Classification of Compact Homogeneous Manifolds with Pseudo-kählerian Structures

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Dans cette note, nous appliquons un théorème de modification pour des “solv-variétés” compactes et homogènes aux variétés compactes complexes equipées d'une structure pseudo-kählérienne. Nous obtenons une classification de ces variétés compactes pseudo-kählériennes sous la forme de certains produits d'espaces projectifs rationnels et homogènes, de tores, et d'espaces pseudo-kählériens réduits et primitifs simples ou doubles.

In this note we apply a modification theorem for compact homogeneous solvmanifolds to compact complex homogeneous manifolds with pseudo-kählerian structures. We are then finally able to classify these compact pseudo-kählerian manifolds, as certain products of projective rational homogeneous spaces, tori, and simple and double reduced primitive pseudo-kähler spaces.

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Compact complex homogeneous spaces with equivariant Kähler structures were classified by Matsushima in [Mt]. Compact complex homogeneous spaces with Kähler structures (not necessary equivariant) were classified by Borel and Remmert in [BR]. Compact complex homogeneous spaces with equivariant pseudo-kählerian structures were classified by Dorfmeister and the author in [DG]. A possible classification of compact complex homogeneous spaces with pseudo-kählerian structures was proposed by the author in [Gu1,2][1]. However, the classification turns out to be much more complicated than suggested therein. See [Gu4], [Ym1, 2], and [Ha]. In this note, I shall give a complete classification. I received a reprint of [Ym2] only after submitting [Gu4], and a preprint of [Ha] only after completing [Gu5][2].

In this paper, I shall provide the last piece of this puzzle, thereby completely solving this problem. What was missing in [Gu1, 2] was the calculation of the cohomology group of a compact solvmanifold. That problem was resolved in [Gu4], using techniques in [Gu3].

The complete proof, which is quite technical, is given in [Gu5].

I already reduced the classification to the case of a solvable parallelizable action in [Gu4] (see the Main Theorem 3 therein), following the plan proposed in [Gu1, 2].

An application of the modification argument in [Gu4] is the so called *complex-parallelizable-right-invariant-pseudo-kählerian algebra* (cf. [Gu4] Corollary 1). This is what has made the present classification possible.

Moreover, at the group level, I proved in [Gu5] that the given group G

[1] The proof in [Gu1] works for the special cases that the manifolds are Kählerian or holomorphic symplectic. In [Gu2], I gave the splitting theorem up to the Mostow condition for which I mistakenly applied a result of Iwamoto from Osaka Journal of Mathematics and a complete proof was given in [Gu4].

[2] As may readily be seen, in the case of complex dimension three considered in [Ha], the results of the present paper are much stronger than those of [Ha].

can be decomposed as $G = AN$ with two abelian subgroups A and N (for partial results, see also [Gu4] Corollary 2 and [Ym2], [DG] p. 509 Corollary 1), and the action of A on N is an algebraic group action of a product of $\mathbf{C}^*$'s.

If the Lie algebra of A acts on N with only one pair of eigenvalue functions k_1 and $k_2 = -k_1$, then we will call the given compact complex parallelizable homogeneous manifold with a pseudo-kählerian structure a *primary pseudo-kähler manifold*.

Theorem A. *Every compact complex parallelizable homogeneous manifold with a pseudo-kählerian structure is a pseudo-kählerian torus bundle over a pseudo-kählerian torus and, up to a finite covering of the fiber, is a torus bundle which is the bundle product of several primary pseudo-kählerian manifolds.*

The bundle product in Theorem A is the fiberwise product. More details concerning the statement of this result can be found in [Gu5].

Let us call a primary pseudo-kählerian manifold a *reduced primary pseudo-kählerian manifold* if the action of A on N is effective. Notice that we may always change a given invariant form ω_1 on A to any other such form. As a consequence:

Theorem B. *After modifying ω_1, if necessary, any primary pseudo-kählerian manifold, up to a finite covering, is the product of a torus and a reduced primary pseudo-kählerian manifold. Moreover,* $\dim_{\mathbf{C}} A = 1$ *and* $\dim_{\mathbf{C}} N = 2m$ *with m the complex dimension of the eigenspaces for a reduced primary pseudo-kählerian manifold. In particular, the index of the pseudo-kählerian structure for a reduced primary pseudo-kähler space is either* 1 *or* -1.

For any reduced primary pseudo-kähler space, we have $k_1(z) = z$ and $k_2(z) = -z$. The fiber torus up to a finite covering can be split into complex irreducible ones with respect to the A action. For a primary pseudo-kähler space, if the fiber is also an irreducible complex torus with respect to the A action, then we will call it a *primitive pseudo-kähler space*. If the A action is also effective, then we will call it a *reduced primitive pseudo-kähler space*. By changing ω_1 on A we can always obtain any primary pseudo-kähler space, up to a finite covering, from a torus bundle product of primitive ones.

For any reduced primitive pseudo-kählerian space, the rational module generated by the discrete subgroup $N_{\mathbf{Z}}$ of N, as a rational representation of $\mathbf{Z} = \Gamma N/N$, can be split into two-dimensional representations, but in this case m can be any positive integer.

In general, set $A_{\mathbf{Z}} = \Gamma N/N$ and $T = A/A_{\mathbf{Z}}$.

Theorem C. *Any compact complex parallelizable homogeneous space with a pseudo-kähler structure is a pseudo-kählerian torus bundle over a pseudo-kählerian torus T and, up to a finite covering of the fiber, is a torus bundle over T which is the bundle product of several primitive spaces. Moreover, any primitive space is, up to a finite covering, a product of a torus and a reduced primitive space. For a primitive pseudo-kählerian manifold, m can be any given positive integer.*

Let us call a primitive pseudo-kähler space a *simple primitive pseudo-kähler space* if $N/N_{\mathbf{Z}}$ is a simple complex torus.

Let us call a primitive pseudo-kähler space a *double primitive pseudo-kähler space* if $N/N_{\mathbf{Z}}$ is isogenous to the product of two identical complex tori.

The example in [Ym1] is a double reduced primitive pseudo-kählerian space.

Theorem D. *Let M be a reduced primitive pseudo-kähler space. If M is not a simple primitive pseudo-kähler space as defined above, then it is a double reduced primitive pseudo-kähler space.*

Theorem E. *The generic reduced primitive pseudo-kähler spaces are simple reduced primitive pseudo-kähler spaces.*

Combining these results with our splitting theorem (the Main Theorem 3 in [Gu4]) and our results in [Gu2], we have:

Theorem F. *Every compact complex homogeneous manifold with a pseudo-kählerian structure is the product of a projective rational homogeneous space and a solvable compact complex parallelizable pseudo-kähler space. In particular, any compact complex parallelizable pseudo-kähler space M is solvable. Moreover, any compact complex parallelizable pseudo-kähler space is a pseudo-kählerian torus bundle over a pseudo-kählerian torus T and, up to a finite covering of the fiber, is a torus bundle over T which is the bundle product of several simple and double primitive pseudo-kähler spaces.*

References

[BR] A. Borel & R. Remmert: Uber Kompakte homogene Kählersche Mannigfaltigkeiten, Math. Ann. **145** (1961), 429–439.

[DG] J. Dorfmeister & Z.-D. Guan: Classification of compact homogeneous pseudo-kähler manifolds, Comment. Math. Helv. **67** (1992), 499–513.

[Gu1] Z. Guan: Examples of compact holomorphic symplectic manifolds which admit no Kähler structure. In *Geometry and Analysis on Complex Manifolds—Festschrift for Professor S. Kobayashi's 60th Birthday*, World

Scientific, Singapore, 1994, pages 63–74.

[Gu2] D. Guan: A splitting theorem for compact complex homogeneous spaces with a symplectic structure, Geom. Dedicata **67** (1996), 217–225.

[Gu3] D. Guan: On compact symplectic manifolds with Lie group symmetries, Trans. Amer. Math. Soc. **357** (2005), 3359–3373.

[Gu4] D. Guan: Modification and the cohomology groups of compact solvmanifolds, Electron. Res. Announc. Amer. Math. Soc. **13** (2007), 74–81.

[Gu5] D. Guan: Classification of compact complex homogeneous spaces with pseudo-kählerian structures, preprint 2007.

[Ha] K. Hasegawa: Complex moduli and pseudo-kähler structure on three-dimensional compact complex solvmanifolds, Electron. Res. Announc. Math. Sci. **14** (2007), 30–34.

[Mt] Y. Matsushima: Sur les espaces homogènes kählériens d'un groupe de Lie réductif, Nagoya Math. J. **11** (1957), 53–60.

[Ym1] T. Yamada: A pseudo-kähler structure on a nontoral compact complex parallelizable solvmanifold, Geom. Dedicata **112** (2005), 115–122.

[Ym2] T. Yamada: A structure theorem of compact complex parallelizable pseudo-kähler solvmanifolds, Osaka J. Math. **43** (2006), 923–933.

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