Classification of Compact Homogeneous Manifolds with Pseudo-kählerian Structures

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In this paper, we apply a modification theorem for a compact homogeneous solvmanifold to compact complex homogeneous manifolds with pseudo-kählerian structures. We are able to classify these manifolds as certain products of projective rational homogeneous spaces, tori, simple and double reduced primitive pseudo-kähler spaces.

1 Introduction

Let M be a complex manifold, ω be a closed differential 2-form representing a class in $H^{1,1}(M) \cap H^2(M, \mathbf{R})$. If $\dim_{\mathbf{C}} M = n$ and ω is nondegenerate at every point, i.e., $\omega^n \neq 0$ at every point, we call ω a pseudo-kählerian structure of M. In particular, if ω is the Kähler form of a hermitian metric, that is, $h(\ , \ ) = \omega(\ , J \ )$ is positive definite, we call ω a Kählerian structure on M.

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A compact complex homogeneous space with an invariant Kählerian structure was classified by Matsushima in [Mt2]. A compact complex homogeneous space with a Kählerian structure (not necessary invariant) was classified by Borel and Remmert in [BR]. A compact complex homogeneous space with an invariant pseudo-kählerian structure was classified by Dorfmeister and the author in [DG]. A solution of the classification of compact complex homogeneous space with a pseudo-kählerian structure was proposed in [Gu1,2]. However, the classification turns out to be much more complicated than we suggested therein.

In this paper, we shall put the last piece of the puzzle into the solution of this problem and completely solve it. The major piece of the puzzle which was missing in [Gu1,2] is the calculation of the cohomology group of a compact solvmanifold.

A compact real homogeneous manifold $M = G/H$ is called a solvmanifold if G is solvable and H is discrete. Here we always assume that G is connected and *simply connected.* If $\mathrm{Ad}(G)$ has the same real algebraic closure as that of $\mathrm{Ad}(H)$, we say that M has the Mostow condition with respect to G and H. When M is a solvmanifold and satisfies the Mostow condition, the cohomology of M can be calculated by the cohomology of the Lie algebra (see [Mo], [Rg Corollary 7.29]). But in general, it is very difficult to calculate the cohomology for a general compact solvmanifold. In this paper we solve the problem of for which solvmanifold we can apply Mostow's Theorem and first prove the following:

Modification Theorem. *If $M = G/H$ is a compact real homogeneous solvmanifold, there is a finite abelian covering space $M' = G/H'$, i.e., H/H' is a finite abelian group, such that there is another simply connected solvable* *real Lie group G' which contains H' and is diffeomorphic to G such that (1) $M' = G'/H'$, (2) M' satisfies the Mostow condition with respect to G' and H'.*

From our proof we can easily see that we can let $H' = H$ if and only if $\mathrm{Ad}H$ is in the identity component of the algebraic closure of $\mathrm{Ad}H$. We shall see later on that this is the case for the pseudo-kähler manifolds in this paper.

In particular, we partially solve the problem posted in [Mt1], [No], [Mo] and have:

Cohomology Theorem. *If G is solvable and H is discrete, $M = G/H$ is compact, then we have $H^*(M') = H^*(\mathcal{G}')$, where $\mathcal{G}'$ is the Lie algebra of the Lie group G' in the Modification Theorem.*

For the case in which $H \neq H'$, we need a little bit more work. We have that $H^*(M) = H^*_{H/H'}(M')$. Major refinements have been found at the end of the second section. But this is already good enough for this paper.

There is also a simpler proof for the modification theorem. Since G is solvable and simply connected, G has a faithful linear representation $\pi(G)$. We can just use the algebraic closure of $\pi(G)$ and use it in the place of $\mathcal{A}(G)$ in the next section. But we still do not know how to get the refinement with this new argument. If ρ is a finite dimensional representation, we can use $\pi + \rho$ in the place of π to get a similar result of the cohomology group with respect to ρ.

A smooth $2n$-dimensional manifold M equipped with a smooth transitive action of a Lie group is what we call a homogeneous space. If in additional M is a symplectic manifold, we refer to it as a *homogeneous space with a symplectic structure* and, if the structure is invariant, a *homogeneous space* *with an invariant symplectic structure*.

Recently there has been much progress in the area of symplectic manifolds and group actions. I was interested in the *classical* problem of classifying compact homogeneous spaces with symplectic structures. The difficulty is that we do not know anything about the transitive group and the isotropy group (Cf. [DG], [Hk], [Gu3]).

The case with invariant symplectic structure was solved (Cf. [Gu3]).

I am also interested in the structure of compact homogeneous manifold with any symplectic structure (which might not be invariant under the group action)

We also posted following conjecture therein [Gu3]:

CONJECTURE. *If G/H is a compact homogeneous space with a symplectic structure, then G/H is diffeomorphic to a product of a rational projective homogeneous space and another homogeneous space N, where N up to a finite quotent is a compact quotient of a compact locally flat parallelizable manifold with a symplectic structure.*

Our Modification Theorem will be a major step toward a proof of this conjecture. Here we call a manifold N *locally flat parallelizable* if $N = G/H$ for a simply connected Lie group G which is diffeomorphic to $\mathbf{R}^k$ for some integer k and H is a uniform (i.e., cocompact) discrete subgroup.

In this paper, we shall finish the complex case and prove:

Splitting Theorem. *If $M = G/H$ is a compact complex homogeneous space with a symplectic structure and G is complex (we can always assume this), then M is a product of a rational projective homogeneous space and a complex solvmanifold $N = N_G(H^0)/H$. Moreover, if we let $G_1 = N_G(H^0)/H^0$ and $\Gamma = H/H^0$, then there is a subgroup Γ' of Γ such* *that Γ/Γ' is finite and N up to a finite quotent is a solvmanifold G_2/Γ' with a right invariant symplectic structure of a real Lie group G_2 which contains the nilradical of G_1 and is possible different from G_1. In particular, if M has a pseudo-kählerian structure, so is N and if the pseudo-kählerian structure is also right invariant by G_1 action, N is a torus. And, if M has a holomorphic symplectic structure, then $M = N$.*

We notice that for the splitting theorem of [Gu1,2] to hold we only need a real symplectic structure which is invariant under the maximal compact subgroup and this is provided by the existence of G' in our Modification Theorem. Also, the complex structure is right invariant under the action of G_2, see Lemma 3 in the fourth section.

Some examples with holomorphic symplectic structures will be addressed also at the end of this paper.

The case when $M = N$ with a right invariant pseudo-kählerian metric was proven in [Gu1,2] (see [Gu2 proof of Lemma 1] or [DG 3.3] in 1989). The theorem in [DG] follows also from our splitting theorem. Applying the same method in [DG] to G_2 one can easily prove that if N is pseudo-kählerian, then the nilradical N_{G_1} of G_1 has at most two steps. An application of the Leray spectral sequence to the fiber bundle $G_2/\Gamma' \to G_2/N_{G_2}\Gamma'$ shows that the derivator $[N_{G_2}, N_{G_2}] = 0$, see Lemma 4 in the fifth section.

Moreover, with a breakthrough in Lemma 7 we prove in Corollary 1 and Theorem 2 the following:

Chevalley Theorem. *Let $M = G/\Gamma$ be a compact complex solvmanifold with a pseudo-kähler structure. The Lie algebra of G is a direct orthogonal sum of two abelian Lie subalgebras A and V such that $[A, V] = V$, $V \subset N_G$, with respect to a pseudo-kählerian structure ω which is right invariant on* *the universal covering under the modified Lie group G_2. In particular, G is Chevalley in the terminology of [Nk]. We also have that both $\omega|_A$ and $\omega|_V$ are closed, and are nonzero cohomology classes.*

This is done with a new algebra called *complex-parallelizabe-right-invariant-pseudo-kählerian algebra*, which is similar to what we used in [DG, Gu3] (see our proof of the Lemma 3). The first proof of this Theorem was done with a complicated calculation, as we did in [DG], in the March of 2007. Then I received a paper from Yamada with an elegant proof of the Lemma 4 in April 2007. Yamada's proof implied some rich structures of the manifolds, especially for the cohomologies. Then I refined his result to give a shorter proof. After we understand better the relation between this Theorem and the Leray spectral sequences, I obtain an even shorter proof, without using [Ym2] and hence [Wk], in the Aug. 2009. The third proof follows a simple argument as in [DG 3.3], which was also one of the fundamental arguments of [DG]. I believe that the new (third) proof catches the essence of the first proof. This paper was written with the second proof. We added the third proof accordingly. With proving of this theorem, one can give a proof for the whole paper basically without [Ym2] and [Wk]. See the remark after Lemma 7. Therefore, *the readers who are not familiar with [Wk] might carry out what I said in the remark as exercises.*

We also deal with the pseudo-kählerian case in the last four sections and give a classification, see our Theorem 6, and Theorem 9. The Borel-Remmert Theorem can also be a corollary.

We also give yet another proof of the Borel-Remmert Theorem. When the pseudo-kählerian structure is Kähler, the original average process in [Gu1,2] works and gives an averaged Kähler structure. Then the splitting theorem holds there and an application of our argument there produces another proof of the Borel-Remmert Theorem.

An announcement [Gu4] of our modification theorem (which is a little weaker than Theorem 1) was published with complete but sketch proofs. The classification has been announced in [Gu5].

As an application of our work on the compact complex homogeneous manifolds with pseudo-kähler structures we obtain many example of compact holomorphic symplectic solvmanifolds which have obstructed deformations:

Holomorphic Symplectic Theorem. *Every complex even dimensional compact solvable complex homogeneous manifold with pseudo-kählerian structure admits a right invariant holomorphic symplectic structure. In particular, every product of two complex odd dimensional compact solvable complex homogeneous manifolds with pseudo-kählerian structures admits a right invariant holomorphic symplectic structure.*

In the last part of this Theorem, one can pick one of the manifold to be a complex torus. A similar argument in [Nk] shows that the nonabelian ones have obstructed deformations.

Our classification also shows that the pseudo-kähler manifold is far from being Kähler manifolds as complex manifolds but is very close to Kähler manifolds as symplectic manifolds.

A classification of compact solvable complex parallelizable manifolds with holomorphic symplectic structures is overdue.

2 The Modification

1. Here we collect some results we need from the splitting theory of the solvable Lie group (see [Gb1]). Let $G = R$ be a simply connected Lie group. We call G a *splittable Lie group* if $R = TU$ with $T \cap U = \{e\}$ such that T acts semisimplely and U acts unipotently on the Lie algebra $\mathcal{G}$. We call a Lie group embedding $\alpha: G \to M(G)$ from G to a splittable simply connected solvable Lie group $M(G) = T \cdot S \cdot U$ a *Mal'cev splitting* or *M-splitting* if $\alpha(G)$ is a normal subgroup of $M(G)$ and $M(G)$ is a semidirect product of T and $\alpha(G)$, and $\alpha(G) \cdot U = M(G)$.

Lemma 1. *For any simply connected solvable Lie group G there is a unique Mal'cev splitting.*

The Mal'cev splitting can be constructed as following:

For the material of algebraic group, one might look at [Ch]. Especially, section 14 of chapter 2

Consider the adjoint representation $\mathrm{Ad}_G: G \to GL(\mathcal{G})$; put $G^* = \mathrm{Ad}_G(G)$, and let $< G >$ be the algebraic closure of G^* in $GL(\mathcal{G})$. Since $< G >$ is algebraic, it has a Chevalley decomposition

$$< G > = T^* U^*,$$

where U^* is the unipotent radical and T^* is abelian and consists of semisimple (i.e., completely reducible) elements. As a discrete algebraic group, $T^* \cap U^*$ is finite. Let $t^*: T^*U^* \to T^*/T^* \cap U^*$ be the natural epimorphism, with kernel U^*. Writing $\hat{T} = T^*/T^* \cap U^*$, we have clearly $t^*(\mathrm{Ad}G) \subset (\hat{T})^0$, since G is connected. If for the connected abelian Lie group $(T^*)^0$ we consider the universal covering for $\pi_T: \tilde{T} \to (T^*)^0$, it is obvious that $t^* \cdot \pi_T: \tilde{T} \to (\hat{T})^0$ is the universal covering for $(\hat{T})^0$. Since G is connected and simply connected, there exists a unique homomorphism $\tilde{t}: G \to \tilde{T}$ such that $t^* \cdot \pi_T \cdot \tilde{t} = t^* \cdot \mathrm{Ad}_G$. Put $T = \tilde{t}(G)$, $T^*_G = \pi_T \cdot \tilde{t}(G)$; then T is a connected simply connected abelian Lie group covering of T^*, while $T_G^* \subset < G >$. We see that T_G^* can be regarded as a subgroup of Aut$G$. The imbedding $T_G^* \to$ Aut$G$ and the homomorphism π_T induce a homomorphism $\phi :\ T \to \mathrm{Aut}G$, with $\ker\phi = \ker\pi_T \cap T$ discrete. Then we can get the Mal'cev splitting $M(G) = T \times_\phi G$ and $M(G) = TU$ for a unipotent group U such that

$$\dim U = \dim R,\ \dim U/N_G = \dim T,$$

where N_G is the nilpotent radical of G.

Now we see that $\mathrm{Aut}(U)$ and the semidirect product $\mathrm{Aut}(U) \propto U$ are prealgebraic groups, i.e., identity components of algebraic groups. We can regard T_G^* as a subgroup of $\mathrm{Aut}(U)$. Let $a(T_G^*)$ be the prealgebraic hull of T_G^* in $\mathrm{Aut}(U)$, and

$$\mathcal{A}_l(G) = a(T_G^*) \propto U.$$

We see that $\mathcal{A}_l(G)$ is prealgebraic. Let $M_l(G) = T_G^* U$ as a quotient of $M(G)$, we see that there is a natural embedding $G \subset M_l(G)$. Then:

Lemma 2. *The group $\mathcal{A}_l(G)$ is prealgebraic, and there exists an imbedding $\beta :\ M_l(G) \to \mathcal{A}_l(G)$ such that the following properties hold:*

1) *$\mathcal{A}_l(G)$ is splittable, and if $\mathcal{A}_l(G) = T'U'$, where U' is unipotent and T' a prealgebraic torus, then $\beta(M_l(G)) \supset U'$, where $U' = U$.*

2) *The prealgebraic closure of each of the subgroup $\beta(G_l)$ and $\beta(M_l(G))$ in $\mathcal{A}_l(G)$ is $\mathcal{A}_l(G)$ itself.*

Here we like to give a very simple example: Let $G = G_1 \times G_2,\ G_1 = TN$ with T, N, G_2 abelian and T acts on N almost faithfully and as a compact torus without any eigenvector. Then $< G >= \mathrm{Ad}_G(T)N,\ U^* = N$,

$$t^* :\ \mathrm{Ad}_G(T)N \to \mathrm{Ad}_G(T) = \hat{T}$$

$$\pi_T: \ T \to \mathrm{Ad}_G(T)$$

$$\tilde{t}: \ TN \times G_2 \to T$$

$$T^*_G = \mathrm{Ad}_G(T), \ \phi: \ T \to \mathrm{Ad}_G(T)$$

$$M(G) = T \times_\phi G = TU, \ U = \{(t, t^{-1}, n, g)|_{t\in T \ n\in N \ g\in G_2}\}$$

$$\mathcal{A}_l(G) = M_l(G) = \mathrm{Ad}_G(T)U.$$

2. Here we prove the *modification* for a compact homogeneous solvmanifolds. This method was first used in [Gb2]. Similar construction can be found in the study of homogeneous Kähler manifolds, e.g., [Dm], [DN].

In this subsection we only deal with the case when G is solvable and H is discrete, in particular H^0 is also the identity.

Let $M = G/H$ be a compact homogeneous solvmanifold of a simply connected solvable real Lie group G. We go through the proof with the similar notations as in [Gu3] 3.1.7., that might help us understand the both constructions here and therein. In our case, we set $G_* = G_l = G/l(S) = G = R$ be the image of $G(H_* = H/H \cap l(S) = H$ be the image of $H)$ in $\mathcal{A}_l(G)$. We also set P_* be the algebraic closure of $\beta(H)$ in $\mathcal{A}_l(G)$. Then the nilradical $N_R \subset P_*$ by a theorem from Mostow [Rg Theorem 3.3] , see also [Rg Corollary 8.27, 8.28] (in [Gu3] 3.1.7. this condition was true by the fact that the nilradical is in the normalizer of H^0, which of course is trivial in our case here since H^0 is the identity and we see that our modification is quite different from the one we mentioned in [Gu3]). Since the subgroup P_* is an algebraic group, the group $\pi_0(P_*)$ is finite. Passing from H to the subgroup $H_1 = H \cap \pi^{-1}(P^0_* \cap H_*)$ of finite index, where $\pi : M(G) \to M_l(G)$

is the natural epimorphism, we might assume that $H_* \subset P_*^0$ by considering a finite covering M' of M. This inclusion will be assumed to hold in what follows.

We consider the natural epimorphism $\gamma : \mathcal{A}_l(G) \to \mathcal{A}_l(G)/N_G$. We have $\mathcal{A}_l(G)/N_G = T_* \times \pi(U)/N_G$ with T_* is a prealgebraic torus; $\pi(U)/N_G = U/N_G$. So $\mathrm{Im}\gamma = T_* \times U/N_G$, we denote it by A. A is connected and Abelian. There is a natural embedding of the group $G_*/N_G = G/N_G$ in $M_l(G)/N_G$ which is contained in A.

We denote the image of G/N_G by B. We have that U/N_G is the projection of B to the second factor U/N_G. By our construction we have $\dim U/N_G = \dim G/N_G$. The composition of γ and the projection restricted on G/N_G is an onto linear map between two Eucliden spaces with same dimensions, and therefore is an isomorphism. That implies $B \cap T_* = \{e\}$. We see that the projection $\mu : T_* \times U/N_G \to U/N_G$ to the second factor is an isomorphism on B, i.e., B is closed in A. Now we consider the subgroup $H_*/H_* \cap N_G$ of A and its closure $\overline{H_*/H_* \cap N_G}$ (in the Euclidean topology) which we denote by A_1. Since $H_*/H_* \cap N_G \subset B$ we have $A_1 \subset B$. Since the group B is simply connected and Abelian, A_1 is a closed subgroup of it, A_1 is torsion free and isomorphic to $\mathbf{R}^p \times \mathbf{Z}^q$ for some $p, q \geq 0$.

Finally we consider the subgroup $\gamma(P_*) \subset A$. The subgroup $\mathrm{Ker}\gamma = N_G$ is closed in the "Zariski topology" on $\mathcal{A}_l(G)$, so does the Lie group P_*, therefore $\gamma(P_*)$ is a closed subgroup of A. But $H_* \subset P_*$, so $H_*/H_* \cap N_G \subset \gamma(P_*)$ and hence $A_1 \subset \gamma(P_*)$, i.e., $A_1 \subset \gamma(P_*^0)$ by our convention. The group $\gamma(P_*^0)$ is connected and Abelian and hence $\gamma(P_*^0) = K \times V$, where K is a maximal compact subgroup of $\gamma(P_*^0)$ (which is a torus), and V is simply connected. Since A_1 is closed in A and torsion free, $A_1 \cap K = \{e\}$.

Hence the projection $K \times V \to V$ onto the second direct factor on A_1 is a monomorphism. Now it follows from this that there exists a closed simply connected subgroup $C \subset \gamma(P_*^0)$, such that $A_1 \subset C$ and A_1 is uniform in C (we notice that C is not always in B). We set $\Phi_l = \gamma^{-1}(C)$. Then $\Phi_l \subset P_*^0$ and therefore has the same algebraic closure as H_l. And Φ_l is a closed connected subgroup of $\mathcal{A}_l(G)$. To it corresponds a closed connected subgroup Φ of $\mathcal{A}(G)$ the universal covering of $\mathcal{A}_l(G)$.

Let $H' = \beta(H) \cap \Phi$ and $\mathrm{Ad}_\Phi : \mathcal{A}(G) \to Aut(\Phi)$, then the pull back of the algebraic closure of $\mathrm{Ad}(\beta(H))$ contains P_* and hence Φ. That is, the algebraic closure of $\mathrm{Ad}(\beta(H))$ contains $\mathrm{Ad}(\Phi)$. We also notice that the algebraic closure of H' is a normal subgroup of $\mathcal{A}(G)$. Therefore, the algebraic closures of $\beta(H)$ and H' have the same identity component.

We say that an abelian subgroup is toric if its action is semismple. We also use $a(\mathrm{Ad}H)$ to denote the algebraic closure of the $\mathrm{Ad}H$.

With this construction at hand, we have the following theorem:

Theorem 1.: *Let $M = G/H$ be a compact homogeneous solvmanifold of a simply connected Lie group G. Then there exists a normal subgroup H' of finite index in H and a subgroup Φ of $\mathcal{A}(G)$, such that:*

(*a*) *Φ is a connected, simply connected, closed subgroup of $\mathcal{A}(G)$, containing H' and N_G,*

(*b*) *$U_\Phi = U_G$ (although $M(\Phi)$ and $M(G)$ are not generally isomorphic),*

(*c*) *for the decomposition $\mathcal{A}(G) = TU$ with T a prealgebraic toric Abelian subgroup of $\mathcal{A}(G)$ we have $\Phi \subset TG$, $G \subset T\Phi$, where $\Phi \cap T = G \cap T = \{e\}$,*

(*d*) *there exists a diffeomorphism $\eta : \Phi \to G$ which is the identity on the* *subgroup H' and induces a diffeomorphism $\Phi/H' \to G/H'$, Actually, it induces a diffeomorphism of the torus bundles $G/H' \to G/H'N_G$ and $\Phi/H' \to \Phi/H'N_G$.*

(e) $\mathrm{Ad}\Phi$ *has the same algebraic closure as that of* $\mathrm{Ad}H'$.

Moreover, $H' = H$ if and only if $\mathrm{Ad}H$ is in the identity component of the algebraic closure of $\mathrm{Ad}H$. In general, H/H' is the abelian group

$$a(\mathrm{Ad}H)/a(\mathrm{Ad}H)^0.$$

Proof: The diffeormorphism is induced from the torus isomorphism

$$(\Phi/N)/(H'/N \cap H') \to (G/N)(H'/N \cap H').$$

By our condtruction, we also see that Φ_l is simply connected since C is. Therefore, $\Phi = \Phi_l$. For the last sentences we notice that we have the maps:

$$H/H \cap N_G \to G/N_G \to M_l(G)/N_G \to \mathcal{A}_l(G)/N_G \to < G > /\mathrm{Ad}_G(N_G).$$

According to [Rg] p.11, the algebraic closure $\mathbf{H}_{\mathcal{A}_l}$ of $H/H \cap N_G$ in $\mathcal{A}_l(G)/N_G$ is a product of a subgroup $A_{\mathcal{A}_l(G)}$ of $\alpha(T^*_G)$ and an unipotent subgroup of U. The latter is connected. The connected components of $\mathbf{H}_{\mathcal{A}_l}$ is determinated by those of $A_{\mathcal{A}_l(G)}$. Similarly, the connected components of algebraic closure $\mathbf{H}_{\mathrm{Ad}}$ of $\mathrm{Ad}(H)/\mathrm{Ad}(H) \cap \mathrm{Ad}(N)$ is determinated by the those of the corresponding $A_{\mathrm{Ad}} \subset T^*_G$. But with the induced maps

$$T^*_G \to \alpha(T^*_G) \to T^*_G$$

we see that the connected components of $A_{\mathcal{A}_l(G)}$ are one to one corresponded to those of A_{Ad}.

Q. E. D.

We obtained our Cohomology Theorem by applying the Mostow Theorem (see [Rg Corollary 7.29]). One might notice that our proof here is almost word to word identical to the corresponding parts 3.1.6 and 3.1.7 in [Gu3], but the purposes are quite different.

The existing of H/H' is not too bad. Once we understand how H/H' acts on H', we should be able to calculate $H^*_{H/H'}(M')$. Those hyperelliptic surfaces at the end of page 586 in [GH] are good examples. Since $\mathcal{A}(G) = T\Phi$, the semisimple action of T on the Lie algebra $\mathcal{G}_2$ of Φ induces a T action Φ. Therefore, $\mathcal{A}(G)$ acts on Φ with isotropic group T. In particular, H acts on Φ and induces an action on $\mathcal{G}_2$. That is, H acts on $\mathcal{G}_2$ and H' is in the kernel of the action. We have

$$H^*(M) = H^*_{H/H'}(M') = H^*_{H/H'}(\mathcal{G}_2).$$

The reason that this is true is that

$$\mathbf{G} = \mathrm{Aut}(G_2) \propto G_2 \to G_2$$

induces a homogeneous structure on G_2 with an action of $\mathbf{G}$. Regarding H having a group homomorphism onto a subgroup of $\mathbf{G}$, it sends the left invariant forms to the left invariant forms.

3 Splitting of the Manifolds

For the proof of the Splitting Theorem, one could easily apply the cohomology Theorem to G_1/Γ and follow the proof of our previous work in [Gu2].

For the convenience to the reader, here I should give a sketch of the proof:

The method in [Gu2] is to prove that the fiberation

$$G/H \to G/N_G(H^0)$$

is a product.

First, we use the Leray spectral sequence to prove that the fiber $N_G(H^0)/H$ is solvable. To see this, we let K be the maximal connected semisimple subgroup of $G_1 = N_G(H^0)/H^0$. Then K acts on M from the right side. We have the quotient $M//K$, and we want to get a K bundle to apply the Leray spectral sequence.

For a reference of the spectral sequences, see [GH p.463].

A problem might be that $M_K = M//K$ might have some singularity. However, in [Gu2] we proved that up to a finite covering we can make M_K smooth. By the Leray spectral sequence, the symplectic class is a sum of ω_1, ω_2 and ω_3 with $\omega_1 \in H^0(M_K, H^2(K)) = 0$, $\omega_2 \in H^1(M_K, H^1(K)) = 0$, and $\omega_3 \in H^2(M_K, H^0(K)) = H^2(M_K, \mathbf{R})$. But the symplectic structure is nondegenerate. That forces K to be the identity. We have that G_1 is solvable.

Now, with G_1 solvable we can apply the modification theorem. By the argument in [Gu2], there is a finite covering $M' = G/H'$ of M such that there is a differential form ω_0 in the symplectic class of the original symplectic structure on M' which is invariant under the left action of the maximal compact subgroup K_G of G and invariant under the right action of G_2. Then ω_0 is nondegenerate everywhere and is a symplectic structure itself. Let S_{K_G} be the semisimple part of K_G. Since ω_0 is invariant under the action of S_{K_G}, we can prove as in [Hk] and [Gu2] that the moment map of S_{K_G} gives a trivial bundle over a rational projective homogeneous space. Therefore, M' is a product of a rational projective homogeneous space and a solvmanifold. So is M.

4 Invariance of Complex Structure

Moreover, since G_1 is a complex Lie group, we have at e

$$\mathrm{Ad}(g) \cdot J = J \cdot \mathrm{Ad}(g)$$

for any $g \in G_1$. Therefore, we also have that

$$\alpha \cdot J = J \cdot \alpha$$

for any $\alpha \in < G_1 >$. Now we let

$$j(x+t) = Jx \ (\mathrm{mod}\ \mathrm{Lie}(< G_1 >)),$$

where $\mathrm{Lie}(< G_1 >)$ is the Lie algebra of $< G_1 >$. One could easily check that

$$\begin{aligned}
&[j(x+t), j(y+s)] = -[x,y] \\
&= -[x,y] + jt(jy) + t(y) - s(x) - js(jx) + [t,s] \ (\mathrm{modLie}(< G_1 >)) \\
&= j[j(x+t), y+s] + j[x+t, j(y+s)] + [x+t, y+s]
\end{aligned}$$

for any $x, y \in \mathcal{G}_1,\ s, t \in \mathrm{Lie}(< G_1 >)$. That is, the complex structure on G_1 is always invariant under the action of $\mathcal{A}(G_1)$, see [Dm, DN, DG] for this formula. This is because of that the action of $\mathcal{A}(G_1)$ on G_1 is factored through the map:

$$\mathrm{Ad}_{G_1} : \mathcal{A}(G_1) \to < G_1 > .$$

Lemma 3.: *The complex structure on N of the Splitting Theorem is right invariant under the action of G_2.*

5 The Two Steps Theorem

For the pseudo-kähler case, we can use the Leray spectral sequence argument again. We have a fiber bundle

$$G_2/\Gamma' \to G_2/\Gamma' N_{G_1}.$$

Now any element α in $H^1(N_{G_1}/\Gamma \cap N_{G_1}, \mathbf{R})$ is $\beta + \bar{\beta}$ with a holomorphic right invariant form β since N_{G_1} is nilpotent and complex. By α being closed, we have that β is closed. This means that $d\beta = 0$ and β is not from the derivation $[N_{G_1}, N_{G_1}]$ but from $N_{G_1}/[N_{G_1}, N_{G_1}]$ since N_{G_1} is complex (see [Gu1]). **We actually shall prove in Lemma 7 that $\beta = 0$ for those β involved in the ω below. This can be done without Lemma 6. That implies in Theorem 2 that G_2 is Chevalley. This is the major breakthrough in this paper, which leads to the classification.**

Let $\omega_F = \omega|_{N_{G_1}/\Gamma \cap N_{G_1}} \in H^2(N_{G_1}/\Gamma \cap N_{G_1}, \mathbf{R})$. By $d\omega_F = 0$ and $\omega_F = \sum_i a_i \beta_i \wedge \bar{\beta}_i$ with β_i a basis of holomorphic differential form on $N_{G_1}/\Gamma \cap N_{G_1}$ we see that $d\beta_i = 0$ if $a_i \neq 0$. That is, ω_F comes from $N_{G_1}/[N_{G_1}, N_{G_1}]$ also.

Let A be a complement of the Lie algebra of N_{G_1} in the Lie algebra of G_1 with respect to the modification of the torus action. For convenient, we also denote the Lie algebra of N_{G_1} by N_{G_1} when there is not confusion. Then $\omega \in \wedge^2 A^* + A^* \wedge N^*_{G_1} + \wedge^2 N^*_{G_1}$. We write $\omega = \omega_0 + \omega_1 + \omega_2$ correspondingly. We want to see that $\omega_1 \in A^* \wedge N^*_{G_1}$ is in $H^1(B, H^1(F))$. $\omega_1 = \sum_j (\alpha_j \wedge \bar{\beta}_j + \bar{\alpha}_j \wedge \beta_j)$ with $\beta_j \in N^{\mathbf{C},*}_{G_1}$ being holomorphic. We notice that by our assumption $\partial \bar{\beta}$ can not have any term of $\bar{\alpha} \wedge \beta'$ for any $\beta, \beta' \in N^{\mathbf{C},*}_{G_1}$ being holomorphic (see also the last statemnet of Lemma 5). Therefore, by closedness of ω we have $\beta_j \in H^1(F)$. We can identify ω_2 with ω_F.

A similar argument as in the proof of Lemma 7 later without

using Lemma 6 will also imply that $\omega_1 = 0$ and A is an abelian Lie algebra itself.

So, a right invariant pseudo-kähler form on G_2/Γ' actually comes from $M_1 = G_2/[N_{G_2}, N_{G_2}]\Gamma'$ since all the three parts in the Leray spectral squences come from M_1. By the nondegeneracy of the pseudo-kählerian form we see that $G_2/\Gamma' = M_1$.

Lemma 4.: *If M is a pseudo-kählerian manifold in the Splitting Theorem, then N is a complex torus bundle over a complex torus and up to a finite covering has a right G_2 invariant pseudo-kählerian structure. G_1 and G_2 have abelian nilradicals.*

For a weaker version of this Lemma, see also [Ym2] for a confirmation, with a different proof.

We should also prove later on that the torus bundle can be chosen to be a pseudo-kählerian torus bundle over a pseudo-kählerian torus. That is, the pseudo-kählerian structure splits also. These compact complex homogeneous parallelizable manifolds also show that the similar result of the Mostow Theorem for the Dolbeault cohomology does not work in general, since the holomorphic 1-forms (as elements in $H^{1,0}(N)$) are generally not right invariant under the modified Lie group which satisfies the Mostow condition.

6 The Corresponding Lie Algbras

A further application of the argument of the modification theorem is the so called *complex-parallelizable-right-invariant-pseudo-kählerian algebra*.

Now we start from the good solvable Lie group Φ in the second section, i.e., we assume the Mostow condition. We denote Φ by G instead. We investigate the opposite question of second section, that we modified G by a torus T such that the modified group is a complex group.

Lemma 5.: *Let $N = [\mathcal{G}, \mathcal{G}]$ and A be the complement of N in $\mathcal{G}$ with respect to the prealgebraic toric abelian group T, Then for any $x, y \in A^{\mathbf{C}}$ such that $jx = ix$, $jy = -iy$, we have $[x, y] = 0$. Similarly, if $z \in N^{\mathbf{C}}$ is holomorphic, so is $[w, z]$ for any $w \in \mathcal{G}$.*

Proof: Since $T(A) = 0$, we have that $t(y)(x) = 0 = t(x)(y)$. Moreover,

$$j[x + t(x), y + t(y)] = [j(x + t(x)), y + t(y)] = i[x + t(x), y + t(y)],$$

and

$$j[x + t(x), y + t(y)] = [x + t(x), j(y + t(y))] = -i[x + t(x), y + t(y)].$$

That is, $[x + t(x), y + t(y)] = 0$, we have $[x, y] = t(y)x - t(x)y = 0$.

Q. E. D.

This is true for any modification of a solvmanifold of a complex solvable Lie group.

Lemma 6.: *The action of the elements in $\mathcal{G}$ on N are semisimple with real eigenvalues, The action of A on N is isogent to a product of several algebraic G_m's modulo the kernel.*

Proof: Here we use the argument in [Wk] Proposition 7.6.8. When we calculate the $H^{1,0}(M)$ by the Leray spectral sequence, there is one factor from $H^0(T, H^{1,0}(F))$ and another factor from

$$H^1(T, H^{0,0}(F)) = H^1(T, \mathbf{C}) = A,$$

where $T = G/\Gamma N$ and $F = N/N \cap \Gamma$. Now F is a torus and we have that $\dim_{\mathbf{C}} H^{1,0}(F) = \dim_{\mathbf{C}} F$. If X is a holomorphic vector field, we let

$\alpha = \omega(X, \ ) \in H^{1,0}(M)$. If $X_1, \cdots X_n$ is a basis, then $\alpha_1 \wedge \cdots \wedge \alpha_n \wedge \Omega = a\omega^n$ is a nonzero volume form with $a \in \mathbf{C}$, where Ω is the dual of $\Omega^* = \bigwedge_i X_i$. That is, α_i is nonzero in $H^{1,0}(M)$, and α_i are linear independent in $H^{1,0}(M)$. Here $H^{1,0}(F)$ can be regarded as a holomorphic vector bundle with an antiholomorphic A action. By [Wk] Proposition 6.2.5 we only need to consider the sections of a trivial subbundle E over T. That is,

$$\dim_{\mathbf{C}} H^0(T, H^{1,0}(F)) = \dim_{\mathbf{C}} H^0(T, E) = \dim_{\mathbf{C}} E.$$

But

$$\dim_{\mathbf{C}} T + \dim_{\mathbf{C}} E = \dim_{\mathbf{C}} H^{1,0}(M) \geq n = \dim_{\mathbf{C}} T + \dim_{\mathbf{C}} F.$$

Therefore, $\dim_{\mathbf{C}} E = \dim_{\mathbf{C}} F$ and α_i is a basis of $H^{1,0}(M)$. Also, by [Wk] Proposition 7.2.1 there is a holomorphic group action on the fiber extending the action of $\Gamma_1 = \Gamma/\Gamma \cap N$. For our case, we only deal with an abelian Lie group and the proof for the Proposition 7.2.1 should be much easier for our circumstance.

Moreover, $H^{1,0}(F)$ is generated by linear combinations of α_i. By our construction there is also an antiholomorphic action of A on the fiber extending the action of Γ_1. By the argument of [Wk] Proposition 7.6.8 we have the map $\rho_1 : A \to (G_m)^k$. In [Wk] $(G_m)^k$ is the algebraic closure of the image ρ_1. We claim that one actually have that ρ_1 is onto. We only need to prove for the case when ρ_1 is locally effective. In that case we only need to prove that the rank of $\Gamma_2 = \Gamma_1 \cap \ker \rho_1$ is k. If the rank of Γ_2 is $l < k$, we let A_1 be the complex space generated by Γ_1. Then there is a homomorphism $A \to A/A_1 = \mathbf{C}^{k-l}$. Then the argument in the proof of [Wk] Proposition 7.6.8 shows that $k - l = 0$, a contradiction.

Q. E. D.

For a weaker result, see [Ym2].

Lemma 7.: *The factor of ω that comes from $H^1(T, H^1(N, \mathbf{R}))$ in the Leray spectral sequences is zero.*

Proof: Let $N_1 \subset N$ be the subalgebra consist of all the elements in N which are perpendicular to N. Then N_1 is an ideal since $\omega(n, [y, n_1]) = \omega([n, y], n_1) = 0$ for any $y \in \mathcal{G}, n \in N, n_1 \in N_1$. We only need to prove that $N_1 = 0$.

Otherwise, ω is trivial on N_1 and we let n_1 be an eigenvector. Let $z = n_1 - ijn_1$, then $jz = jn_1 + in_1 = iz$. Let $N_0 \subset N$ be the subalgebra generated by a set of linearly independent eigenvectors in N such that N_0 is complement of N_1, then N_0 is also an ideal by our construction and ω is nondegenerate on N_0.

Let B_0 be the subset perpendicular to N_0, then B_0 is a subalgebra by N_0 being an ideal. Let B_1 be the subset perpendicular to N_1 and B_1 is also subalgebra. The intersection $C = B_0 \cap B_1$ is then a subalgebra. C is an ideal of B_0. Let C_0 be a complement of N_1 in C with zero $t(x)$ eigenvalues. Let $C_1 \subset B_0$ be the subset perpendicular to C_0, then C_1 is an ideal of B_0 and ω are nondegenerate on B_0, C_0 and C_1.

Modify A of Lemma 5 if necessary such that A is perpendicular to N_0 and $C_0 \subset A$. This is possible since $t(x)B_0 \subset N_1$ and the proof of Lemma 5 still go through. We still denote it by A. Then $A \subset B_0$.

Let $A_0 = C_0$ be all the elements in A which is perpendicular to N, $A_1 \subset A$ all the elements perpendicular to A_0. By a proper modification of A, we can let $A_1 \subset C_1$. This is possible since $t(x)C_1 \subset N_1$ and the proof of Lemma 5 still go through.

We have

$$\omega \in \sum_{i=1}^{2} \wedge^2 A_i^* + A_1^* \wedge N_1^* + \wedge^2 N_0^*.$$

Then there is one term $z^* \wedge w^*$ with $w = y + ijy$ with a $y \in A_1$ and it is the only one with w^*. We have $dw^* = 0$. Then there is either a term $z^* \wedge a^* \wedge w^*$ with $a \in A^{\mathbf{C}}$ and $ja = ia$, or a term $a_1^* \wedge a_2^* \wedge w^*$ with $a_1, a_2 \in A^{\mathbf{C}}$ and $ja_k = ia_k, k = 1, 2$ in $\partial(z^* \wedge w^*)$. The latter can not come from another this kind of term since we assume that there is only one of them with w^*, and it can not come from the $\partial(a^* \wedge n^*)$ with $n \in N_1,\ a \in A$ and $jn = -in,\ ja = ia$ by lemma 5. But the first term can not come from another term.

Q. E. D.

Remark: Similar arguments can also give Lemma 7 with only Lemma 4 and 5 but without Lemma 6. With Lemma 7, a similar argument as in the proof of the Lemma 7 implies A is an abelian subgroup. Further manipulation of the pseudo-kähler form implies that A acts semisimplely on N_G and the eigenvalues come out as pairs $\alpha, -\bar{\alpha}$. The unimodular property of the action of A then implies that all the α's must be real. All these therefore can be done without [Ym2] and [Wk].

Corollary 1.: *A is perpendicular to N.*

Theorem 2.: *A is an abelian Lie subalgebra and the elements in A acting on N are semisimple with real eigenvalues. Therefore, G is Chevalley.*

Proof: For any $a_1, a_2 \in A$, we have that $\omega([a_1, a_2], n) = \omega(a_1, [a_2, n]) + \omega(a_2, [n, a_1]) = 0$ for any $n \in N$. Therefore, $[a_1, a_2] = 0$.

Q. E. D.

Theorem 3.: *If $\omega(n_1, n_2) \neq 0$ for eigenvectors n_1, n_2 with real eigenvalue functions $k_1(a), k_2(a)$, then $k_2(a) = -k_1(a)$. $\omega(n_1, jn_1) = 0$ always.*

Proof: $k_1\omega(n_1, n_2) = \omega([a, n_1], n_2) = \omega(n_1, [n_2, a]) = -k_2\omega(n_1, n_2)$.

By $[a + t(a), jn_1] = j[a + t(a), n_1]$ and $jt(a)n_1 = t(a)jn_1$ we have that $[a, jn_1] = j[a, n_1] = k_1(a)jn_1$. Therefore, $\omega(n_1, jn_1) = 0$ since $k_1 \neq 0$.

Q. E. D.

The argument in the Lemma 6 now shows that if we write A and N by complex coordinates $z_1, \cdots, z_k$; $w_1, w_2, \cdots, w_{2l-1}, w_{2l}$ such that $\omega_2 = dw_1 \wedge d\bar{w}_2 - dw_2 \wedge d\bar{w}_1 + \cdots + dw_{2l-1} \wedge d\bar{w}_{2l} - dw_{2l} \wedge d\bar{w}_{2l-1}$, then the eigenvalue functions are $k_{2s-1}(z) = \mathrm{Re}l_s(z), k_{2s}(z) = -k_{2s-1}(z)$, where l_s are complex linear function of z.

Let $L(z)$ be the diagonal matrix with l_s as the nonzero elements. We have product

$$(z, w)(z', w') = (z + z', \exp(\mathrm{Re}L(z))w' + w).$$

The corresponding complex Lie group is

$$(z, w)(z', w') = (z + z', \exp(L(z))w' + w).$$

The right invariant holomorphic differential forms are $dz_k, \exp(-l_s(z))dw_s$. The basis for $H^1(G/\Gamma, \mathcal{O})$ are $d\bar{z}_k, \exp(-l_s(z))d\bar{w}_s$. The basis for $H^{1,1} \cap H^2$ are $dz_k \wedge d\bar{z}_l, dw_s \wedge d\bar{w}_t - dw_t \wedge d\bar{w}_s, i(dw_s \wedge d\bar{w}_t + dw_t \wedge d\bar{w}_s)$ with $l_s = -l_t$. In particular, the dimension of the Dolbeault cohomology is much bigger than that of the de Rham cohomology.

We actually always have:

$$\dim_{\mathbf{C}}(H^1(M, \mathbf{C}) \cap H^{1,0}(M)) = \dim_{\mathbf{C}}(H^1(M, \mathbf{C}) \cap H^{0,1}(M)) = \dim_{\mathbf{C}} A,$$

and

$$\dim_{\mathbf{C}} H^{1,0}(M) = \dim_{\mathbf{C}} H^{0,1}(M) = \dim_{\mathbf{C}} M = \dim_{\mathbf{C}} A + \dim_{\mathbf{C}} N.$$

7 Compact complex homogeneous manifold with a pseudo-kählerian structure

After our understanding of the Lie algebra in the last section, we deal with the original manifold with the original complex Lie group.

It will be convenient for us that we use both A and N for both the subalgebras and their corresponding subgroups if there is no confusion.

From the group level, the action of A on N is an algebraic group action of a product of $\mathbf{C}^*$'s. The characters of $\mathbf{C}^*$ are just the integer powers. Therefore, $L(z)$ is a matrix of integer numbers, i.e., the coefficients of l_s can be chosen to be integers. Moreover, if we write $z_k = x_k + iy_k$ then in the modified group the action is only related to the integer linear combinations of x_k.

$\Gamma N/N$ acts on $N_{\mathbf{Z}} = \Gamma \cap N$ as a rational representation of $\mathbf{Z}^n$. Therefore, applying [Ti] Theorem 7.2 to our circumstance we see that the representation is isogent to a product of representations with only pair of eigenvalues k_{2s-1} and $k_{2s} = -k_{2s-1}$. This is also comparable with the complex decompositon. If the action of A on N only have one pair of eigenvalue functions k_1 and $k_2 = -k_1$ we call the compact complex parallelizable homogeneous manifolds with a pseudo-kählerian structure a *primary pseudo-kählerian manifold*.

Theorem 4.: *Every compact complex parallelizable homogeneous manifold with a pseudo-kählerian structure is a pseudo-kählerian torus bundle over a pseudo-kählerian torus which, up to a finite covering of the fiber, is a bundle product of primary pseudo-kählerian manifolds.*

We call a primary pseudo-kählerian manifold a *reduced primary pseudo-kählerian manifold* if the action of A on N is almost effective. We could always modify ω_1 to whatever invariant form on A. We then have:

Theorem 5.: *After modifying of* ω_1 *any primary pseudo-kählerian manifold is, up to a finite covering, a product of a torus and a reduced primary pseudo-kählerian manifold. Moreover,* $\dim_{\mathbf{C}} A = 1$ *and* $\dim_{\mathbf{C}} N = 2m$ *with* m *the complex dimension of the eigenspaces for a reduced primary pseudo-kählerian manifold. In particular, the index of a reduced primary pseudo-kähler space is either* 1 *or* -1.

For the reduced primary space, we have $l_1(z) = z$ and $l_2(z) = -z$. The fiber torus up to a finite covering can be splitted into complex irreducible ones with respect to the A action. For a primary pseudo-kähler space, if the fiber is also an irreducible complex torus with respect to the A action, we call it a *primitive pseudo-kähler space*. If the A action is also almost effective, we call it a *reduced primitive pseudo-kähler space*. By modifying ω_1 on A we can always obtain any reduced primary space, up to a finite covering of the fiber, from a torus bundle product of primitive ones.

For a primitive pseudo-kählerian space, the rational representation $N_{\mathbf{Z}}$ of $\mathbf{Z} = \Gamma N/N$ can be splitted into two dimensional spaces, but m above can be any positive integer.

Theorem 6.: *Any compact complex parallelizable homogeneous space with a pseudo-kähler structure as a torus bundle over* T *is, up to a finite covering of the fiber, a bundle product of primitive pseudo-kählerian spaces. Moreover, a primitive pseudo-káhlerian space is, up to a finite covering, a product of a torus and a reduced primitive pseudo-kähler space. For a primitive pseudo-kählerian manifold,* m *can be any given positive integer.*

Proof: We only need to prove the last statement. Using the construction of Yamada, we notice that construction of the Γ action has nothing relate to the complex structure. We now want to construct a real lattice on $\mathbf{C}^m$

which does not have any subgroup which is a nontrivial lattice of a lower dimensional complex subspace. That is, the corresponding torus is simple (does not have any proper subtorus). Then we apply Yamada's action on the direction sum of two copies of our $\mathbf{C}^m$. Actually, we shall see in the next section that any primitive pseudo-kählerian manifold has this form if $N/N \cap \Gamma$ is not simple as a complex torus.

Q. E. D.

8 Constructions and Classification

In this section, we shall reconstruct Yamada's example and construct more examples of reduced pseudo-kähler space of any complex odd dimension.

We recall that the fiber $F = N/\Gamma \cap N$.

We call a reduced primitive pseudo-kähler space a *simple reduced primitive pseudo-kähler space* if F is a simple complex torus.

We call a reduced primitive pseudo-kähler space a *double reduced primitive pseudo-kähler space* if F is isogent to a product of two identical complex torus.

We now construct some new three dimensional reduced primitive pseudo-kähler spaces by constructing new lattice in $\mathbf{C}^2$ which is invariant under an element of $SL(2, \mathbf{Z})$. Let $B = \begin{bmatrix} 0 & 1 \\ -1 & n \end{bmatrix}$ with $n > 2$ a integer, $\gamma_1 = (1, 0), \gamma_2 = B\gamma_1, \gamma_3 = i(1, t), \gamma_4 = B\gamma_3$. Let L_t be the lattice in $\mathbf{C}^2$ generated by $P\gamma_i$ $i = 1, 2, 3, 4$ with PBP^{-1} diagonal as it is in [Ym]. Then L_0 is the L_2 in [Ym p.117]. We can use L_t instead of L_2 in Yamada's construction. We obtain the reduced primitive pseudo-kähler space M_t. When $t \in \mathbf{Z}$, M_t is just the Yamada's example. When $t \in \mathbf{Q}$, M_t is a double reduced primitive space. When $t \in \mathbf{R} - \mathbf{Q}$, M_t is a simple reduced primitive space.

In the proof of Theorem 6, we constructed the $2m+1$ complex dimensional double reduced primitive pseudo-kähler space by letting $L = P(\Gamma, \Gamma)$ with Γ a simple lattice in $\mathbf{C}^m$.

Theorem 7. *Let M be a reduced primitive pseudo-kähler space, if F is not simple, then M must be a double reduced primitive pseudo-kähler space constructed above.*

Proof: Let α be a generator in the corresponding lattice of G_m (e.g., $\alpha = PBP^{-1}$ in the examples above. In general, as an action on vector space over $\mathbf{R}$ the matrix representation of α is just a product B's). If F_1 is a proper subtorus of F, then $F_0 = F_! \cap \alpha F_1$ is a proper subtorus and it is invariant under α. Therefore, F is isogent to a product of two proper A invariant complex subtorus and M is not primitive. We have that F_0 is just finite points. $F^1 = F_1(\alpha F_1)$ must be F. Otherwise we apply our argument for F_0 above to F^1 and have a contradiction.

Q. E. D.

Theorem 8. *The generic reduced primitive pseudo-kähler spaces are simple primitive pseudo-kähler spaces.*

Proof: With a fixed eigenspace $\mathbf{C}^m$ in N, the moduli of the $2m+1$ complex dimensional double reduced primitive pseudo-kähler spaces constructed above has complex dimension $2m^2$.

If F is a simple complex torus with a latice Γ in $\mathbf{C}^{2m}$ and Γ is invariant under the A with a corresponding action of $(\mathbf{C}^m, \mathbf{C}^m)$, then we can construct a simple reduce primitive pseudo-kähler space as above.

Therefore, the construction has a moduli space of complex dimension $4m^2 > 2m^2$. Thus, the generic example constructed in this way is simple.

Q. E. D.

All the reduced primitive pseudo-kähler spaces can be constructed in this way.

Let us recall that a projective rational homogeneous space is a complex manifold G/P with G complex semisimple and P a parabolic subgroup of G, that is, P contains a Borel subgroup B of G which is generated by a Cartan subgroup and all the positive root vectors.

Finally, with the splitting theorem we have:

Theorem 9.: *Every compact complex homogeneous manifold with a pseudo-kählerian structure is a product of a projective rational homogeneous space and a solvable compact parallelizable pseudo-kähler space. In particular, any compact parallelizable pseudo-kähler space is solvable. Moreover, any compact parallelizable pseudo-kähler space M is a pseudo-kählerian torus bundle over a pseudo-kählerian torus T such that M, up to a finite covering of the fiber, is a bundle product of several simple and double primitive pseudo-kähler spaces.*

9 Further Results

Now, we consider the case $m = 1$. That is, $\dim_{\mathbf{C}} M = 3$. In this case A is locally isomorphic to $G_m = \mathbf{C}^*$. $F = T^2$ is a complex two dimensional torus. Let Λ be the lattice for F, i.e., $F = \mathbf{C}^2/\Lambda$. Let $\alpha \in G_m$ be a generator of the infinite part of the corresponding lattice of G_m (the lattice might have a finite subgroup which acts on $\mathbf{C}^2$ as $\mathbf{Z}_2$ generated by -1), then $\alpha\Lambda \subset \Lambda$ and $\alpha \neq 1$.

Let $\gamma \in \Lambda$ be a prime element. Since the elementary polynomial of α has degree 2 and α has eigenvalues e^k with $k \in \mathbf{R}$, $e^{k_1} = \alpha$ and $k_2 = -k_1$,

γ and $\alpha\gamma$ consist of a basis of $\mathbf{C}^2$. The representation matrix of α must be

$$\begin{bmatrix} 0 & 1 \\ -1 & n \end{bmatrix}$$

with an integer $n > 2$ and $\alpha = \frac{n}{2} + \sqrt{\frac{n^2}{4} - 1}$. Therefore, we classify all the possible α. This is the same for any positive integer m above.

We see that there is a $\delta \in \Lambda$ such that Λ is generated by $\gamma, \alpha\gamma, \delta, \alpha\delta$.

Moreover, γ is not an eigenvector of α. Therefore, $\mathbf{C}^2$ is generated by γ and $\alpha\gamma$. We could just let $\gamma = (1, 0)$ and $\alpha\gamma = (0, 1)$. We write $\delta = (d_1, d_2)$. By picking up the right basis of $\mathbf{C}^2$ we could always assume that $\mathrm{Im} d_1 > 0$ and $0 \leq \mathrm{Re} d_1, \mathrm{Re} d_2 < 1$. Therefore, obviously we can classify all the pseudo-kähler three spaces. Similar things could be done for higher dimensional spaces.

For a much weaker version of the complex three dimensional case, see [Ha] for a confirmation.

One more observation is that:

Let Z represent the coordinate for one of the eigenvector space for a primary space and W for the other, then $dZ \wedge dW$ is closed. Therefore, we have the holomorphic symplectic theorem:

Theorem 10. *Every complex even dimensional compact complex homogeneous solvmanifold with pseudo-kählerian structures admits a holomorphic symplectic structure. Moreover, it is a holomorphic symplectic torus bundle over a holomorphic symplectic torus. In particular, every product of two complex odd dimensional compact complex homogeneous solvmanifold with pseudo-kählerian structures admits a holomorphic symplectic structure.*

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