On Classification of Compact Complex Solvmanifolds

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In this paper, we complete Nakamura's classification of compact complex parallelizable solvmanifolds up to the complex dimension five. We find that the holomorphic symplectic ones are either nilpotent or pseudo-kähler-like, i.e., with a complex solvable Lie group as that of a compact complex solvable pseudo-kähler space in [Gu1]. We also found that, for any even complex dimension, all the compact complex pseudo-kähler solvmanifolds are hypersymplectic. Therefore, for compact complex solvmanifolds, hypersymplectic is as general as pseudo-kähler.

1 Introduction

Let M be a complex manifold, ω be a closed differential 2-form representing a class in $H^2(M, \mathbf{R})$. If $\dim_{\mathbf{C}} M = n$ and ω is nondegenerate at every point, i.e., $\omega^n \neq 0$ at every point, we call ω a symplectic structure. If ω is also in $H^{1,1}(M)$, we call it a pseudo-kählerian structure of M. If, at the other end,

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ω is in $H^{2,0}(M)+H^{0,2}(M)$, we call it a holomorphic symplectic structure. In the latter case, ω is just the real part of the classical holomorphic symplectic 2-form.

A compact complex homogeneous space with a pseudo-kählerian structure (not necessary invariant) was classified in [Gu1]. It is a product of a classical projective homogeneous space and a pseudo-kähler complex solvmanifold.

Here, we say that a compact complex manifold is a compact complex solvmanifold if it is a quotient of a complex solvable Lie group over a cocompact discrete subgroup.

It turns out that all the pseudo-kähler complex compact solvmanifold have holomorphic symplectic structures (actually a hypersymplectic structure) when it has an even complex dimension. When the complex dimension is odd, we can always make it even by product with a complex torus. Actually we prove in [Gu1], see also Theorem 1, that if a compact complex solvmanifold is pseudo-kähler-like, i.e., if the complex Lie group is the same as some of the pseudo-kähler ones, then the manifold has a right-invariant holomorphic symplectic structure coming from the universal covering. These are a little bit more than those manifolds which are actually pseudo-kählerian. For example, The examples III-(3a) in [Nk] are pseudo-kähler-like but not pseudo-kähler (notice that the concrete construction given by them does not really work. One might choose other complex algebraic unite α of order four, e.g., the square root of $2^{-1}(-3+\sqrt{5})$). In general, the pseudo-kähler-like ones are not pseudo-kähler and therefore are not hypersymplectic. But a finite covering of them are pseudo-kähler and hypersymplectic.

On the other hand, by the method in [Gu2], it is easy to construct compact complex nilmanifold with holomorphic symplectic structures. Therefore, one has the following natural question:

Question 1: *Are all the compact holomorphic symplectic solvmanifolds from some kind of combination of these two classes of holomorphic symplectic manifolds?*

From all the information we already have, this might be true in certain sense (see the last section). In this paper, we shall see that this is true for the case in which the complex dimension is ≤ 5. And we shall see that this is also true for the six dimensional case in a different paper.

By [Gu1], any pseudo-kähler compact complex solvmanifolds are Chevalley. Therefore, one might have following question:

Question 2: *Let $M = G/\Gamma$ be a compact complex solvmanifold with a holomorphic symplectic structure. Could the Lie algebra of G be a direct sum of two Lie subalgebras A and N such that $[A, N] \subset N$ with A abelian, N nilpotent? That is, could G be Chevalley in the terminology of [Nk]?*

We shall see that this is true for those cases in which the complex dimension is ≤ 5. In a different paper, we shall see that this is true for the six dimensional case. We expect that this is true for any dimension compact complex solvmanifolds with holomorphic symplectic structures.

The reason that we choose to deal with these cases in which the complex dimension ≤ 5 is not because that is how far we can go. It is because that it is convenient for us since there was already a (long) list of the possible ones in [Nk] for the cases in which the complex dimension is ≤ 5. For the higher dimensional case, the work would be a little bit tedious if we do not apply a more systematic argument. Therefore, we shall deal with the higher dimensional cases only in the future.

The classification of compact complex solvmanifolds is also important for the classification of compact complex homogeneous spaces as it was shown in [Gu3]. In this paper we obtain a much shorter list and completely solve the classification problem for the compact complex solvmanifolds of dimension ≤ 5. Therefore, we have:

Classification Theorem. *There are only seven types of compact complex non-nilpotent solvmanifolds of complex dimension* ≤ 5*. They are:*

(1) III-(3) in [Nk] in complex dimensional 3*. See the manifold at the beginning of the third section. A product with a 1-dimensional complex torus is a holomorphic symplectic solvmanifold.*

(2) IV-(4) and IV-(6) in [Nk] in complex dimension 4*. Only IV-(4) has a holomorphic symplectic structure. That is the same as the one given by III-(3) in the universal covering. For the manifold IV-(6), see the middle of the third section.*

(3) V-(7), V-(12), V-(15), V-(17) with $\alpha = -1$ *in [Nk] in complex dimension* 5*. Only V-(7) and V-(17) have holomorphic symplectic structures after producting with a 1-dimensional complex torus. The one from V-(7) is a product of IV-(4) with a 2-dimensional complex torus in the universal covering. The one from V-(17) is a torus bundle product of two (possibly different) copies of VI-(4). The description of these manifolds can be found in the fourth section.*

We note here that H. C. Wang studied the compact complex parallelizable manifolds, i.e., compact complex manifolds with trivial tangent bundles. He proved in [Wa] that these manifolds are complex homogeneous manifolds, i.e., compact quotients of complex Lie groups by their cocompact discrete subgroups. In [Nk], the author dealt with the case in which the big complex

Lie groups are solvable. Therefore, our classification theorem also applies to compact complex paralizable solvmanifolds of complex dimension ≤ 5.

Combining with [Nk] and [Gu3], it will be easy to classify compact complex homogeneous spaces of complex dimension ≤ 5. To make this paper easier to the reader, we shall treat this in another paper. The classification of compact complex homogeneous spaces of complex dimension three is due to Tits [Ti2]. One note I like to make here is that by [Gu3] for a compact complex homogeneous manifold, the nonzero part of the first Chern class only come from the factors with the trivial actions of the semisimple part of the bigger group on the their radicals. That is, the calculation reduces to the one similar to that in [Gu8] since the given Cartan subalgebras for the other factors only have zero dimensional intersections with the isotropy subalgebra.

In [Gu4], we prove that any compact complex homogeneous manifold with a holomorphic symplectic structure is actually a complex solvmanifold. Although the argument for the pseudo-káhler case and the general real symplectic case had a gap (it was fixed in [Gu5, 6, 1]), the argument worked well for the holomorphic symplectic case, which was our major purpose there. A classification of compact solvable complex parallelizable manifolds with holomorphic symplectic structures is overdue.

Here, I thank Professor Salamon for telling me the Schanuel's conjecture which led me to the solution of complex dimension five and led my attention to Alan Baker's solution [Ba] (Theorem 1 there) on Gelfond conjecture, which led Baker to his reward of the Fields Medal in 1970. I also like to express my thank to Professor Bogomolov for mentioning Gelfond's solution of the Hilbert seventh problem, i.e., the Euler-Hilbert conjecture. That led me to the reference [Ge] in the references. I also thank T. Yamada for drawing my attention to Nakamura's paper [Nk]. I also thank A. Fino for mentioning Witte's work [Wi] to me. It turns out that our modification in [Gu6] is already in [Wi], e.g., Proposition 8.2. He just used a different terminology. He actually modified the original Lie group by a compact torus. That is, the modification has only pure imaginary eigenvalues. Actually, we can use Hattori's reult in [Ha] instead of the Mostow's result in [Gu5]. See the remark after the proof of Corollary 3. I also take this chance to thank the editors and the referees for their useful comments.

2 The Compact Complex Pseudo-kähler-like and Hypersymplectic Solvmanifolds

In this section, we shall quickly review the pseudo-kähler-like case, using the third proof mentioned in [Gu1]. Let $M = G/\Gamma$ be a compact complex solvmanifold with a pseudo-kählerian structure. In [Gu1], we found a real solvable Lie group G_1 such that there is a cofinite subgroup Γ' of Γ and the algebraic closure of G_1 is the same as that of Γ'. We proved in [Gu1] that the complex structure is right invariant under G_1 and the nilradicals N_G does not affected by the modification, $M' = G_1/\Gamma'$ is a finite covering of M. By Mostow's theorem, we can assume that the pseudo-kähler structure ω is right invariant under the action of G_1.

We consider a fiber bundle

$$G_1/\Gamma' \to G_1/\Gamma' N_G = B,$$

with a fiber F.

Now, any element α in $H^1(N_G/\Gamma' \cap N_G, \mathbf{R})$ is $\beta + \bar{\beta}$ with a holomorphic right invariant form β since N_G is nilpotent and complex. By α being closed, we have that β is closed. This means that $d\beta = 0$ and β is not from the derivation $[N_G, N_G]$ but from $N_G/[N_G, N_G]$ since N_G is complex (see [Gu1]).

Let $\omega_F = \omega|_{N_G/\Gamma' \cap N_G} \in H^2(N_G/\Gamma' \cap N_G, \mathbf{R})$. By $d\omega_F = 0$ and $\omega_F = \sum_i a_i \beta_i \wedge \bar{\beta}_i$ with β_i a basis of holomorphic differential form on $N_G/\Gamma \cap N_G$ we see that $d\beta_i = 0$ if $a_i \neq 0$. That is, ω_F comes from $N_G/[N_G, N_G]$ also.

Let N_1 be the kernel of ω on N_G. N_2 be a complement of N_1 in N_G. Let A be a complement of the Lie algebra of N_G in the Lie algebra of G_1 which is orthogonal to N_2 with respect to ω. Here, for convenient, we also denote the Lie algebra of N_G by N_G when there is not confusion. Then $\omega \in \wedge^2 A^* + A^* \wedge N_G^* + \wedge^2 N_G^*$. We write $\omega = \omega_0 + \omega_1 + \omega_2$ correspondingly.

As in [Gu1], we see that for any $a_1, a_2 \in A$ (being (1,0) vectors)

$$[a_1, n], [\bar{a}_2, n] \in N_G$$

(being (1,0) vectors) for any $n \in N_G$ (being (1,0) vectors).

There is a relation between the differential of the differential 1-forms and the Lie bracket. See (1.1) in [Nk] for example. See also [Nk] Lemma 1.1 (2) for another interpretation for a 2-form to be closed. This works both for the complex case in this paper and the real case. e.g., for the Kodaira-Thurston surfaces.

We want to see that $\omega_1 \in A^* \wedge N_G^*$ is in $H^1(B, H^1(F))$. $\omega_1 = \sum_j (\alpha_j \wedge \bar{\beta}_j + \bar{\alpha}_j \wedge \beta_j)$ with $\beta_j \in N_G^{\mathbf{C},*}$ being holomorphic.

$$\partial \omega_1 = \sum_j (\alpha_j \wedge \partial \bar{\beta}_j + \bar{\alpha}_j \wedge \partial \beta_j).$$

We notice that by our assumption $\partial \bar{\beta}$ can not have any term of $\bar{\alpha} \wedge \beta'$ for any $\beta, \beta' \in N_G^{\mathbf{C},*}$ being holomorphic by our result in the last paragraph (see also the last statement of Lemma 5 in [Gu1]). We have

$$\partial \bar{\beta}_j = \sum a_j^{kl} \alpha_k \wedge \bar{\beta}_l + \sum A_j^{kl} \alpha_k \wedge \bar{\alpha}_l,$$

$$\partial\beta_j = \sum b_j^{kl}\alpha_k \wedge \alpha_l + \sum c_j^{kl}\alpha_k \wedge \beta_l + \sum d_j^{kl}\beta_k \wedge \beta_l.$$

If d_j^{kl} is not zero, one can not cancel out the similar terms from a different $\bar{\alpha}_{j'} \wedge \beta_{j'}$ by α_j being linearly independent. It can not cancel out with those terms from ω_2 neither. Therefore, by closedness of ω, all the d_j^{kl} must be zeros. We have $\beta_j \in H^1(F)$. We can identify ω_2 with ω_F.

So, a right invariant pseudo-kähler form on G_1/Γ' actually comes from $M_1 = G_1/[N_G, N_G]\Gamma'$ since all the three parts in ω come from M_1. By the nondegeneracy of the pseudo-kählerian form we see that $G_1/\Gamma' = M_1$.

Lemma 1.: *If M is a compact complex pseudo-kählerian solvmanifold, then M is a complex torus bundle over a complex torus and up to a finite covering has a right G_1 invariant pseudo-kählerian structure. G and G_1 have abelian nilradicals.*

To go further, we now replace N_G by $N = [\mathcal{G}, \mathcal{G}]$. That is, we let A be a complement of N. We use β_j for a basis of the dual of N and α_i for a basis of the dual of A. Again we have the decomposition of ω into three parts. Similarly, we can define N_1 and N_2.

Now, let b_j be the dual of $\beta_j \in N_1^*$, then for any $n \in N$, $\omega(b_j, n) = 0$. Therefore, for any $a \in A$,

$$\omega([a, b_j], n) = -\omega(b_j, [n, a]) = 0.$$

That is, N_1 is an ideal.

Notice that by our construction of the modification group G_1, we only modified the action by a semisimple torus. We might assume that both N_1 and N_2 are invariant under the action of the modification torus and all the b_k are eigenvectors. Then the only nonzero a_j^{kl} are a_j^{kj}. Again, by the linearly independent property we obtain that all a_j^{kl} are zeros. That is, N_1 is also in the kernel of the modification torus action. All the nonzero eigenvectors of the modification torus action are in N_2 (Cf [Wi] Proposition 8.2).

Now, we might assume that A is invariant under the modification torus. Then as in the proof of the Lemma 5 in [Gu1] we have:

Lemma 2.: *Let $N = [\mathcal{G}, \mathcal{G}]$ and A be the complement of N in $\mathcal{G}$ with respect to the prealgebraic toric abelian group T, Then for any $x, y \in A^{\mathbf{C}}$ such that $jx = ix$, $jy = -iy$, we have $[x, y] = 0$. Similarly, if $z \in N^{\mathbf{C}}$ is holomorphic, so is $[w, z]$ for any $w \in \mathcal{G}$.*

Let A_1 be all the elements in A generated by the dual of the differential forms in A^* involved in ω_1 which is the part of ω in $A^* \wedge N_1^*$. A_2 be a complement of A_1 in A including the subspace generated by the dual of the differential forms in A^* involved in ω_1' which is the part of ω in $A^* \wedge N_2^*$. This can be down by the definition of N_1. Therefore,

$$\omega \in \wedge^2 A^* + A_1^* \wedge N_1^* + A_2^* \wedge N_2^* + \wedge^2 N_2^*.$$

We have

$$\partial(\bar{\alpha}_j \wedge \beta_j) \in \bar{\alpha}_j \wedge (A^* \wedge (A^* + N^*))$$

and

$$\partial(\alpha_j \wedge \bar{\beta}_j) \in \alpha_j \wedge A^* \wedge \bar{\beta}_j,$$

$$\partial(\beta_j \wedge \bar{\beta}_k) \in A^* \wedge (A^* + N^*) \wedge \bar{\beta}_k.$$

Therefore, the first one must be zero. The modification only changes the imaginary part of the A action. Therefore, if the A action is trivial after modification, it should be trivial for the original action. By our assumption the modified group has the same algebraic closure as the cocompact subgroup, we see that the second one must be zero also. Another agument is that in the case that the first one is zero, even if the second one might not be zero, we can modified ω by an exact form to make the second one to be zero. That is $N_1 = 0$ and A is orthogonal to N with respect to ω.

Lemma 3.: *$N_1 = 0$ and A is orthogonal to N with respect to ω.*

Corollary 1.: *A is an abelian Lie subalgebra.*

Proof: For any $a_1, a_2 \in A$, we have that

$$\omega([a_1, a_2], n) = \omega(a_1, [a_2, n]) + \omega(a_2, [n, a_1]) = 0$$

for any $n \in N$. That is, $[a_1, a_2] = 0$.

Q. E. D.

Lemma 4.: *The action of A on N is semisimple.*

Proof: Since A is abelian, N is decomposed into Jordan blocks. Let J_1 and J_2 be two Jordan blocks such that $\omega(J_1, J_2) \neq 0$. Now, $\bar{A}$ acts semisimply and ω is (1,1), we can assume that $J_2 \subset \bar{N}_2$ is an eigenvector b of A. then $J_1 \subset N$.

Let $b_1, \cdots, b_s$ be a basis of J_1. We might assume that $\omega(b_s, b) \neq 0$ by choosing the right J_2. Let $k_1(a)$ and $k_2(a)$ be the eigenvalues, we have:

$$k_1(a)\omega(b_s, b) = \omega([a, b_s], b) = \omega(b_s, [b, a]) = -k_2(a)\omega(b_s, b).$$

Therefore, $k_1(a) = -k_2(a)$.

We also have

$$\omega([a, b_{s-1}], b) = \omega(b_{s-1}, [b, a]).$$

Therefore, $[a, b_{s-1}] = k_1(a)b_{s-1}$. This implies that $s = 1$. J_1 has complex dimension 1.

Q. E. D.

Proposition 1.: *If $\omega(n_1, n_2) \neq 0$ for eigenvectors n_1, n_2 with eigenvalue functions $k_1(a), k_2(a)$, then $k_2(a) = -\bar{k}_1(a)$. Moreover, all $k_j(a)$ are real. $\omega(n_1, jn_1) = 0$ always.*

Proof: Since ω is a (1,1) form, one of them can be chosen as antiholomorphic and the other one as holomorphic.

$$k_1\omega(n_1, n_2) = \omega([a, n_1], n_2) = \omega(n_1, [n_2, a]) = -\bar{k}_2\omega(n_1, n_2).$$

Let $k_j(a) = \sum(a_i\alpha_i + b_i\bar{\alpha}_i)$, then

$$\partial(\beta_j \wedge \bar{\beta}'_j) = (-\sum_i(a_i - \bar{b}_i)\alpha_i) \wedge \beta_j \wedge \bar{\beta}'_j = 0.$$

That is $b_i = \bar{a}_i$. Therefore, $k_j(a) = \sum_i(a_i\alpha_i + \bar{a}_i\bar{\alpha}_i)$ is real.

By $[a + t(a), jn_1] = j[a + t(a), n_1]$ and $jt(a)n_1 = t(a)jn_1$ we have that $[a, jn_1] = j[a, n_1] = k_1(a)jn_1$. Therefore, $\omega(n_1, jn_1) = 0$ since $k_1 \neq 0$.

Q. E. D.

Now, since $\Gamma' \in G_1$, we see that Γ' also acts semisimply with pairs of real eigenvalues.

Proposition 1 gives us a good picture for the Lie algebra of the compact complex solvmanifold with pseudo-kählerian structures. And therefore, it gives a good classification for the compact complex pseudo-kähler-like solvmanifolds.

What we have for G_1 is only the modified Lie group. The original complex Lie group G has the structure equations: $d\alpha_i = 0$, $d\beta_{2j-1} = -k_j \wedge \beta_{2j-1}$, $d\beta_{2j} = k_j \wedge \beta_{2j}$ where we regard k_j as (1,0) forms on A.

There are natural closed holomorphic 1-forms α_i and 2-forms:

$$\beta_{2j-1} \wedge \beta_{2j}.$$

Theorem 1.: *All the pseudo-kähler-like compact complex solvmanifolds are holomorphic symplectic, up to product with a torus of complex dimension one if it is needed.*

Actually, we see that the pseudo-kähler manifolds with even complex dimensions are complex hyper-quaternion-symplectic manifolds. That is there are three linear transformation $T_i,\ i = 1, 2, 3$ such that (1) $T_i^4 = I$ the identity, $T_iT_j = -T_jT_i = \epsilon_{ij}T_k$ where ϵ_{ij} are signs; (2) T_1 is a complex structure; (3) there is a symmetric two form h such that $h(\ ,T_i\ )$ are symplectic structures. (1) is called hyperquarterionic. (2) is called complex. (3) is call symplectic. We note that in our definition, we do not assume that T_2, T_3 have any integrability.

Especially, if $T_2^2 = -I$, we call it a hypercomplex-symplectic structure. We call it a *hypersymplectic* structure if $T_2^2 = I$ and it was first defined by Hitchin [Hi] in 1990. I heard about Hitchin's definition in a talk given by Gueo Grantcharov in 2011. To find hypersymplectic structures on compact solvmanifolds, it will be very natural to find the pseudo-kähler ones. Our results show that the hypersymplectic ones are as general as the pseudo-kähler ones at least for the compact complex solvmanifolds. We call a hypercomplex-symplectic manifold *hyperkähler* if h is positive definite. The hypercomplex-symplectic manifolds are very rare. But we do not have much control of the index of h. Actually, we can use both h or $-h$ from a hyperkähler manifold. And a product of hyperkähler manifolds with different chosen signs for h can have a big range of the index of the h. We shall see that the index of h for any hypersymplectic manifold must be 0. One might ask:

Question 3: Are all the compact simply connected hypercomplex-symplectic manifolds hyperkähler?

Question 4: Are all the compact simply connected holomorphic symplectic manifolds hypersymplectic up to a lower dimensional subset?

Any product of hyperkähler manifolds, hypercomplex-symplectc and hypersymplectic manifolds are both holomorphic symplectic and hyper-quaternion-symplectic manifolds.

Of course, in our case, the complex hyper-quarterion-symplectic structures are usually not right invariant. Actually, any compact complex solvmanifold with a pseudo-kähler structure is hypersymplectic if it has a complex even dimension.

If only (1) and (2) are true with $T_2^2 = I$, according to [AS], we have an almost complex product structure. The almost complex product structure is very general. Let $x + iy$ be the differential form generating $\mathbf{C}$. We define T_1 to be the standard complex structure, and T_2 to be defined by $T_2(x) = x$ and $T_2(y) = -y$. Then we have an almost complex product structure on the standard $\mathbf{C}$.

On the other hand, if there is an almost complex structure on a vector space, the T_1, T_2, T_3 generate an $sl(2, \mathbf{R})$ action on the vector space. T_2 generates a Cartan subalgebra. By the eigenvalues of T_2, we see that all the irreducible representations of $sl(2, \mathbf{R})$ in this vector space are the standard $\mathbf{C}$. That is, the vector space is $\mathbf{C}^n$ with the standard almost complex product structure.

Now, assume that (h, T_1, T_2, T_3) is a hypersymplectic structure on a vector space. By $h(x, T_2 y)$ skewsymmetric, we have

$$h(T_2 x, T_2 y) = -h(y, T_2(T_2 x)) = -h(y, x) = -h(x, y).$$

Therefore, both eigenspaces of T_2 are in the nullcone of h. Moreover, by

$h(x, T_1 y)$ skewsymmetric, we have that $h(x, T_1 x) = 0$. That is, any standard copy of $\mathbf{C}$ is also in the nullcone of h. Now, if we let $\beta_1 = x_1 + iy_1$ generate a copy of the standard $\mathbf{C}$, by the nondegeneracy of h, there is a $\beta_2 = x_2 + iy_2$ such that the h is nondegenerate on the space generated by β_1, β_2. In particular, we have that h is proportional to $x_1 \cdot y_2 - x_2 \cdot y_1$ since $h(x_2, y_1) = h(T_1 x_2, T_1 y_1) = -h(y_2, x_1) = -h(x_1, y_2)$. The corresponding pseudo-kähler metric $h(x, y) + ih(x, T_1 y)$ is proportional to $\beta_1 \bar{\beta}_2 - \beta_2 \bar{\beta}_1$. The corresponding holomorphic symplectic form $\omega = h(x, T_2 y) + ih(x, T_3 y)$ is proportional to $\beta_1 \wedge \beta_2$. This fits quite well with the compact complex solvmanifolds with pseudo-kähler structures (see [Gu1] or Proposition 1).

Actually, this also fits with the Kodaira-Thurston surface. Of course, the Kodaira-Thurston surface is not a compact complex solvmanifold in this paper, instead it is only a compact quotient of a *real* Lie group with a cocompact discrete subgroup. The structure equations are

$$dz = dx_1 = dx_2 = 0, \qquad dy = x_1 \wedge x_2.$$

We just let $\beta_1 = x_1 + ix_2, \beta_2 = z + iy$, which also defind a right-invariant complex structure. Then everything go through. This was also shown much earlier by Kamada in [Ka].

Corollary 2.: *All the even dimensional compact complex solvmanifolds with pseudo-kähler structures has a hypersymplectic structure. The same is true for the Kodaira-Thurston surface.*

Corollary 3.: *For any pseudo-kähler-like compact complex solvmanifolds, there is a finite covering which admits a hypersymplectic structure.*

Proof: From [Gu1], the manifold is pseudo-kähler if and only if the discrete subgroup only has real eigenvalues. This can be also observed by the fact that the modification does not affect the discrete group at all. This can be achieved by the fact that every algebraic unit has a finite power which is real.

If a is an algebraic unit, so is $b = a\bar{a}^{-1}$. By the Dirichlet's Theorem on the units, seee [BS] page 112, we see that b is a root of 1.

Q. E. D.

The argument in the proof also implies that for any compact real solvmanifold, there is a finite covering such that the discrete subgroup has only real eigenvalues. This also implies that in [Gu5], we can apply Hattori's result in [Ha] instead of the Mostow's result therein. Also, in the proof of [Gu1], we have $A_1 = \overline{H_*/H_* \cap N_G} = H/H \cap N_G \subset V$ and $p = 0$, $C \subset V$ in the proof of the theorem 1 there in **2.** of the second section. The modification can be obtained by just the projection $K \times V \to V$, that is, forgetting the K effects. Therefore, the modification only modifies the imaginary part.

In [Gu9, 10], we see that Question 4 is positive for the examples there including the K3 surfaces. We now have the following questions which is related to the fourth question:

Question 5. Are some of the Hilbert scheme of the Kodaira-Thurston surface hypersymplectic?

Question 6. Are all the simply connected holomorphic symplectic manifolds birational to some quotient of hypersymplectic solvmanifold?

We noticed that for the K3 surface, Proposition 1 in [Ko] implies that there is no nonkählerian pseudo-kähler structure.

3 Compact complex solvmanifold of complex dimension three and four

In [Nk], Nakamura and the Kodaira group classified all the three dimensional compact complex solvmanifolds. The only non-nilpotent one they obtained is the III-(3) case in [Nk].

It is exactly what we obtained in the last section with the Lie algebra generated by a_1, b_1, b_2 and $a = 1$. In the dimension three case we can always make $a = 1$ by choosing a_1 properly.

To construct the solvmanifold, we need to make a lattice in A. If we regard the A action as a $G_m = \mathbf{C}^*$ action on N, there is a natural generator $2i\pi$ which gives $1 \in G_m$ by the exponential map. The other generator might come from an algebraic unit number α such that $\log \alpha$ is linearly independent of $2i\pi$. If it is real, then we have a pseudo-kähler solvmanifold, the III-(3b) in [Nk]. If α is not a real number, then we only have a pseudo-kähler-like solvmanifold III-(3a) in [Nk] (notice that the example they gave does not work since one of the eigenvalue should be α and $\alpha + 3$ is a cubic root of -1, that is, up to finite covering the action on the nilradical is trivial). Both of them have a holomorphic symplectic structure

$$\beta_1 \wedge \beta_2 + \alpha_1 \wedge \gamma$$

after product with a one dimensional complex torus, where γ comes from the torus. III-(3b) has an obstructed deformation but not for III-(3a).

For complex dimension four, [Nk] gave four possible Lie algebras: IV-(4), IV-(5), IV-(6), IV-(7).

IV-(4) is the same as III-(3) product with a torus in the universal covering. We just let $\alpha_2 = \gamma$ from above.

They proved in [Nk] page 110, the paragraph after Lemma 6.2, that IV-(7) does not exist (not IV-(6), there was a typo).

IV-(6) has the structure equations: $d\alpha_1 = 0$, $d\alpha_2 = \alpha_1 \wedge \alpha_2$, $d\alpha_3 = -\alpha_1 \wedge \alpha_3$ then $d\alpha_4 = \alpha_2 \wedge \alpha_3$. It is a central extension of III-(3) with the closed holomorphic 2-form $\alpha_2 \wedge \alpha_3$. There is no holomorphic symplectic structure, but it is more like a holomorphic contact solvmanifold in a certain sense. By modification from [Gu5](see also [Gu1]), we also see that there is not real symplectic structure. By the argument in the proof of corollary 3 in the last section, we could assume that the discrete subgroup only has real eigenvalues. By writing $\alpha_k = x_k + iy_k$ we have that the structure equation for the modified Lie group is: $dx_1 = dy_1 = 0$, $dx_2 = x_1 \wedge x_2, dy_2 = x_1 \wedge y_2$, $dx_3 = -x_1 \wedge x_3, dy_3 = -x_1 \wedge y_3$, $dx_4 = x_2 \wedge x_3 - y_2 \wedge y_3$, $dy_4 = x_2 \wedge y_3 + y_2 \wedge x_3$. Assume that ω be a real symplectic structure, X_k, Y_k are the dual of x_k and y_k. Then

$$\omega(X_4, X_k) = \omega(X_4, Y_j) = \omega(Y_4, X_k) = \omega(Y_4, Y_j) = 0$$

for all $k \neq 1$ and j. For example,

$$\omega(X_4, X_2) = \omega(X_4, [X_1, X_2]) = \omega([X_4, X_1]) + \omega([X_2, X_4], X_1) = 0$$

and

$$\omega(X_4, Y_1) = \omega([X_2, X_3], Y_1) = 0.$$

Therefore, there is a nonzero 2-vector (a, b) such that $aX_4 + bY_4$ is in the kernel of ω, a contradiction.

Now, we deal with IV-(5) with structure equations: $d\alpha_1 = 0$, $d\alpha_2 = \alpha_1 \wedge \alpha_2$ then $d\alpha_3 = a\alpha_1 \wedge \alpha_3$ and $d\alpha_4 = -(1+a)\alpha_1 \wedge \alpha_4$ with $a(1+a) \neq 0$.

They were not able to determine the existence of this one (see [Nk] page 110 last paragraph, not IV-(5), there was a typo. See also page 86, the paragraph right before the Prelimilaries and page 94, page 98 Case 3 and page 100, 110, etc.).

Now, by applying [Ti1] Theorem 7.2 to our circumstance we see that the representation of the algebraic G_m is isogent to a product of irreducible $\mathbf{Q}$ representations. Let $\log \alpha$ be a generator which is linearly independent with $2i\pi$. One of the other eigenvalues should be either α or α^{-1}. One might also apply the argument in [Bo] Chapter 7, section 5, no. 9, p.44 that the decomposition is actually produced by polynomials with rational coefficients. But then this contradicts to the condition or it becomes IV-(7). This is impossible.

We have:

Theorem 2.: *The only possible compact complex non-nilpotent solvmanifolds of dimension three or four are III-(3), IV-(4), IV-(6) in [Nk]. All the holomorphic symplectic related ones are pseudo-kähler-like.They are III-(3) and IV-(4). Moreover, IV-(6) does not admit any real symplectic structures.*

The nilradical of IV-(6) has two steps and therefore, it is clear not pseudo-kähler-like by Proposition 1.

4 Compact complex solvmanifolds of complex dimension five and the Hilbert seventh problem

In [Nk], Nakamura and the Kodaira group had classified the possible structure equations for compact complex solvmanifolds of complex dimension five. There is always no problem for the existence of the nilpotent ones, once all the coefficients are integers.

The non-nilpotent solvable Lie algebras are: V-(7), V-(11) to V-(20).

V-(7) can be a product of VI-(4) with a torus. It is therefore pseudo-kähler-like.

They proved that V-(14) and V-(18) can not exist. But they can not determine the existence of V-(11), V-(13), V-(16), V-(19), V-(20) (see [Nk] page 110 the last paragraph).

V-(11) has the structure equations: $d\alpha_i = 0$ for $i = 1, 2$, $d\alpha_3 = \alpha_1 \wedge \alpha_3$, $d\alpha_4 = -\alpha_1 \wedge \alpha_4$, $d\alpha_5 = \alpha_1 \wedge \alpha_2$.

If V-(11) exist, its center is generated by a_5 the dual of α_5. Therefore, it has a quotient compact complex solvmanifold IV-(4).

IV-(4) has a nilradical generated by $\alpha_i, i \neq 1$. The G_m action of α_1 splits into two representations. One is generated by α_2 and the other, we denote it by N, is generated by α_3 and α_4. Therefore, α_2 corresponding to a lattice in $\mathbf{C}$ generated by γ_1, γ_2. The G_m action on N induces a lattice in $\mathbf{C}$ related to α_1 generated by $\log \alpha, 2i\pi$. α is an (could be quadratic) algebraic integer.

Similarly, in V-(11), the nilradical splits into two representations of the Lie group $\mathbf{C}$ corresponding to α_1. One is generated by α_2 and α_5. The other is just the preimage of N above, we also denote it by N. Now, it is not difficult to see that N is an ideal. Combinning all what we have above, we see that G/N induces a compact complex solvmanifold of complex dimension three. It is a III-(2) manifold in [Nk]. It is also called the Iwasawa manifold. We denote it by I.

Lemma 5. *For any α and $\gamma_i,\ i = 1, 2$, the induced Iwasawa manifold I does not exist.*

Proof: Let

$$\Pi = \begin{bmatrix} 1 & 2i\pi & a_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 1 & 0 & c_1 \\ 0 & 1 & \gamma_1 \\ 0 & 0 & 1 \end{bmatrix}$$

be two elements in Γ. Then $\Pi\Gamma_1^{-1}\Pi^{-1}\Gamma_1 \in \Gamma$. That is, $2i\pi\gamma_1$ is in the lattice in the $\mathbf{C}$ related to α_5. Similarly, so are $2i\pi\gamma_2$ and $\log(\alpha)\gamma_i,\ i = 1, 2$. Therefore, by $\log(\alpha)\gamma_1$, $2i\pi\gamma_1$ being linearly independent, there are rational numbers q_{ij} for $i, j \in \{1, 2\}$ such that $\log(\alpha)\gamma_2 = q_{11}\log(\alpha)\gamma_1 + q_{12}2i\pi\gamma_1$ and $2i\pi\gamma_2 = q_{21}\log(\alpha)\gamma_1 + q_{22}2i\pi\gamma_1$.

Therefore, let $x = \frac{\log(\alpha)}{i\pi}$, then x is a quadratic number. This contradict to the solution of the seventh Hilbert problem by Gelfond (1934) and Schneider (1935), which was also called the Euler-Hilbert conjecture.

Actually, we only need a special case of the Gelfond-Schneider Theorem obtained by Gelfond in 1929. Since x is not real and $x^2 + px + q = 0$ for two rational numbers p and q. we have $x = -p_1 + i\sqrt{q_1}$ for two rational numbers p_1 and q_1.

$$\alpha = e^{\log\alpha} = (-1)^x = (-1)^{-p_1}(-1)^{i\sqrt{q_1}}$$

being algebraic implies that $(-1)^{i\sqrt{q_1}}$ is algebraic. This contradicts to the statement in [Ge] p.102 if q_1 is not a square of another rational number. Now, if $q_1 = q_2^2$ with another nonzero rational number q_2, then $e^{\pi} = (-1)^{-i}$ is algebraic, a contradiction to another statement in the same page.

Q. E. D.

Corollary 4. *V-(11) does not exist. That is, the corresponding complex Lie group does not have any cocompact discrete subgroup.*

Now, the next one, V-(12) has the structure equation $d\alpha_i = 0,\ i = 1, 2$, $d\alpha_3 = \alpha_1 \wedge \alpha_3$, $d\alpha_4 = \alpha_2 \wedge \alpha_4$, $d\alpha_5 = -(\alpha_1 + \alpha_2) \wedge \alpha_5$.

This does exist, e.g., see [Ah] p.95 example 2. One might choose the totally real number field over $\mathbf{Q}$ extended by, e.g., the equation

$$x^3 + x^2 - nx - 1$$

with $n > 1$.

But there is no holomorphic symplectic structure. There is a closed (actually exact) holomorphic 2-form

$$\alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4 + (\alpha_1 + \alpha_2) \wedge \alpha_5$$

with a kernel of complex dimension one.

V-(13) does not exist by the same reason as IV-(5).

V-(15) is a product of IV-(6) and a torus. It does not admit any holomorphic symplectic structure. By the same argument as we discussed for IV-(6), there is no real symplectic structures on it.

V-(16) has a structure equations: $d\alpha_i = 0,\ i = 1, 2,\ d\alpha_3 = \alpha_1 \wedge \alpha_3$, $d\alpha_4 = -\alpha_1 \wedge \alpha_4$ and $d\alpha_5 = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4$.

We notice that the nilradical N is generated by $\alpha_i,\ i \neq 1$. There is a natural map

$$G/\Gamma \to T$$

with the fiber generated by N. $[N, N]$ is generated by α_5. There is another fibration

$$G/\Gamma \to M_1 = G/[N, N]\Gamma.$$

M_1 is a IV-(4) manifold and the G_m action has two representations. One generated by α_2 and the other by α_3 and α_4. Quotient by the first rational representation, take a finite covering if it is necessary, we obtain a fibration

$$M_1 \to M_2.$$

This induces, after taking a finite covering, a fibration

$$M \to M_2.$$

The fiber is generated by $\alpha_i, i = 2, 5$. We denote the fiber by F and the cocompact subgroup by Γ_F, the group by G_F. Now, G_F is an ideal of G, we can consider the adjoint action of G on G_F which induces the adjoint action of Γ on Γ_F. The kernel of this adjoint action is exactly N. Therefore, we can construct an Iwasawa manifold $(G/N\Gamma) \times_{Ad_{G_F}(G)} F$.

Applying our Lemma 5 we have a contradition.

Corollary 5.: *V-(16) does not exist.*

Next, we consider V-(17). The structure equations are: $d\alpha_1 = 0$, $d\alpha_2 = \alpha_1 \wedge \alpha_2$, $d\alpha_3 = a\alpha_1 \wedge \alpha_3$ and $d\alpha_4 = b\alpha_1 \wedge \alpha_4$, $d\alpha_5 = -(1 + a + b)\alpha_1 \wedge \alpha_5$.

Again by Tits' result, one of $a, b, -(1 + a + b)$ should be 1 or -1.

If $a = 1$, one of $b, -(2 + b)$ must be 1 or -1. Otherwise, we have a rational representation with eigenvalues $\{1, 1\}$. This can not be true since we need the trace to be zero. If $b = 1$ also, we get V-(18), which can not happen by [Nk] Lemma 6.2. If $b = -1$, we might just assume that $a = -1$ and $b = 1$.

Lemma 6. *In the case V-(17), $a = -1$.*

Now we consider the last two cases.

For the case V-(19), the structure equations are: $d\alpha_1 = 0$, $d\alpha_2 = \alpha_1 \wedge \alpha_2$, $d\alpha_3 = -\alpha_1 \wedge \alpha_3$, $d\alpha_4 = \alpha_1 \wedge \alpha_4 + \alpha_1 \wedge \alpha_2$ and $d\alpha_5 = -\alpha_1 \wedge \alpha_5 - \alpha_1 \wedge \alpha_3$.

$N = [G, G]$ is generated by $\alpha_i,\ i \neq 1$. $[N, G]$ is generated by $\alpha_i,\ i = 4, 5$. $G/[N, G]\Gamma$ is III-(3) (see below for the existence of the quotient manifold). Therefore, the lattice in $\mathbf{C}$ related to α_1 is generated by an algebraic number α and $2i\pi$.

By [Bo] Chapter 7, Section 5, no. 9, Theorem 1, any rational action A on N can be written as sn with s semisimple and n unipotent rational actions. This will induce a V-(17) manifolds with $a = -1$ by the semisimple part.

This is clear. See also [Au] Chapter IV section 2 for a similar construction.

Now, we also try to get another solvmanifold from the unipotent part. This can be done because $\mathrm{Ad}_N(G)$ is abelian. In this way, we get a V-(4) compact complex nilmanifold. See [Au] chapter IV section 2 for a similar construction. The structure equations of V-(4) are: $d\alpha_i = 0,\ i = 1, 2, 3$, $d\alpha_4 = \alpha_1 \wedge \alpha_2$ and $d\alpha_5 = -\alpha_1 \wedge \alpha_3$.

The semisimple model induces the so called Mostow or nilradical fibration (see [Nk] for example) and the nilpotent model induces a commutator fibration (see [Rg] for example).

As above, let

$$\gamma_1 = (g_1, g_2), \gamma_2 = (\alpha g_1, \alpha^{-1} g_2), \gamma_3 = (h_1, h_2), \gamma_4 = (\alpha h_1, \alpha^{-1} h_2)$$

be a basis which generates the lattice in $\mathbf{C}^2$ related to $\alpha_i,\ i = 2, 3$. Then

$$(2i\pi g_1, 2i\pi g_2), (\log(\alpha) g_1, \log(\alpha) g_2), (2i\pi\alpha g_1, 2i\pi\alpha^{-1} g_2), \log(\alpha)(\alpha g_1, \alpha^{-1} g_2)$$

are in the lattice Γ_0 of $\mathbf{C}^2$ related to $\alpha_i,\ i = 4, 5$ and they are linearly independent by the Gelfond-Schneider solution of the Euler-Hilbert conjecture in [Ge] p.104 or Theorem II in p.106..

Similarly, $2i\pi\gamma_3, \log(\alpha)\gamma_3 \in \Gamma_0$. As above we have

$$x = \frac{\log\alpha}{i\pi} = \frac{n_{21} + n_{22}x + n_{23}\alpha + n_{24}\alpha x}{n_{11} + n_{12}x + n_{13}\alpha + n_{14}\alpha x}$$

with some rational numbers n_{ij}. Again, by the Gelfond-Schneider Theorem ([Ge] p.106 Theorem II) we have a contradiction.

Lemma 7. *V-(19) does not exist.*

Now, we look at the possible V-(20) compact complex solvmanifold in [Nk] p.109. The structure equations are: $d\alpha_1 = 0$, $d\alpha_2 = \alpha_1 \wedge \alpha_2$, $d\alpha_3 = \alpha_1 \wedge (\alpha_3 + \alpha_2$, $d\alpha_4 = a\alpha_1 \wedge \alpha_4$ and $d\alpha_5 = -(2+a)\alpha_1 \wedge \alpha_5$ with $a(2+a) \neq 0$.

Again, we can pass it into the related semisimple and the nilpotent models as above by [Bo] Chapter 7, Section 5, no. 9, Theorem 1. The semisimple model is a V-(17) solvmanifold, we have that $a = -1$ by Lemma 6. From the nilpotent model we get a Lie group G_n. $N_n = [G_n, G_n]$ is generated by α_3. The center C_n of G_n is generated by $\alpha_i,\ i = 3, 4, 5$. Therefore, $G_n/\Gamma_n \to G_n/C_n\Gamma_n$ is a fiber bundle and the fiber has a subtorus $N_n/N_n \cap \Gamma_n$, which induces a torus bundle over $G_n/C_n\Gamma_n$. The latter is an Iwasawa manifold and we can apply Lemma 5. A contradiction.

Therefore, we have:

Corollary 6. *V-(20) does not exist.*

Combining all of what we have in this section, we get:

Theorem 3. *The five dimensional non-nilpotent compact complex solvmanifolds are V-(7), V-(12), V-(15), V-(17). The holomorphic symplectic related ones are pseudo-kähler-like. They are V-(7) and V-(17).*

V-(15) is not pseudo-kähler-like with the same reason as that for IV-(6). V-(12) is not pseudo-kähler-like because of Proposition 1. Again, as what we did for IV-(6), we see that V-(12) and V-(15) do not have any *real* symplectic structures. See also the comments at the end of the next section for V-(12). Therefore, we have following natural question:

Question 7: Are all the compact complex homogeneous solvmanifolds with real symplectic structures holomorphic symplectic?

5 Further comments

In general, our methods can reduce the classification of compact complex solvable manifold to the case in which the complex Lie group has a Chevalley decompositon $G = AN$ as in the question 2 such that A acts on N semisimply. We shall do this in a different paper. The symplectic form, after a series of modifications, has the form

$$\omega = \omega_0 + \sum \beta_{2i-1} \wedge \beta_{2i},$$

where ω_0 comes from those closed 1-forms and $\beta_{2i-1}, \beta_{2i} \in N^*$ are pairs of holomorphic 1-forms which corresponding to the pairs of eigenvectors with eigenvalues different by a sign. Different from the pseudo-kähler-like case, the 1-forms involved in ω_0 might also correspond to pure nilpotent elements with nontrivial adjoint actions.

To see some examples, we could just take any example in [Ya1] with real symplectic structures, then we complexify them by the principle of Proposition 4 in [Gu7] similar to what Yamada did in [Ya2]. For the semisimple actions, we just extend the action naturally. The $2i\pi$ with $e^{2i\pi} = 1$ will give the other generators we need in the lattice. For the nilpotent actions, we simply complexify the action as Yamada did.

Once we have some examples, we can always use the Proposition 4 in [Gu7] to construct more examples.

Another example comes from [BG] example 3. By our argument in the second section, it is not difficult to see that the example 2 there does not exist. But we can easily see that example 3 does exist. Let α be a root of the equation:

$$x^2 - nx + 1 = 0$$

with $n > 2$. Let $A = \mathrm{diag}(\alpha, \alpha^{-1})$ be the lattice for the $\mathbf{R}^2$ generated by $X_i,\ i = 1, 2$ be generated by $\gamma_1 = (1, 1), \gamma_2 = A\gamma_1 = (\alpha, \alpha^{-1})$. Similarly for the $\mathbf{R}^2$ generated by $Y_i,\ i = 1, 2$. Then, $A\gamma_2 - n\gamma_2 + \gamma_1 = 0$ and we have the action of a generator a we need for both X_i and Y_i. a acts on the first by A and the second by A^{-2}. For the lattice related to $Z_i,\ i = 1, 2$ we use $\gamma_1^2 = (1, 1), \gamma_1\gamma_2 = A\gamma_1^2$. We notice that $\gamma_2^2 = A\gamma_1\gamma_2$ and action of a on Z_i by A^{-1}. See also the construction in [SY] (I was told by C. Benson about this paper after I told him our construction. However, their further construction related to the example 2 can not have any compact complexification). After complexifying this example, we obtain an example of compact holomorphic symplectic solvmanifold such that the Lie group has three steps. This means the relation between the pseudo-kähler-like ones and the holomorphic symplectic ones in our theorems does not extend to the complex eight dimensional cases, actually not even to the complex seven dimensional cases since the same example can be obtained by a product of a complex one dimensional torus and a seven dimensional compact complex solvmanifold. But in a different paper we shall see that the relation is still true for the complex six dimensional compact complex solvmanifolds. Therefore, the answer for our question 1 is yes up to complex dimension 6 and partially true for higher dimensions. These examples also show that the question 3 is not true without simply connectedness even up to a finite covering for the non-nilpotent solvmanifolds.

However, we believe that the answer for our question 2 is yes always.

The major different from the pseudo-kähler-like case is that Lemma 1 and Lemma 4 do not work in general.

One may make other further and different modifications such that all the 1-forms involved in ω_0 are closed, semisimple and the β_j generate a subspace of the abelian nilradical. That is, it has a pseudo-kähler-like modification. This is also even true for the real symplectic solvmanifolds after modification. One consequence of this is that, again, V-(12) does not have any *real* symplectic structure by the opposite signs for the eigenvalues similar to the statement in Proposition 1. We shall deal with these progresses in another paper.

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