G/H$ be a compact homogeneous solvmanifold of a simply connected Lie group G. Then there exists a subgroup H' of finite index in H and a subgroup Φ of $\mathcal{A}(G)$, such that:*

(a) Φ *is a connected, simply connected, closed subgroup of* $\mathcal{A}(G)$*, containing* H' *and* N_R*,*

(b) $W_\Phi = W_G = U_\Phi = U_G$ *(although* $M(\Phi)$ *and* $M(G)$ *are not generally isomorphic),*

(c) for the decomposition $\mathcal{A}(G) = TW_G = TU$ *with* T *an Abelian subgroup*

of $\mathcal{A}(G)$ *we have* $\Phi \subset TG,\ G \subset T\Phi$, *where* $\Phi \cap T = G \cap T = \{e\}$,

(d) *there exists a diffeomorphism* $\eta: \ \Phi \to G$ *which is the identity on the subgroup* H' *and induces a diffeomorphism* $\Phi/H' \to G/H'$, *Actually, it induces a diffeomorphism of the torus bundles* $G/H' \to G/H'N_R$ *and* $\Phi/H' \to \Phi/H'N_R$.

(e) $\mathrm{Ad}\Phi$ *has the same algebraic closure as that of* $\mathrm{Ad}H'$.

We obtained our Main Theorem 2 by applying the Mostow Theorem (see [Rg Corollary 7.29]). One might notice that our proof here is almost word to word identical to the corresponding parts 3.1.6 and 3.1.7 in [Gu3], but the purposes are quite different. For the proof of the Main Theorem 3, one could easily apply the Main Theorem 2 to N and follow the proof of our previous work in [Gu2].

Also, we shall point out that in real practice one could use $< G >$ instead of $\mathcal{A}(G)$, but any argument for a mathematical proof using $< G >$ is not available at the present time. The proof of the next result also shows that the using of $\mathcal{A}(G)$ is more favorable in the mathematical arguments.

Moreover, since G_1 is a complex Lie group, we have at e

$$\mathrm{Ad}(g) \cdot J = J \cdot \mathrm{Ad}(g)$$

for any $g \in G_1$. Therefore, we also have

$$\alpha \cdot J = J \cdot \alpha$$

for any $\alpha \in < G_1 >$. Now we let

$$j(x+t) = Jx \ (\mathrm{mod}\ \mathrm{Lie}(< G_1 >)),$$

where Lie($< G_1 >$) is the Lie algebra of $< G_1 >$. One could easily check that

$$\begin{aligned}
&[j(x+t), j(y+s)] = -[x,y] \\
&= \quad -[x,y] + jt(jy) + t(y) - s(x) - js(jx) + [t,s] \ (\text{mod Lie}(< G_1 >)) \\
&= \quad j[j(x+t), y+s] + j[x+t, j(y+s)] + [x+t, y+s]
\end{aligned}$$

for any $x, y \in \mathcal{G}_1,\ s, t \in$ Lie($< G_1 >$). That is, the complex structure on G_1 is always invariant under the action of $M(G_1) = T \times_\phi G$ and hence its prealgebraic closure $\mathcal{A}(G_1)$, see [Dm, DN, DG] for this formula.

Corollary 1.: *The complex structure on N of the Main Theorem 3 is right invariant under the action of G_2.*

Now any element in $H^1(N_{G_1}/\Gamma \cap N_{G_1}, \mathbf{R})$ is $\beta + \bar{\beta}$ with a holomorphic left invariant form β which is closed, that is, $d(\omega) = 0$. This means that β is not from the derivation $[N_{G_1}, N_{G_1}]$ but from $N_{G_1}/[N_{G_1}, N_{G_1}]$ (see [Gu1]). So a right invariant pseudo-kähler form on G_2/Γ' actually come from $M_1 = G_2/[N_{G_2}, N_{G_2}]\Gamma'$. Therefore, combine with what we have for the part from $H^2(G_2/\Gamma', \mathbf{R})$ in [DG] we get all the three parts in the Leray spectral squence come from M_1 (see a similar argument in [Gu2]). By the nondegeneracy of the pseudo-kählerian form we see that $N = M_1$.

Corollary 2.: *If M is a pseudo-kählerian manifold in the Main Theorem 3, then N is a complex torus bundle over a complex torus and up to finite covering has a left G_2 invariant pseudo-kählerian structure. G_1 and G_2 have abelian niradicals.*

These compact complex homogeneous parallelizable manifolds also shows that the similar result of the Mostow Theorem for the Dolbeault cohomology does not work in general, since the holomorphic 1-forms (as elements in

$H^{1,0}(N)$) are generally not right inariant under the modified Lie group which satisfies the Mostow condition.

After a further labor we can prove that all the compact complex homogeneous spaces with pseudo-kählerian structures have the similar form as Yamada's example [Gu4].

References

[Dm] J. Dorfmeister: Homogeneous Kähler Manifolds Admitting a Transitive Solvable Group of Automorphisms, Ann. Scient. Ec. Norm. Sup., 4 Serie, vol 18 (1985), 143–180.

[DN] J. Dorfmeister & K. Nakajima: The Fundamental Conjecture for Homogeneous Kähler Manifolds, Acta Math. 161(1988), 23–70.

[DG] J. Dorfmeister & Z.-D. Guan: Classification of Compact Homogeneous Pseudo-Kähler Manifolds, Comment. Math. Helv. 67 (1992), 499–513.

[Gb1] V. V. Gorbatservich: Splittings of Lie Groups and Their Application to the Study of Homogeneous Spaces, Math. USSR Izvestija. vol 15 (1980), 441–467.

[Gb2] V. V. Gorbatservich: Plesiocompact Homogeneous Spaces, Siber. Math. J. 30 (1989), 217–226.

[Gu1] Z. Guan: Examples of compact holomorphic symplectic manifolds which admit no Kähler structure. In *Geometry and Analysis on Complex Manifolds—Festschrift for Professor S. Kobayashi's 60th Birthday*, World Scientific 1994 63–74.

[Gu2] D. Guan: A Splitting Theorem for Compact Complex Homogeneous Spaces with a Symplectic Structure. Geom. Dedi. 67(1996), 217–225.

[Gu3] D. Guan: On Compact Symplectic Manifolds with Lie Group Symmetries, Transactions of AMS vol. 357 (2005), 3359–3373.

[Gu4] D. Guan: Classification of Compact Complex Homogeneous Manifolds with Pseudo-Kählerian Structures, in reprint 2007.

[Hk] A. Huckleberry: Homogeneous Pseudo-Kählerian Manifolds: A Hamiltonian Viewpoint, Note Mat. 10 (1990), 337–342.

[On] A. L. Onishchik: On Lie Groups Transitive on Compact Manifolds II, Math. USSR Sbornik. vol. 3 (1967), 373–388.

[Rg] M. S. Raghunathan: *Discrete Subgroups of Lie Groups,* Springer-Verlag, Belin, 1972.

[Ym] T. Yamada: A Pseudo-Kähler Structure on a Nontoral Compact Complex Parallelizable Solvmanifold, Geom. Dedicata 112 (2995), 115–122.

Author's Addresses:

Zhuang-Dan Guan

Department of Mathematics

University pf California at Riverside

Riverside, CA 92507 U. S. A.

e-mail: zguan@math.ucr.edu