Type I Almost-Homogeneous Manifolds of Cohomogeneity One—III

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Abstract: This paper is one of a series of papers in which we generalize our results in [Gu5] on the equivalence of existence of Calabi extremal metrics to the geodesic stability for any type I compact complex almost homogeneous manifolds of cohomogeneity one. In this paper, we actual carry all the results in [Gu5] to the type I cases. As requested by earlier referees of this series of papers, in this third part, we shall first give an updated description of the classification of compact almost homogeneous Kähler manifolds of cohomogeneity one. Then, we shall give a proof of the equivalence of the geodesic stability and the negativity of the integral in the first part. Finally, we shall address the relation of our result to Ross-Thomas version of Donaldson's K-stability. One should easily see that their result is a partial generalization of our integral condition in the first part. And we shall give some further comments on the Fano manifolds with the Ricci classes. The similar proofs of the results other than the existence for the type II cases are more complicated and will be done in [Gu6].

1 Introduction

This paper is one of a series of papers in which we finished the project of the existence of extremal metrics in any Kähler class on any compact almost homogeneous manifolds of cohomogeneity one.

In [Gu8,9] we proved that for the type I compact almost homogeneous Kähler manifolds of cohomogeneity one, the existence of Calabi extremal metrics is the same as the negativity of a topological integral. We also proved in [Gu9] that for any two Káhler metrics in the Mabuchi moduli

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space of Kähler metrics there is a smooth geodesic connecting them. That is, the geodesic principle I is true for these manifolds.

In this paper, we shall prove that the negativity of the integral is actually the same as the geodesic stability.

A classification which we refer to in this paper can be found in [Gu5 section 12].

2 Preliminary

Here we summarize some known results about compact complex almost-homogeneous manifolds of cohomogeneity one. In this paper, we only consider manifolds with a Kähler structure. For earlier results one might check with [Ak] and [HS].

We call a compact complex manifold an almost homogeneous manifold if its complex automorphism group has an open orbit. We say that a manifold is of cohomogeneity one if the maximal compact subgroup has a (real) hypersurface orbit. In [GC] and [Gu5], we reduced the compact complex almost homogeneous manifolds of cohomogeneity one into three types of manifolds.

We denote the manifold by M and let G be a complex subgroup of its automophism group which has an open orbit on M.

Let us assume first that M is simply connected. Let the open orbit be G/H, K be the maximal connected compact subgroup of G, L be the generic isotropic subgroup of K, i.e., K/L be a generic K orbit. We have [GC Theorem 1]:

Proposition 1. *If G is not semisimple, then M is a completion of a $\mathbf{C}^*$ bundle over a projective rational homogeneous space,*

If a compact almost homogeneous Kähler manifold is a completion of a $\mathbf{C}^*$ bundle over a product of a torus and a projective rational homogeneous space, we call it a *manifold of type III*. We have dealt with this kind of manifolds in our dissertation [Gu1,2]. There always exists an extremal metric in any Kähler class. Recently, we generalized this existence result to a family of metrics, which connects the extremal metric in [Gu2] and the generalized quasi-einstein metric [Gu11], called the extremal-soliton metrics in [Gu12]. The existence of the extremal-soliton is the same as the geodesic stability with respect to a generalized Mabuchi functional.

In general, if M is a compact almost homogeneous Kähler manifold and O is the open orbit, then $D = M - O$ is a proper close submanifold. Moreover, D has at most two components. We call each component of D an end. If D has two components (or one component), we say M is an almost homogeneous manifold with two ends (or one end). We have [HS Theorem 3.2]:

Proposition 2. *If M is a compact almost homogeneous Kähler manifolds with two ends, then M is a manifold of type III.*

Therefore, we only need to deal with the case with one end. Again, in the case of M being simply connected, we only need to take care of the case in which G is semisimple. If G is semisimple and M has two G orbits, one open and one closed, and moreover if the closed orbit is a complex hypersurface, there are two possibilities. Let k, l be the Lie algebras of K, L. Then the centralizer of l in k is a direct sum of l and a Lie subalgebra $\mathcal{A}$ with $\mathcal{A}$ being either one dimensional or a 3-dimensional Lie algebra $su(2)$. If $\mathcal{A}$ is one dimensional, we call M a *manifold of type I*. If $\mathcal{A}$ is $su(2)$, we call M a *manifold of type II*.

In general, if the closed orbit has a higher codimension, we can always blow up the closed orbit to obtain a manifold $\tilde{M}$ with a hypersurface end. we call the manifold M a *manifold of type I (or II)* if $\tilde{M}$ is of type I (or II).

There is a special case of the type II manifolds. If the open orbit is a $\mathbf{C}^k$ bundle over a projective rational homogeneous manifold, we call M an *affine type manifold* (not to be confused with the closed complex submanifolds of $\mathbf{C}^m$).

Then we have (see [Gu5 section 12]):

Proposition 3. *Any compact almost homogeneous Kähler manifold M of cohomogeneity one is an $Aut_0(M)$ equivariant fibration over a product of a rational projective homogeneous manifold Q and a complex torus T with a fiber F. Therefore, M can be regarded as a fiber bundle over T with a simply connected fiber M_1. One of following holds:*

(i) M is a manifold of type III.

(ii) M_1 is of type II but not affine.

(iii) M_1 is affine.

(iv) M_1 is of type I.

We say that M is a *manifold of type I (or type II, affine)* if M_1 is a manifold of type I (or type II, affine).

We actually can also obtain a structure of the M_1 bundle over T from [HS]. We only need to understand the bundle structure for the open orbit. By [HS Corollary 4.4] we have that the bundle structure is a product unless when we apply Proposition 3 to $\tilde{M}$ $F = Q^k$. In the latter case, there is an unbranched double covering $\bar{M}$ of M such that the bundle structure is a product. We have:

Proposition 4. *The M_1 bundle over T is a product except in the case with which the open orbit is a F_0 bundle over $Q \times T$ such that F_0 is in the second, sixth and eighth cases in [Ak p.67]. In the latter cases, the M_1 bundle has an unbranched double covering which is a product of M_1 and T.*

In [Gu8,9], we dealt with the type I cases.

3 More

This is only an announcement. If you need the whole paper, please send me an email.

[Ak] D. Akhiezer: Equivariant completions of homogeneous algebraic varieties by homogeneous divisors, *Ann. Glob. Analysis and Geometry,* vol. 1 (1983) 49–78.

[GC] D. Guan & X. Chen: Existence of extremal metrics on almost homogeneous manifolds of cohomogeneity one. *Asian J. of Math.* 4 (2000), 817–830.

[Gu1] Z. Guan: On certain complex manifolds, Dissertation, University of California, at Berkeley, Spring 1993.

[Gu2] Z. Guan: Existence of extremal metrices on almost homogeneous spaces with two ends. *Transactions of AMS,* 347 (1995), 2255–2262.

[Gu3] D. Guan: Examples of holomorphic symplectic manifolds which admit no Kähler structure II. *Invent. Math.* 121 (1995), 135–145.

[Gu4] D. Guan: Existence of extremal metrics on almost homogeneous manifolds of cohomogeneity one—II, *J. of Geom. Anal.* 12 (2002), 63–79.

[Gu5] D. Guan: Existence of extremal metrics on almost homogeneous manifolds of cohomogeneity one—III, *Intern. J. of Math.* 14 (2003), 259–287.

[Gu6] D. Guan: Jacobi fields and geodesic stability, in preparation.

[Gu7] D. Guan: Classification of compact complex homogeneous spaces with invariant volumes, *Transactions of AMS,* 354 (2003), 4493–4504.

[Gu8] D. Guan: Type I Almost Homogeneous Manifolds of Cohomogeneity One—I, accepted and to appear in *Pacific J. of Applied Mathematics*, preprint 2010.

[Gu9] D. Guan: Type I Almost Homogeneous Manifolds of Cohomogeneity One—II, accepted and to appear in *Pacific J. of Applied Mathematics*, preprint 2010.

[Gu10] D. Guan: On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundles, Math. Res. Lett 6 (1999), 547–555.

[Gu11] Z. Guan: Quasi-einstein metrics, Intern. J. Math. 6 (1995), 371–379.

[Gu12] D. Guan: Exponential C^∞ convergence of the modified Calabi flow, preprint 2005.

[HS] A. Huckleberry & D. Snow: Almost Homogeneous Kähler Manifolds with Hypersurface Orbits, Osaka J. Math. 19 (1982), 763–786.

[PS] F. Podesta & A. Spiro: New examples of almost homogeneous Kähler-Einstein manifolds, *Indiana Univ. Math. J.* 52 (2003), 1027–1074.

[Si] A. Spiro: The Ricci tensor of an almost homogeneous Kähler manifold, *Adv. Geom.* 3 (2003), 387–422.

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