On the Nonexistence of S^6 type Complex Threefolds in Any Compact Homogeneous Complex Manifolds with the Compact Lie Group G_2 as the Base Manifold

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In a recent preprint Professor Etesi asked a question: Could one find a complex three dimensional submanifold S in a compact complex seven dimensional homogeneous space with the compact real 14 dimensional Lie group G_2 as the base manifold, such that S is diffeomorphic to the six dimensional sphere S^6? We apply a result of Tits on compact complex homogeneous space, or of H. C. Wang and Hano-Kobayashi on the classification of compact complex homogeneous manifolds with a compact reductive Lie group to give an answer to his question. In particular, we show that one could not obtain a complex structure of S^6 in his way.

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1 Introduction

Let M be a complex manifold, h be an Hermitian metric. For a compact complex manifold, there is always some Hermitian metric h by the partition of the unity argument. If h is an Hermitian metric and G is a compact Lie group acting on M biholomorphically, then by taking average on G, we can always assume that h is invariant under G.

A compact complex homogeneous space with an invariant Hermitian structure was classified by H. C. Wang in [W], see also [HKo]. In fact, they classified the compact complex homogeneous spaces with compact Lie groups. In particular, an Hermitian manifold is a Riemannian manifold. The identity component of the Riemannian isometric group for a compact Riemannian manifold is a compact Lie group. So is the identity component of the Hermitian isometric group for a compact Hermitian manifold.

Therefore, we have:

Lemma 1. *If $M = G/H$ is a compact homogeneous Riemannian manifold with G connected, then G is a subgroup of a compact Lie group. In particular, both G and H are reductive with compact semisimple parts.*

We then have (see [HKo] Theorem B):

Lemma 2. *Any compact Hermitian homogeneous manifold is a complex torus bundle over a rational (therefore simply connected) projective homogeneous space.*

One could also see [Gu2] page 66, Remark for a detail understanding of this fibration.

There is also a similar fibration [Ti] closely related to Lemma 2 for any general compact complex homogeneous space:

Proposition 1. *Let $M = G/H$ be a compact complex homogeneous space such that G is a complex Lie group, H a complex Lie subgroup and $M = G/H$ is the complex quotient. Then there is a complex fibration $G/H \to G/N$ such that $N = Norm_G(H^0)$ and G/N is a rational projective homogeneous space.*

Here, H^0 is the identity component of H and $Norm_G(H^0) = \{g \in G|_{gH^0g^{-1}\subset H^0}\}$ is the normalizer of H^0 in G.

Let $G = G_2$ be the compact real 14 dimensional Lie group of type G_2, $G^{\mathbf{C}}$ be the complex simple Lie group such that the Lie algebra of $G^{\mathbf{C}}$ is the complexification of the Lie algebra of G. Let B be the Borel subgroup such that B is connected and the Lie algebra of B contains all the negative root vector spaces and the given Cartan subalgebra. Then $G^{\mathbf{C}}/B$ has a complex dimension 6. We notice that the Cartan subalgebra has a complex dimension 2. So, let U be the maximal nilpotent subgroup of B, which is generated by the all the negative root spaces, then B/U is a complex two dimensional algebraic torus $(\mathbf{C}^*)^2$. Let $\pi : B \to B/U$ be the quotient map. Given any complex one dimensional line subspace $l = \mathbf{C}$ in B/U, $S_l = \pi^{-1}(l)$ is a complex codimension one normal subgroup of B. Then $G^{\mathbf{C}}/S_l$ is diffeomorphic to G_2. Therefore, it gives a complex structure on G_2.

On the other hand, if $G = G_2$ is a complex homogeneous space itself, then its complexification $G^{\mathbf{C}}$ acts on G also. And $G = G^{\mathbf{C}}/H$. By Lemma 2, the Hano-Kobayashi fiber bundle is $G^{\mathbf{C}}/H \to G^{\mathbf{C}}/B$. The torus is just the Cartan subgroup of G. By earlier works, e.g., [Gu1], we know that the Tits fibration and the HK fibration are the same and therefore, $B = Norm_{G^{\mathbf{C}}}(H^0)$. But $\dim_{\mathbf{C}} B/H = 1$ and H^0 is normal in B. We see that H

contains all the subgroups generated by the negative root spaces. That is, $U \subset H$.

Proposition 2. *Let G be a complex manifold with the base manifold being G_2 such that $G = G_2$ acts on itself biholomorphically. Then there is a holomorphic fibration from G into a projective space $F : G \to G^{\mathbf{C}}/B$ such that each fiber is a complex one dimensional complex torus. If S is a complex three dimensional compact submanifold of G, then either $F(S)$ is complex three dimensional or $F(S)$ is complex dimensional two and $S = F^{-1}(F(S))$.*

Therefore, if S is diffeomorphic to S^6, by the result in Campana et al [CDP1, 2], the algebraic dimension of complex structure on S is zero. This implies that S can not be a complex submanifold of G.

However, in this paper, we would like to give an argument which is mildly independent of the result from Campana et al.

We shall prove:

Main Theorem *The conjugate orbit described in both [Et] and [CR] can not be a complex submanifold of G_2.*

Recall that $S^6 = G_2/SU(3)$. The Cartan subgroup of $SU(3)$ is also a Cartan subgroup of G_2. The Cartan subgroup of $SU(3)$ has elements $\mathrm{diag}(a, b, c)$ with $abc = 1$. The center of $SU(3)$ has elements $aI = \mathrm{diag}(a, a, a)$ with $a^3 = 1$. This implies the center of $SU(3)$ has the structure $C = \mathbf{Z}/(3\mathbf{Z}) = \mathbf{Z}_3$. Let A be one of the generator of $C = \mathbf{Z}_3$. Then $S = \{gAg^{-1}|_{g \in G_2}\}$. Since A is in the center of $SU(3)$, it is not difficult to see that $SU(3)$ fixs A.

A description of the Lie algebra of G_2 can be also found in the section 4 of [Gu3], in which the Lie algebra of $SU(3)$ is simply generated by the root spaces with the short roots $e_i - e_j$ with $\{i, j\} \in \{1, 2, 3\}$.

2 Proof

Proof of the Proposition 2: If $F(S)$ in the Proposition 2 is of two complex dimension, let $p \in F(S)$ be a regular point for F, then $F^{-1}(p) \cap S$ is a complex one dimensional manifold. As a closed complex one dimensional submanifold of $F^{-1}(p)$, it can only be the whole complex torus. Since the fiber bundle is locally free and S a complex submanifold, the regular points for F are a dense set, by the continuity we have the last part of the Proposition 2.

Q. E. D.

Proof of the The Main Theorem: Assume that S is a complex submanifold, we shall get a contradiction.

If $F(S)$ is complex three dimensional, by $F(G)$ is projective, the pullback of the Kähler form ω is a nonzero H^2 class since it introduces a nonzero measure ω^3 on S. A contradiction.

If $F(S)$ is of complex dimension two, the conjugate orbit of Λ in [Et] or [CR] is S. Here, Λ is one of the two nonidentity elements in the center of $SU(3)$ and therefore is in the given Cartan subgroup of $SU(3)$, which is exactly the Cartan subgroup T of G_2. But $S = F^{-1}(F(S))$, we have that the whole Cartan subgroup of G_2 is $F^{-1}(F(\Lambda))$. Therefore, S could be described as $S_t = \{gtg^{-1}|_{g \in G_2}\}$ for any $t \in T$. But $S_t = G/G_t$ with G_t the centralizer $\{g \in G|_{gt=tg}\}$. However, in general, $G_t = T$. That is S_t is real 12 dimensional. A contradiction.

Q. E. D.

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