Toward A Classification of Real Compact Solvmanifolds with Real Symplectic Structures

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In this paper, we deal with the problem of classifying compact complex solvmanifolds with holomorphic symplectic structures and obtain some structure results which make the classification possible. In particular, we reduce the classification to the nilpotent case with the same dimension, which we call the nilpotent reduction. The same method also works for the real compact solvmanifolds with real symplectic structures. The real six dimensional case was treated completely. This is one of the major steps to obtain further examples of compact holomorphic symplectic manifolds. For example, the Kodaira-Thurston surface is NOT a complex homogeneous manifold with a transitive Lie group action which keeps the complex structure invariant, but a real solvmanifold with a complex structure.

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1 Introduction

Let M be a complex manifold, ω be a closed differential 2-form representing a class in $H^2(M, \mathbf{R})$. If $\dim_{\mathbf{C}} M = n$ and ω is nondegenerate at every point, i.e., $\omega^n \neq 0$ at every point, we call ω a symplectic structure. If ω is also in $H^{1,1}(M)$, we call it a pseudo-kählerian structure of M. If, at the other end, ω is in $H^{2,0}(M) + H^{0,2}(M)$, we call it a holomorphic symplectic structure. In the latter case, ω is the real part of the usual holomorphic 2-form.

In the case in which M is also Kähler as a compact complex manifold, M to be holomorphic symplectic is the same as it is hyperkähler. There are a lot of interests in the relation between hyperkähler and holomorphic symplectic manifolds. Although there have been a lot of efforts to find new examples of the compact simply connected ones of both of them, the examples are still very few. Therefore, it is essential for us to see how far we could go in the direction of [Gu10]. Some effort has also been done earlier in [GK] (also [Gu2]) for the nilpotent case. In this paper, we shall also address the non-nilpotent solvable case.

A compact complex homogeneous space with a pseudo-kählerian structure (not necessary invariant) was classified in [Gu1]. It is a product of a classical projective homogeneous space and a pseudo-kähler complex solvmanifold.

It turns out that all the pseudo-kähler complex compact solvmanifolds have holomorphic symplectic structures when they have even complex dimensions. Actually they are hypersymplectic with a non-right invariant involution which is anti-commutative with the symplectic structure. When the complex dimension is odd, we can always make it even by product with a complex torus. Actually we proved that if a compact complex solvmanifold is pseudo-kähler-like, i.e., if the complex Lie group is the same as some of the pseudo-kähler ones, then the manifold has a right-invariant holomorphic symplectic structure coming from the universal covering. These are a little bit more than those manifolds which are actually pseudo-kählerian. For example, the examples III-(3a) in [Nk] are pseudo-kähler-like but not pseudo-kähler.

On the other hand, by the method in [Gu2], it is easy to construct compact complex nilmanifolds with holomorphic symplectic structures.

Question 1: *Are all the compact holomorphic symplectic solvmanifolds from some kind of combination of these two classes of holomorphic symplectic manifolds?*

Most people might give a negative guess. But against all odds, from all the information we already have, this might be true in certain sense. In an earlier paper [Gu9], we saw that this is true for the case in which the complex dimension is at most 5. We should see in the last section that in the complex dimension eight, the complexification of the Benson-Gordon manifold has a two steps nilpotent radical with non-nilpotent actions at each level. Therefore, it is not the product in the strict sense. However, the non-nilpotent actions are: A. Semisimple; B. With pairs of eigenvectors of opposite eigenvalues. Therefore, after modification of forgetting the non-commutative products in the nilradical, it is pseudo-kähler-like.

In general, so far we could not get A. yet. However, we can get B.:

Theorem A. *Let G/Γ be a compact complex solvmanifold with a holomorphic symplectic structure, a be a non-nilpotent element in the Lie algebra. Then the semisimple part of* $\mathrm{ad}a$ *has pairs of eigenvectors with opposite eigenvalues.*

Now, the following question is very natural:

Question 2: *Let $M = G/\Gamma$ be a compact complex solvmanifold with a holomorphic symplectic structure. Could the Lie algebra of G be a direct sum of two Lie subalgebras A and N such that $[A, N] \subset N$ with A abelian, N nilpotent? That is, could G be Chevalley in the terminology of [Nk][1]?*

We saw that this is also true for those cases in which the complex dimension is at most 5 in [Gu9]. Moreover, we see that the action of A on N is always semisimple. One might conjecture that this is true for any compact complex solvmanifold. That is, we have the **Semisimple Chevalley Conjecture** that any compact complex solvmanifold is Chevalley with A acts on N semisimply.

In [Gu4], we prove that any compact complex homogeneous manifold with a holomorphic symplectic structure is actually a complex solvmanifold. Although the argument for the pseudo-kähler case and the general real symplectic case had a gap (it was fixed in [Gu5, 6, 1]), the argument worked well for the holomorphic symplectic case, which was our major purpose there. Notice that all the holomorphic forms are right invariant. A classification of compact solvable complex parallelizable manifolds with holomorphic symplectic structures is overdue.

In this paper we shall deal with the general case and obtain some structure results. In the process, we see that our methods actually also work for the compact real solvmanifolds with real symplectic structures. This is essential if we really try to apply the method of [Gu10]. Notice that the Kodaira-Thurston surface is not a complex homogeneous space.

In section 2, we apply our modification method again. This time, instead

[1] It is easy to check that our condition implies the existence of a Chevalley decomposition in [Nk] p.91.

of using a finite covering and changing the Lie group structure as we did in [Gu5], we have to change the manifold completely. We first tried to obtain a semisimple Chevalley modification with the same dimension. It does not always work. The reason is that we could not get a nondegenerate symplectic form on it in general. Therefore, we got a **semisimple Chevalley modification** of complex dimension $n + b_1$, where b_1 is the first Betti number of M. To our surprise, in the third section, we found that the restriction of the symplectic form (after a modification) on the nilshadow is nondegenerate. Therefore, we have a nilpotent modification of the same complex dimension instead. We call the nilmanifold modification the **nilpotent reduction**. We have:

Theorem B. *For any compact complex holomorphic symplectic solvmanifold of complex dimension n, there is a holomorphic symplectic semisimple Chevalley modification of complex dimension $n + b_1$ and a corresponding holomorphic symplectic nilpotent reduction of complex dimension n.*

Notice that in the complex case b_1 is always even.

In the fourth section, using the Dirichlet Theorem for the algebraic units, we proved that:

Theorem C. *For any compact solvmanifold $M = G/\Gamma$, there is a finite covering $M' = G/\Gamma'$ such that $\mathrm{Ad}\Gamma'$ has only real eigenvalues on the Lie algebra.*

Using Theorem C, we can assume that after modification through the original finite covering method as we did in [Gu5], the Lie algebra only have real eigenvalues. That is, if we assume that $\mathrm{ad}a = R + I + N$ with R the real semisimple part, I the pure imaginary semisimple part, and N the nilpotent part, then the original modification through the finite covering as we did in

[Gu5] can be achieved by forgetting I.

We carry all our arguments to the real compact solvmanifolds with real symplectic structures. The modification of this paper is adding the element R and a companying trivial element T. If the same dimensional semisimple Chevalley modification is successful, it is doing so by forgetting N. But in general, we have to use the higher dimensional semisimple Chevalley modification. The dimension is $n+2b_1$ instead of $n+b_1$ in the complex case. Surprisingly, the restriction of the symplectic structure (after a modification) on all the N including the original nilpotent ones, i.e., the nilshadow, is nondegenerate. Therefore, we have the nilpotent reduction again.

In particular, we classified the real compact six dimensional solvmanifolds with real symplectic structures, following a suggestion from Professor Anna Fino. There are only six corresponding nilpotent reductions. Our classification fits quite well with the classification in Bock's dissertation [Bk] later told by professor Salamon.

For the complex case, there are more obstructions by the cocompact discrete subgroups (Cf. [Gu9], [Ba]). The only survivor for the nilpotent reduction is the last one, which is abelian. Therefore, we have:

Theorem D. *All the complex six dimensional compact holomorphic symplectic solvmanifolds are pseudo-kähler-like.*

It is plausible that our results made a computer program for the classification of real compact solvmanifold with real symplectic structures possible.

2 Semisimple Chevalley modifications of compact complex solvmanifolds with holomorphic symplectic structures

In this paper, *if there is no confusion we shall use the same notation, e.g., G, for both the Lie group and its Lie algebra*—just as what we did in [Gu3]. The reason is: the Lie algebra is part of the Lie group. And it is very convenient for us.

Now, we assume that the solvable manifold is not nilpotent. Therefore, there is at least a non-nilpotent element $\gamma_1 \in \Gamma$ in the Lie group which is not unipotent. It corresponds to a linear action A_{γ_1} on the Lie algebra by the induced inner automorphism. Let N be the commutator of G. A_{γ_1} actually acts on the rational Lie algebra of $N \cap \Gamma$, which we also denote by N by abusing the notation. By [Bo] Chapter 7, Section 5, no. 9, Theorem 1, any rational action A on the N can be written as sn with s semisimple and n unipotent rational actions. Actually, by our construction, all the eigenvalues of A are algebraic units. Therefore, when we apply the Chinese Reminder Theorem in [Hu p.18], we use it for the algebraic numbers field generated by the eigenvalues and obtain a polynomial with algebraic numbers. By the coefficients being invariant under the Galois group we have that the coefficients are rationals. To go one step further, we notice that s acts on the lattice in the Lie algebra, which is the preimage of the lattice through the exponential map (see also [Au] page 246). The reason is that the lattice is a $\mathbf{Z}$ module which, after extension, is a direct sum of submodules which are the modules of the ring generated by $\mathbf{Z}$ and the eigenvalues (and its conjugates, see [BS p.83]), which are algebraic units. The original lattice is a subset invariant under the Galois group action. By the construction of the Jordan decomposition, we see that s also acts on the lattice. We denote the map from A to s by φ, i.e., $s = \varphi(A)$. For many mathematicians, this decomposion is called the Jordan decomposition. It is related to the Jordan decompositon of the linear algebra. It is called Chevalley-Jordan decomposition in [Hu]. Personally, I was taught by Professor Dorfmeister in the Lie algebra level. This induces an extension of the Lie group G by s. We can do this for all the possible non-unipotent elements in Γ. Since $\Gamma/\Gamma \cap N_G$ is finite generated abelian. This can be done in finite steps. Therefore we see that the solvmanifolds can be regarded as a submanifold of a bigger manifold with a complex algebraic group G_1. We need to extend the symplectic form to the extension. Let X be an element in the Lie algebra, X_s its semisimple part which is also in the new extended Lie algebra $\mathcal{G}_1$, then $\mathcal{G}_1$ is generated by X_s eigenvectors. For two elements X_1, X_2 we have:

$$\begin{aligned} \omega(X, [X_1, X_2]) &= \omega([X, X_1], X_2) + \omega([X_2, X], X_1) \\ &= \omega([X, X_1], X_2) + \omega(X_1, [X, X_2]). \end{aligned}$$

Similarly, for any element $[X_1, X_2]$ in the Lie algebra $\mathcal{N}$ of the commutator, we can define the symplectic structure by

$$\omega(X_s, [X_1, X_2]) = \omega([X_s, X_1], X_2) + \omega(X_1, [X_s, X_2]).$$

To make the definition consistent for different expression we only notice that the formula for X is true by the closeness of ω. Then by G and Γ has the same complex algebraic closure (see [Iw]) we see that the formula is true for X_s also. Then we just extend the symplectic form to $\mathcal{G}_1$. This is doable since by our construction the differential forms corresponding to the elements in the given basis which is not in the commutator are closed.

We could always assume that X_s is not in the kernel of the symplectic form. Otherwise, we might just extend G by product a two dimensional torus, then modify the symplectic form. Now we just replace X by X_s with restrict the symplectic structure to the new Lie algebra $\mathcal{G}'$ as a subalgebra of $\mathcal{G}_1$. Therefore, if the restriction is nondegenerate, we obtain a new real compact solvmanifold with a real symplectic structure. We call the new manifold a semisimple modification with respect to γ_1.

Now, by induction, we could obtain a compact complex solvmanifold with a holomorphic symplectic structure such that the Lie group G' has Chevalley decompositon $G' = AN_{G'}$ with a semisimple action of A on the nilradical $N_{G'}$. We obtain the new solvmanifold by a twisted modification. Different from those in [Gu5], we obtain possibly a complete different manifold.

Definition 1. We say that two compact nilmanifolds G/Γ and G'/Γ', with both G and G' simply connected, are twisted equivalent if they have the same dimension. A linear isomorphism between the two Lie algebras is a twisted map if it sends the preimage of the exponential map of Γ to that of Γ'.

Of course, two compact nilmanifolds are twisted equivalent with some twisted map if and only if they have the same dimensions.

Definition 2. We say that two compact solvmanifolds $M = G/\Gamma$ and $M' = G'/\Gamma'$ are **twisted equivalent** if there is a common unipotent subgroup N such that

1. $N \cap \Gamma = N \cap \Gamma'$ is cocompact in N and

2. $(G/N)/(\Gamma/N \cap \Gamma)$ is twisted equivalent to $(G'/N)/(\Gamma'/N \cap \Gamma')$ as nilmanifolds with a twisted map which keep the eigenvalues and eigenvectors

in the Lie algebra of N.

We say that two compact solvmanifolds are **strong twisted equivalent** if G/N is abelian.

Definition 3. We say that two compact complex nilmanifolds G/Γ and G'/Γ', with both G and G' simply connected, are complex twisted equivalent if they are twisted equivalent with a complex twisted map.

Obviously, two compact complex nilmanifolds may not necessary twisted equivalent even if they have the same complex dimension.

Definition 4. We say that two compact complex solvmanifolds are **complex twisted equivalent** (or **strong complex twisted equivalent**) if the common unipotent subgroup N and the twisted map in the definition 2 are both complex.

Therefore, we obtained:

Theorem 1. *Let $M = G/\Gamma$ be a compact complex solvmanifold. Then M is strong complex twisted equivalent to another compact complex solvmanifold $M' = G'/\Gamma'$ such that G' has a Chevalley decomposition $G' = AN_{G'}$ with an abelian A action on the nilradical $N_{G'}$ semisimplely.*

Proof: Let $\gamma_1 \in \Gamma$ be a non-unipotent element in Γ, use the $\gamma_{1,s}$ we have a real compact 1-dimensional solvmanifold extension M_{γ_1} of the original compact complex solvmanifold. Now, the centralizer $C_{\gamma_{1,s}}$ of $\gamma_{1,s}$ in the new solvmanifold induces a compact subsolvmanifold $C_{\gamma_{1,s}}/\Gamma \cap C_{\gamma_{1,s}}$ (see a proof below). $C_{\gamma_{1,s}}$ contains a complement of the nilradical. We pick up a non-unipotent element γ_2 in $C_{\gamma_{1,s}} \cap \Gamma$ which represents a non-unipotent element in the complement of γ_1. Then we use $\gamma_{2,s}$ to get a real compact 1-dimensional solvmanifold M_{γ_1,γ_2} of M_{γ_1}.

To prove the existence of γ_2, we need to prove a similar result of the

Lemma 14 in [Gu3 p.54]. We regard $\gamma_{1,s}$ as an automorphism of G. As in the Lemma 13 in [Gu3 p.53], we have a compact fundamental domain Ω of G/Γ. For any $c \in C_{\gamma_{1,s}}$, there is a $\gamma_c \in \Gamma$ such that $\gamma_c c \in \Omega$. $\gamma_{1,s}(\gamma_c)\gamma_c^{-1} = \gamma_{1,s}(\gamma_c c)c^{-1}\gamma_c^{-1} \in \gamma_{1,s}(\Omega)\Omega^{-1}$, which is compact. That means that there are finite many c_j such that for any c, there is a c_j with

$$\gamma_{1,s}(\gamma_c)\gamma_c^{-1} = \gamma_{1,s}(\gamma_{c_j})\gamma_{c_j}^{-1}.$$

That is, $\gamma_{1,s}(\gamma_{c_j}^{-1}\gamma_c) = \gamma_{c_j}^{-1}\gamma_c$. Therefore, $\gamma_{c_j}^{-1}\gamma_c \in C_{\gamma_{1,s}} \cap \Gamma$. Thus, $C_{\gamma_{1,s}}/\Gamma \cap C_{\gamma_{1,s}}$ is compact.

Then, we have the centralizer $C_{\gamma_{1,s},\gamma_{2.s}}$. Arguing as above and the proof of Lemma 14 in [Gu3 p.54] (by replacing n_i there with $\gamma_{i,s}$) we have that $C_{\gamma_{1,s},\gamma_{2,s}}/\Gamma \cap C_{\gamma_{1,s},\gamma_{2,s}}$ is compact. Eventually, we get $\gamma_1, ..., \gamma_{2k} \in \Gamma$ and $M_1 = M_{\gamma_1,...,\gamma_{2k}}$ such that $\gamma_1, ..., \gamma_{2k}$ induces a complement of the nilradical. We see that M_1 is actually complex. Now, by deleting $\gamma_1, .., \gamma_{2k}$ we proved our theorem.

Q. E. D.

However, for the manifolds with holomorphic symplectic structures, the restriction of the holomorphic 2-form might be degenerated in the above process. Therefore, instead of the modification, we might consider the bigger compact complex solvmanifold $M_1 = G_1/\Gamma_1$. M_1 can be regarded as a twisted product of M and a complex torus induced by the automorphism φ. To obtain a holomorphic symplectic manifold, we can product M_1 with a torus of complex dimension $k = \dim_{\mathbf{C}} M_1 - \dim_{\mathbf{C}} M$. Note that k is the complex dimension of the base of the Mostow fibration. That is, $k = \dim_{\mathbf{C}} G/N_G$, where N_G is the nilradical of G. Let M' be the new manifold, then M' is a twisted product of M and an even dimension complex torus.

We therefore obtained:

Theorem 2. *Let $M = G/\Gamma$ be a compact complex solvmanifold with a holomorphic symplectic structure. Then, after twisted product with a complex torus T of complex dimension $2k$ if it is necessary, $M' = M \times_\varphi T = G'/\Gamma'$ has a holomorphic symplectic structure such that G' has a Chevalley decomposition $G' = AN_{G'}$ with an abelian A acting on the nilradical $N_{G'}$ semisimplely.*

This is the first part of our Theorem B. After finishing our results in this section we found a similar construction (to the Theorem 1) in [Au] chapter IV section 2.

3 Semisimple Chevalley compact complex solvmanifolds with holomorphic symplectic structures

3.1 Definition 5. We say that a solvable Lie group G is Chevalley if there is a decomposition $G = AN_G$ with an abelian subgroup A and the nilradical N_G. we say that G is semisimple Chevalley if A acts on N_G semisimplely. We say that a compact solvmanifold $M = G/\Gamma$ is semisimple Chevalley if G is semisimple Chevalley.

3.2 Now, we assume that our solvmanifold is semisimple Chevalley. Let the Lie algebra have a basis a_i, b_j with a_i semisimple and b_j nilpotent. We assume that b_j are eigenvectors of a_i. Let α_i, β_j be the corresponding holomorphic 1-forms. Consider $\alpha_{i_j} \wedge \beta_j$ be the only one with β_j in ω. If the eigenvalue is not zero for β_j, we denote it by α'_j. then $d(\alpha_{i_j} \wedge \beta_j)$ has a term $-\alpha_{i_j} \wedge \alpha'_j \wedge \beta_j$, which must be zero. therefore α_{i_j} is proportional to α'_j. That is $\alpha_{i_j} \wedge \beta_j$ is exact modulo the wedge products of β_j's. Therefore, we can always get rid of these terms if β_j has nonzero eigenvalues.

3.3 Now, for any term $\beta_j \wedge \beta_{j'}$, the differential has a term

$$(\alpha'_j + \alpha'_{j'}) \wedge \beta_j \wedge \beta_{j'},$$

which must be zero. Therefore, $\alpha'_j = -\alpha'_{j'}$. That is, β_j and $\beta_{j'}$ have eigenvalues negative to each other.

3.4 We now consider the term $\alpha_{i_j} \wedge \beta_j$ again. We now know that β_j has zero eigenvalue. We claim that $d\beta_j = 0$. Otherwise, $d(\alpha_{i_j} \wedge \beta_j) = \alpha_{i_j} \wedge (\sum a_k \beta_{j_k} \wedge \beta_{j'_k})$ with all the β_{j_k} $\beta_{j'_k}$ having eigenvalues negative to each other. They could not come from other term. We have $d\beta_j = 0$ as desired.

3.5 Therefore, we obtained:

Theorem 3. *Let M be a semisimple Chevalley compact complex solvmanifold with a holomorphic symplectic structure. Then the holomorphic symplectic form can have a form*

$$\omega^{2,0} = \omega_0 + \sum_{i=1}^{k} \beta_i \wedge \beta_{k+i}$$

such that ω_0 is a linear combination of products of closed holomorphic 1-forms and β_i, β_{k+i} correspond to eigenvectors with eigenvalues negative to each other.

3.6 Therefore, A is perpendicular to the commutator N of G. This is because of

$$\omega(a, [x, y]) = \omega([a, x], y) + \omega([y, a], x).$$

Assuming x, y are eigenvectors, the right side is zero unless x, y have opposite eigenvalues. But then the right side is also zero.

3.7 Let N_1 be the subspace of N which is perpendicular to N, and N_2 be a subspace of N which is a complement of N_1 and is invariant under the action of A. We see that ω is nondegenerate on N_2.

Since N is perpendicular to A, there is a subspace N_3 in N_G which intersects N only at zero such that ω is nondegeneate on $N_1 + N_3$. We let $N_0 = N_3 + N \subset N_G$. By deleting the possible β_j which involves in ω_0 from ω_0 and modify ω_0 if it is necessary, we can make the 1-forms involved perpendicular to all the β_j in the second term. Then we let A_0 be the orthogonal complement of N_0.

Corollary 1. *$G = A_0 N_0$ with N corresponding to the unipotent subgroup generated by $\beta_1, ..., \beta_{2k}$ and A_0 an abelian subgroup. In particular, if our semisimple Chevalley compact complex solvmanifold is the one we obtained in Theorem 2, then N_0 can have the same dimension as the original manifold there. That is, we can identify N_0 as the nilshadow.*

Proof: Let A_0 be the subgroup generated by those elements in the Lie algebra which are perpendicular to the Lie algebra $\mathcal{N}_0$ of N_0. For any a_1, a_2 being perpendicular to $\mathcal{N}_0$, we have

$$\omega([a_1, a_2], n) = \omega(a_1, [a_2, n]) + \omega(a_2, [n, a_1]) = 0$$

for any $n \in \mathcal{N}_0$. Therefore, $[a_1, a_2]$ is also perpendicular to $\mathcal{N}_0$. But G/N_0 is abelian. we have $[a_1, a_2] = 0$.

The reason for the last statement is because that the element in the center is always perpendicular to N. Therefore, those elements we obtained by product a torus in the Theorem 2 can be also chosen in A_0.

Q. E. D.

This is our Theorem A and the second part of Theorem B.

3.8 Now by the Mostow Fibration or Structure Theorem in [Mo] page 6 to 7, section 5 Theorem, also [Au] page 249 or Chapter IV, section 3, we see that our manifold is a nilmanifold bundle over a torus. By our structure of the symplectic structure, if the nilmanifold has an even complex dimension, we can always change the symplectic structure such that it is a holomorphic symplectic nilmanifold bundle over a holomorphic symplectic torus. If the nilmanifold has an odd complex dimension, we can always product a copy of complex torus of dimension 1 such that the new manifold is a holomorphic symplectic nilmanifolds bundle over a holomorphic symplectic torus. Then the method in [Gu2] (Cf. [GK]) gives a way to classify the compact holomorphic symplectic solvmanifolds by inductions.

3.9 Actually, after applying Theorem 2 and the last statement of the Corollary 1, we can have nilmanifold bundle over $(G/N_G)/(\Gamma/\Gamma \cap N_G) \times T^k$ and the fiber is the *unipotent modification* (or nilpotent reduction as we mentioned before) of M, that is, in the construction of Theorem we replace the elements by its unipotent part instead of the semisimple part. Therefore, to classify the compact complex solvmanifolds with holomorphic symplectic structures, we can just start from the compact complex nilmanifolds with holomorphic structures of the same dimension.

In particular, we see that the unipotent modification is always holomorphic symplectic, while in general the semisimple modification of the same dimension in the Theorem 1 might not be holomorphic symplectic. *In this special situation, the modification of the symplectic structure in 3.8 is unnecessary although the modification of 3.2 might be needed.*

Corollary 2. *If M is a compact complex solvmanifold with a holomorphic symplectic structure, then it is strongly twisted equivalent to a compact complex nilmanifold with a holomorphic symplectic structure.*

However, the unipotent modification lost a lot of information from the original manifold. Therefore, Theorem 2 is more important than Corollary 2.

But we do need to understand the unipotent modification first to understand the semisimple Chevalley modification. We shall see in the next section that actually, we get a lot of more for the structure of N_0 and very few of symplectic nilmanifolds can be N_0.

In the next section, we shall see that the same method works for the compact real solvmanifold with real symplectic structures although in that general case we have to use the result in [Gu5].

4 Compact real solvmanifolds with real symplectic structures

Similarly, we can also work on the compact real solvmanifolds with real symplectic structures. By [Gu5], we can always reduce to the situation, up to a finite covering, such that the symplectic structure is right invariant and the group and the uniform discrete subgroup have the same (real) algebraic closure. Then the same argument as in the earlier sections reduces the situation to the semisimple Chevalley modification. Then we can apply the argument in the previous section.

For the convenience of the readers, suggested by the referees, we explain a little bit about [Gu5] here. In general, for many years, people did not know how to calculate the cohomology of the compact solvmanifolds. It was known that, the right invariant cohomology might be strictly smaller than the actual cohomology, e.g., see [Rg], [Ya2]. This made the classification of compact solvmanifolds with real symplectic structures unreachable, see [BG] and [Bk] for examples. However, if the big (Lie) group G and the uniform subgroup Γ have the same algebraic closure in the representation Ad_G, then Mostow proved that cohomology can be calculated by the right invariant cohomology. In [Gu5], the author used our earlier method to prove:

Proposition 1. *For any compact solvmanifold $M = G/\Gamma$, there is a cofinite subgroup Γ' of Γ and another solvable Lie group G' such that: (A) $G/\Gamma' = G'/\Gamma'$; and (B) Γ' and G' have the same algebraic closure in the representation of $\mathrm{Ad}_{G'}$.*

Therefore, according to Mostow, the cohomology of the finite covering $M' = G'\Gamma' = G/\Gamma'$ can be calculated by the G' right invariant cohomology. If G/Γ is symplectic, then G'/Γ' is also symplectic and the symplectic form on G'/Γ' is right invariant. This made a classification of compact symplectic solvmanifolds possible.

Therefore, we obtained:

Theorem 4. *Let $M = G/\Gamma$ be a real compact solvmanifold. Then, up to a finite covering, it is strong twisted equivalent to a semisimple Chevalley compact solvmanifold.*

Theorem 5. *Let $M = G/\Gamma$ be a real compact solvmanifold with a real symplectic structure. Then, up to a finite covering and after twisted product with a torus, it is a semisimple Chevalley real compact solvmanifold with a real symplectic structure.*

Again, we could apply the same argument as above and obtain a similar result as Theorem 3 and therefore the Corollary 1. But in this case, to write a form as in the Theorem 3 in the real case, we need all the eigenvalues to be real. This actually can be achieved after a finite covering. This is a result from the algebraic units theory. See [BS] page 105 Theorem 2, which is a special case of the Dirichlet Theorem on the group of units—the Theorem 5 in page 112. Let a be an algebraic unit, then $b = \bar{a}a^{-1}$ is also an algebraic unit. Then by Theorem 2 there, it is a root of 1. That is after a finite power,

a became a real number.

The same argument also implies that for any compact solvmanifold, up to finite covering, we might assume that all the eigenvalues are real. This is our Theorem C, which actually already appeared in [Gu9]. This is related to the totally real algebraic number fields.

The same argument also implies that the eigenvalues in Theorem 3 can be real up to a finite covering. And any compact complex solvmanifold with a holomorphic symplectic structure is complex twisted equivalent to that with a pseudo-kähler structure in a more general sense.

Even in the real case, the possible N_0 are very few. For example, the b_1 for N_0 can not be 2. Otherwise, all the eigenvalues for the first level of the descending central series of N_0 are zeros and the original manifold must be nilpotent.

We notice that all the modified elements in the basis of N_0 has zero eigenvalues. Therefore, there is at least one zero eigenvalue in the first level of the descending central series of N_0. Therefore, if $b_1 = 3$, then the eigenvalues for the first level have to be $0, 1, -1$. And so on.

Therefore, if the dimension is 2 and 4, N_0 is abelian. The reason is that we can not have $(0, 1, -1), 0$. Otherwise, we let the first level be generated by z, x_1, x_{-1} and $dx_0 = x_1 \wedge x_{-1}$ for the second level. But $\omega(Z, X_0) = \omega(Z, [X_1, X_{-1}]) = 0$, a contradiction.

If dimension is 6, then N_0 must be one of $n_{6,12}$, $n_{6,21}$, $n_{4,2} \oplus \mathbf{R}^2$, $n_{5.5} \oplus \mathbf{R}$, $H_3 \oplus \mathbf{R}^3$, $\mathbf{R}^6$. For the notation of N_0, we follow [GK] page 307 to 311. They are the number 20 with structure equation $(0, 0, 0, 12, 13 + 14, 24)$, number 24 with the structure equation $(0, 0, 0, 12, 13, 23)$, number 26, 31, 33, 34(the last one) in the Table A.1 of [Sl]. Here, we use the notation of Salamon in [Sl] that the i-th coordinate $kl + mn$ means $dx_i = x_k \wedge x_l + x_m \wedge x_n$ so on. $n_{4,2}$ has the structure equation $(0,0,12,13)$ and $n_{5,5}$ has the structure equation $(0,0,0,12,13)$. H_3 has the structure equation $(0,0,12)$. The corresponding eigenvalues are $(1,-1,0),0,(1,-1)$ for $n_{6.12}$. The brackets denote each level of the descending central series. We get nothing from $(0,1,-1),0,0,0$ and $(0,1,-1),0,(1,-1)$ as well as $(0,1,-1),(1,-1),0$. The corresponding eigenvalues are $(0,1,-1),(0,1,-1)$ for $n_{6.21}$. We also have $(0,0),0,0$ for $n_{4,2}$ and $1,-1$ for $\mathbf{R}^2$. Nothing is for $(0,0,1,-1),(0,0)$. They are $(0,1,-1),(1,-1)$ for $n_{5.5}$ and 0 for $\mathbf{R}$. Nothing for $(0,1,-1,k,-k),0$ if $k \neq 0$. For $(0,1,-1,0,0),0$, we have $(0,0),0$ for H_3 and $(0,1,-1)$ for $\mathbf{R}^3$. And they are $0,0,l,-l,k,-k$ for $\mathbf{R}^6$.

This offshoot was motivated by a question from Anna Fino in 2011.

This fits quite well with the recent work of Bock [Bk], about which I was told by Professor Salamon in 2011. Let me explain a little bit our result and his results: Theorem 3.8.3.2 there gave a manifold with $n_{6.21}$. In Proposition 3.8.3.3, the discrete subgroup has pure imaginary eigenvalues and the manifolds are nilpotent after a finite covering. For example, he put $t_1 = \frac{\pi}{3}$, then a six to one covering has only the identity 1 as the eigenvalues for the cocompact discrete subgroup. By [Gu5] (and [Gu1]) we can modify the big group such that the big group only has the identity 1 as the eigenvalues. That is, after a six to one covering, it is unipotent. For Theorem 3.8.3.4 there, the nilpotent modification is $n_{6,12}$ in [GK]. For Proposition 3.8.4.3 and 3.8.4.4, the nilpotent modification is $n_{4,2} \oplus \mathbf{R}^2$. For Proposition 3.8.4.4 and 3.8.4.5, those manifolds are actually nilpotent. For Proposition 3.8.4.6, the nilpotent modification is $n_{5,5} \oplus \mathbf{R}$ with $(0,1,-1),(1,-1)$ for $n_{5,5}$ and 0 for $\mathbf{R}$. Those manifolds in Propositions 3.8.4.8, 3.8.4.9, 3.8.4.10 are actually nilpotent after a finite covering since the eigenvalues are pure imaginary. For the manifolds in Proposition 3.8.4.11, after a finite covering to get rid of the pure imaginary in the eigenvalues of the cocompact discrete subgroup, the nilpotent modification is $\mathbf{R}^6$ with eigenvalues $(0,1,-1,0,1,-1)$.

This is also true for the decomposable manifolds there. They are all in the Table 3.6 in the section 3.9 there. Actually, there are two tables there. The first one lists the possible product of a five dimension manifold with a torus. The second table lists the possible product of two three dimensional manifolds. The $g_{5,7}^{p,-p,-1}$ is the same as $g_{5,13}^{-1,0,r}$. They are with $\mathbf{R}^6$ and $(0,1,-1,p,-p;0)$. $g_{5,8}^{-1}$ is $H_3 \oplus \mathbf{R}^3$ with $(0,0),0$ and $(0,1,-1)$. $g_{5,14}^{0}$ is unipotent after a finite covering. $g_{5,15}^{-1}$ is with $n_{5,5} \oplus \mathbf{R}$. For $g_{5,17}^{p,-p,r}$ with $r=1$ or -1, one can see from Proposition 3.7.2.12 that the cocompact subgroup only has real eigenvalues and therefore, one can use [Gu5] or [Gu1] to modify the Lie group; and according to Propositions 3.7.2.13 and 3.7.2.14, for $g_{5,17}^{0,0,r}$ there, the eigenvalues are pure imaginary and therefore, can be modified into nilmanifolds. Now, finally for $g_{5,18}^{0}$, the eigenvalues are pure imaginary and can be modified into nilmanifolds.

Now, let us look at the second table for the product of two three dimensional manifolds. The first one, $g_{3,4}^{-1} \oplus \mathbf{R}^3$, $\mathbf{R}^6$. For the second one, $g_{3.5}^{0}$, eigenvalues are pure imaginary, a finite covering is unipotent. $g_{3,1} \oplus g_{3,4}^{-1}$, $H_3 \oplus \mathbf{R}^3$. $g_{3,1} \oplus g_{3,5}^{0}$, finite covering is nilpotent. $g_{3,4}^{-1} \oplus g_{3,4}^{-1}$ and $g_{3,4}^{-1} \oplus g_{3,5}^{0}$ are with $\mathbf{R}^6$, with eigenvalues $(0,1,-1,0,1,-1)$ and $(0,1,-1,0,0,0)$. The last one $g_{3.5}^{0} \oplus g_{3,5}^{0}$ is nilmanifold after a finite covering.

Our arguments are much shorter.

For the complex case only the last case related to $\mathbf{R}^6$ occurs according to the methods in [Gu9] and [Ba]. Therefore, we have:

Corollary 3. *Every six complex dimensional non-nilpotent compact complex solvmanifold with holomorphic symplectic structures are pseudo-Kählerian-like and have hypersymplectic solvmanifolds as finite coverings.*

This is our Theorem D. Therefore, as in the last section, the method in [Gu2] and [GK] gives an inductive classification.

5 Further comments and examples

Therefore, in general, our methods could reduce the classification of compact complex solvmanifolds with holomorphic symplectic structures to the situation in which the complex Lie group has a Chevalley decomposition $G = AN$ as in the question 2 such that A acts on N semisimplely. The symplectic form, after a series of modifications, has the form

$$\omega = \omega_0 + \sum \beta_{2i-1} \wedge \beta_{2i},$$

where ω_0 comes from those closed 1-forms and $\beta_{2i-1}, \beta_{2i} \in \mathcal{N}^*$ are pairs of holomorphic 1-forms which corresponding to the pairs of eigenvectors with eigenvalues different by a sign. In particular, we can build up the compact complex solvmanifolds with holomorphic symplectic structures from those compact complex nilmanifolds (of at most the same dimension) with holomorphic symplectic structures of the form

$$\omega_1 = \sum \beta_{2i-1} \wedge \beta_{2i},$$

which could be classified by following the process in [Gu2] (Cf. [GK]) with applying the Mostow Fibration Theorem. Up to finite covering, one can actually see that the eigenvalues for the discrete subgroup could be real algebraic units.

Different from the pseudo-Kähler-like case in [Gu9], the 1-forms involved in ω_0 might also correspond to pure nilpotent elements with nontrivial adjoint actions.

To see some examples, we could just take any example in [Ya1] with real symplectic structures, then we complexify them by the principle of Proposition 4 in [Gu7] similar to what Yamada did in [Ya2]. For the semisimple actions, we just extend the action naturally. The $2i\pi$ with $e^{2i\pi} = 1$ will give the other generators we need in the lattice. For the nilpotent actions, we simply complexify the action as Yamada did.

In all our examples, the group is complex. Therefore, they do not satisfy the Mostow condition. The cohomologies might not be right invariant. For example, the pseudo-kähler forms on the pseudo-kähler solvmanifolds are not right invariant. However, the holomorphic symplectic forms are all right invariant since all the holomorphic forms are right invariant. For the real examples in this paper, they have however right invariant cohomologies since all have only real eigenvalues. That is, Hattori's result applies.

Another example comes from [BG] example 3. By our argument in the second section, it is not difficult to see that the example 2 there does not exist. But we can easily see that example 3 does exist. Let α be a root of the equation:

$$x^2 - nx + 1 = 0$$

with $n > 2$. Let $A = \mathrm{diag}(\alpha, \alpha^{-1})$ and the lattice for the $\mathbf{R}^2$ generated by $X_i,\ i = 1, 2$ be generated by $\gamma_1 = (1, 1), \gamma_2 = A\gamma_1 = (\alpha, \alpha^{-1})$. Similarly for the $\mathbf{R}^2$ generated by $Y_i,\ i = 1, 2$. Then, $A\gamma_2 - n\gamma_2 + \gamma_1 = 0$ and we have the action of a generator a we need for both X_i and Y_i. a acts on the first by A and the second by A^{-2}. For the lattice related to $Z_i = [X_i, Y_i],\ i = 1, 2$

we use $\gamma_1^2 = (1,1), \gamma_1\gamma_2 = A\gamma_1^2$. We notice that $\gamma_2^2 = A\gamma_1\gamma_2$ and action of a on Z_i by A^{-1}. See also the construction in [SY] (I was told by C. Benson about this paper after I told him our construction). After complexifying this example, we obtain an example of compact holomorphic symplectic solvmanifold such that the Lie group has three steps. Their further example in [Sa] similar to the proposed example 2 in [BG] does have our form in the fourth section. However, it can not be complexified as a *compact* complex solvmanifold with holomorphic symplectic structures.

The Lie algebra in this example is generated by a, b, X_i, Y_i, Z_i $i = 1, 2$. We have $[X_i, Y_i] = Z_i, [a, X_i] = (-1)^{i+1}X_i,\ [a, Y_i] = 2(-1)^i Y_i, [a, Z_i] = (-1)^i Z_i$ and other Lie brackets are zeros. The symplectic form is $\omega_{a,b,c,d,e} = a\alpha\wedge\beta + bx_1\wedge z_1 + cx_2\wedge z_2 + dx_1\wedge x_2 + ey_1\wedge y_2$ for nonzero numbers a, b, c, d, e.

We could actually apply the modification backward to obtain the original compact solvmanifolds with symplectic structures. For example, we might assume that there is only one dimensional non-nilpotent elements. Then semisimple part X_s can be just a, we let N be the nilpotent part, and our manifold is obtained by a direct modification as we proposed at the beginning of our section 2. Then we have $NX_1 = kZ_2$ and $NX_2 = lZ_1$, others are zeros. We only need

$$\omega(NX_1, X_2) + \omega(X_1, NX_2) = -ck + bl = 0.$$

Then, we can easily replace a by $a+N$ to get the original compact solvmanifolds with symplectic structures if we could obtain a rational representation of the modified element.

We see easily that there is a subgroup generated by a, b, X_i, Z_i $i = 1, 2$. The action of A on the $\mathbf{R}^4$ is $\operatorname{diag}(\alpha, \alpha^{-1}, \alpha^{-1}, \alpha)$. By $NA = AN$ we see that $N = \begin{bmatrix} 0 & N_1 \\ 0 & 0 \end{bmatrix}$ and $N_1 = \begin{bmatrix} 0 & n_1 \\ n_2 & 0 \end{bmatrix}$. This fits for the compact real solvmanifolds with real symplectic structures. But this does not fit for the complex case by the argument in the proof of the Lemma 7 in [Gu9] by using the (Hilbert Seventh problem related) results in [Ba], [Ge]. Therefore, one might suggest that for the complex case (or at least for the case with holomorphic symplectic structures), the solvmanifold is always semisimple Chevalley. We shall deal with this in a further publication.

The major difference from the pseudo-kähler-like case in [Gu9] is that Lemmas 1 and 4 in [Gu9] do not work in general.

After that, one might make other further and different modifications such that all the 1-form involved in ω_0 are closed, semisimple and β_j has nonzero eigenfunctions. Since the rational structure of the nilpotent group is the same as its Lie algebra through the exponential map (see also [Au] page 246), we can actually have a pseudo-kähler-like modification in a more general sense. By a finite covering, that actually lead to a pseudo-kähler one by the algebraic units theory. This should lead to some kind of classification as certain product, similar to the one in [Gu8], of two types of holomorphic symplectic solvmanifolds. The reason that the pseudo-kähler-like ones only have one type is that all the pseudo-kähler-like nilpotent ones are tori. This also applies to the real symplectic solvmanifolds after modifications as we see in last section.

By our arguments in the last section, we can also avoid the Mostow condition and Mostow Theorem on the cohomology in [Gu5, 6] and modify the Lie group by torus only, i.e., the imaginaries, then use Hattori's result in [Ha] instead. After reading [Au], we found a similar construction to our Theorem 1 in Chapter IV section 2. That is also the foundation of our

Theorem 2.

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