Classification of Compact Homogeneous Manifolds with Pseudo-kählerian Structures

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Dans cette note, nous appliquons un théorème de modification pour des “solv-variétés” compactes et homogènes aux variétés compactes complexes équipées d’une structure pseudo-kählérienne. Nous obtenons une classification de ces variétés compactes pseudo-kählériennes sous la forme de certains produits d’espaces projectifs rationnels et homogènes, de tores, et d’espaces pseudo-kählériens réduits et primitifs simples ou doubles.

In this note we apply a modification theorem for compact homogeneous solvmanifolds to compact complex homogeneous manifolds with pseudo-kählerian structures. We are then finally able to classify these compact pseudo-kählerian manifolds, as certain products of projective rational homogeneous spaces, tori, and simple and double reduced primitive pseudo-kähler spaces.


Key words and phrases: Solvmanifolds, cohomology, invariant structure, homogeneous space, product, fiber bundles, symplectic manifolds, splittings, decompositions, modification, Lie group, compact manifolds, uniform discrete subgroups, pseudo-kählerian structures.
Compact complex homogeneous spaces with equivariant Kähler structures were classified by Matsushima in [Mt]. Compact complex homogeneous spaces with Kähler structures (not necessarily equivariant) were classified by Borel and Remmert in [BR]. Compact complex homogeneous spaces with equivariant pseudo-kählerian structures were classified by Dorfmeister and the author in [DG]. A possible classification of compact complex homogeneous spaces with pseudo-kählerian structures was proposed by the author in [Gu1,2]. However, the classification turns out to be much more complicated than suggested therein. See [Gu4], [Ym1, 2], and [Ha]. In this note, I shall give a complete classification. I received a reprint of [Ym2] only after submitting [Gu4], and a preprint of [Ha] only after completing [Gu5].

In this paper, I shall provide the last piece of this puzzle, thereby completely solving this problem. What was missing in [Gu1, 2] was the calculation of the cohomology group of a compact solvmanifold. That problem was resolved in [Gu4], using techniques in [Gu3].

The complete proof, which is quite technical, is given in [Gu5].

I already reduced the classification to the case of a solvable parallelizable action in [Gu4] (see the Main Theorem 3 therein), following the plan proposed in [Gu1, 2].

An application of the modification argument in [Gu4] is the so called complex-parallelizable-right-invariant-pseudo-kählerian algebra (cf. [Gu4] Corollary 1). This is what has made the present classification possible.

Moreover, at the group level, I proved in [Gu5] that the given group $G$
can be decomposed as $G = AN$ with two abelian subgroups $A$ and $N$ (for partial results, see also [Gu4] Corollary 2 and [Ym2], [DG] p. 509 Corollary 1), and the action of $A$ on $N$ is an algebraic group action of a product of $\mathbb{C}^*$’s.

If the Lie algebra of $A$ acts on $N$ with only one pair of eigenvalue functions $k_1$ and $k_2 = -k_1$, then we will call the given compact complex parallelizable homogeneous manifold with a pseudo-kählerian structure a primary pseudo-kähler manifold.

**Theorem A.** Every compact complex parallelizable homogeneous manifold with a pseudo-kählerian structure is a pseudo-kählerian torus bundle over a pseudo-kählerian torus and, up to a finite covering of the fiber, is a torus bundle which is the bundle product of several primary pseudo-kählerian manifolds.

The bundle product in Theorem A is the fiberwise product. More details concerning the statement of this result can be found in [Gu5].

Let us call a primary pseudo-kählerian manifold a reduced primary pseudo-kählerian manifold if the action of $A$ on $N$ is effective. Notice that we may always change a given invariant form $\omega_1$ on $A$ to any other such form. As a consequence:

**Theorem B.** After modifying $\omega_1$, if necessary, any primary pseudo-kählerian manifold, up to a finite covering, is the product of a torus and a reduced primary pseudo-kählerian manifold. Moreover, $\dim_{\mathbb{C}} A = 1$ and $\dim_{\mathbb{C}} N = 2m$ with $m$ the complex dimension of the eigenspaces for a reduced primary pseudo-kählerian manifold. In particular, the index of the pseudo-kählerian structure for a reduced primary pseudo-kähler space is either $1$ or $-1$.  

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For any reduced primary pseudo-kähler space, we have \( k_1(z) = z \) and \( k_2(z) = -z \). The fiber torus up to a finite covering can be split into complex irreducible ones with respect to the \( A \) action. For a primary pseudo-kähler space, if the fiber is also an irreducible complex torus with respect to the \( A \) action, then we will call it a \textit{primitive pseudo-kähler space}. If the \( A \) action is also effective, then we will call it a \textit{reduced primitive pseudo-kähler space}. By changing \( \omega_1 \) on \( A \) we can always obtain any primary pseudo-kähler space, up to a finite covering, from a torus bundle product of primitive ones.

For any reduced primitive pseudo-kählerian space, the rational module generated by the discrete subgroup \( N \mathbb{Z} \) of \( N \), as a rational representation of \( \mathbb{Z} = \Gamma N / N \), can be split into two-dimensional representations, but in this case \( m \) can be any positive integer.

In general, set \( A \mathbb{Z} = \Gamma N / N \) and \( T = A / A \mathbb{Z} \).

**Theorem C.** Any compact complex parallelizable homogeneous space with a pseudo-kähler structure is a pseudo-kählerian torus bundle over a pseudo-kählerian torus \( T \) and, up to a finite covering of the fiber, is a torus bundle over \( T \) which is the bundle product of several primitive spaces. Moreover, any primitive space is, up to a finite covering, a product of a torus and a reduced primitive space. For a primitive pseudo-kählerian manifold, \( m \) can be any given positive integer.

Let us call a primitive pseudo-kähler space a \textit{simple primitive pseudo-kähler space} if \( N / N \mathbb{Z} \) is a simple complex torus.

Let us call a primitive pseudo-kähler space a \textit{double primitive pseudo-kähler space} if \( N / N \mathbb{Z} \) is isogenous to the product of two identical complex tori.

The example in [Ym1] is a double reduced primitive pseudo-kählerian
space.

**Theorem D.** Let $M$ be a reduced primitive pseudo-kähler space. If $M$ is not a simple primitive pseudo-kähler space as defined above, then it is a double reduced primitive pseudo-kähler space.

**Theorem E.** The generic reduced primitive pseudo-kähler spaces are simple reduced primitive pseudo-kähler spaces.

Combining these results with our splitting theorem (the Main Theorem 3 in [Gu4]) and our results in [Gu2], we have:

**Theorem F.** Every compact complex homogeneous manifold with a pseudo-kählerian structure is the product of a projective rational homogeneous space and a solvable compact complex parallelizable pseudo-kähler space. In particular, any compact complex parallelizable pseudo-kähler space $M$ is solvable. Moreover, any compact complex parallelizable pseudo-kähler space is a pseudo-kählerian torus bundle over a pseudo-kählerian torus $T$ and, up to a finite covering of the fiber, is a torus bundle over $T$ which is the bundle product of several simple and double primitive pseudo-kähler spaces.

**References**


[Gu1] Z. Guan: Examples of compact holomorphic symplectic manifolds which admit no Kähler structure. In *Geometry and Analysis on Complex Manifolds—Festschrift for Professor S. Kobayashi’s 60th Birthday*, World


