Moser Vector Fields and Geometry of the Mabuchi Moduli Space of Kähler Metrics

Daniel Guan[†]

June 14, 2013

Abstract: There is a natural Moser type transformation along any curve in the moduli spaces of Kähler metrics. In this paper we apply this transformation to give an explicit construction of the parallel transformation along a curve in the Mabuchi moduli space of Kähler metrics. This is crucial in the proof of the equivalence between the existence of the Kähler metrics with constant scalar curvature and the geodesic stability for the type II compact almost homogeneous manifolds of cohomogeneity one mentioned in [15]. We also explain a new description of the geodesics and prove a curvature property of the moduli space, called curvature symmetric, which makes it similar to some special symmetric spaces with nonpositive curvatures although the spaces are usually not complete. Finally, we generalize our geodesic stability conjectures in [6] and give several results on the Lie algebra structures related to the parallel transformations. In the last section, we generalize the Futaki obstruction of the Kähler-Einstein metrics to the parallel vector fields of the invariant Mabuchi moduli space. We call the related stability the parallel stability. This includes the toric and cohomogeneity one cases as well as the spherical manifolds.

1 Introduction

In the study of existence of Kähler-Einstein metrics Mabuchi introduced the geodesic equations in [16]. It turns out that the special homogeneous complex Monge-Ampére equation Semmes considered in [18] is exactly the same equation considered by Mabuchi. Let (M, ω) be a compact Kähler manifold, $\varphi(t, m)$ be a real smooth function with variables of time t and

[†]Supported by NSF DMS-0103282

point $m \in M$. Regarding t as the real part of a complex number z_0, φ is a function of z_0 and m. If φ satisfy the complex homogeneous Monge-Ampére equation $\det(\omega + \partial\bar{\partial}\varphi) = 0$ and $\omega_t = \omega + \partial_M \bar{\partial}_M \varphi$ is a Kähler metric for $t \in [0, T]$, then ω_t is a geodesic in the Mabuchi moduli space of the Kähler metrics.

Locally if $F(0, m)$ is the Kähler potential of ω and $F(t, m) = F(0, m) + \varphi$, by letting $F(z_0, m) = F(t, m)$ we see that the equation is the same as $\det(\partial\bar{\partial}F) = 0$.

It was observed that the kernel of $\partial\bar{\partial}F$ induced a vector field of certain Moser type transformation. And the Moser type vector field W can be applied to any general curve in the space of the Kähler metrics. In this paper we shall apply this Moser type transformation to give an explicit construction for the parallel transformation along any curve in the Mabuchi moduli spaces of Kähler metrics. The existence of the parallel transformation was proven in [16 p.234] by integration of a vector field and then in [7], without knowing Mabuchi's first proof, obtained by solving one of the two quasi-linear equations which led to the existence of the geodesics therein. That eventually gave a proof of the opposite direction of [10] and [12] in 2003 for the cohomogeneity one case. We shall give a different proof of the existence of the parallel transformation in this paper. The corresponding quasi-linear equation is actually $L_W h = 0$. Our arguement here is independent of [7]. While we dealt mostly with the case with a flat Mabuchi moduli space of Kähler metrics in a given Kähler class in [7], we deal with the general case here.

One can also easily observe that the geodesic equation is actually $L_W \dot{F} = 0$.

From the properties of the Moser vector fields we also expect that there should be some applications to metrics flows. We shall investigate this in the last three sections and in a near future. In particular, we prove that the curvature is parallel on the Mabuchi moduli space of the Kähler metrics of a given Kähler class.

After the first version of this paper, many things happened. It is evident that to define a generalization of Kähler metrics with constant scalar curvature, instead of the scalar curvature being a potential function of a holomorphic vector field as Calabi did, we could consider with a Kähler metrics of a scalar curvature being a function as the restriction of a parallel function on the equivariant Mabuchi moduli space of Kähler metrics. This is motivated by [5]. For example, in the case of cohomogeneity one with a semisimple automorphism group in [6], [15], the Futaki invariants are zero. Therefore, there is no Calabi extremal metrics in general. However, there are nontrivial parallel functions on the equivariant Mabuchi moduli space and therefore, there are many "generalized" Calabi extremal metrics, see also [13]. In particular, any equivariant Kähler metrics *is* a generalized Calabi extremal metrics in this sense. Of course, we need more restriction to get a more meaningful generalized Calabi extremal metrics. In [11] in 2001, it was proven that Calabi flow was better than the generalized Ricci flow in general. But it seemly was only true for the low dimensional case. When we try to publish [11], we proposed that the Calabi flow always has the long time existence property just as what Cao and Koiso observed in the Kähler-Ricci flow. However, it seems possible that it is only true for the low dimensional case. For example, in the case of $\mathbf{C}P^n$ with the standard cohomogeneity one metrics as in [11], as one end has a high codimension n, the Calabi flow might not always have long time existence. Therefore, it is easier to consider a generalized Calabi metrics called the m-extremal metrics and the related m-Calabi flow [14]. To make the short time existence easier, one might actually consider an exponential Calabi extremal metrics or e-extremal metrics and e-Calabi flow.

2 Preliminaries

Definition: Let ω_t be a family of symplectic forms. If a vector field $X(t,m)$ satisfies $X(\omega_t) = 0$, then we call X a Moser vector field.

See also Moser's original paper [17].

Regarding $\partial\bar\partial F = (F_{i\bar j})$ as a matrix. Let $(A_{i\bar j})$ be the adjoint matrix, then $F_{i\bar j}A_{j\bar k} = \delta_{ik}\det(\partial\bar\partial F)$ for any $i,k \in (0,1,\cdots,n)$. Apply this formula we obtain:

Lemma 1. *$Y = A_{0\bar 0}\frac{\partial}{\partial z^0} + A_{0\bar s}\frac{\partial}{\partial z^s}$ is in the kernel if* $\det(\partial\bar\partial F) = 0$.

Let $W = \mathrm{Re}\frac{Y}{A_{0\bar 0}}$. In general, we let $\omega_t = (\omega_{t,i\bar j}) = \omega_0 + \partial\bar\partial\varphi_t$ be a curve in the space of Kähler metrics even if we do not have $\det(\partial\bar\partial F) = 0$, and we can define Y and W in the same way with $F_t = F_0 + \varphi_t$. Then by direct calculation we obtain:

Lemma 2. *$L_W(\partial_M\bar\partial_M F)|_M = 0$.*

Proof: Let $X = \frac{Y}{A_{0\bar 0}}$ and denote $\partial_i = \frac{\partial}{\partial z^i}$, $\partial_{\bar i} = \frac{\partial}{\partial \bar z^i}$. Then

$$L_X = L_{\partial_0} + L_{\frac{A_{0\bar s}}{A_{0\bar 0}}\partial_s}.$$

By

$$\begin{aligned} L_{a_i\partial_i}(dz^k)(\partial_l) &= L_{a_i\partial_i}(\delta_{kl} - dz^k([a_i\partial_i, \partial_l]) \\ &= dz_k(\partial_l(a_i)\partial_i) \\ &= \partial_l(a_k) \\ &= a_{k,l} \end{aligned}$$

and similarly we have

$$L_{a_i\partial_i}(dz^k)(\partial_{\bar{l}}) = a_{k,\bar{l}},$$

that is, $L_{a_i\partial_i}(dz^k) = a_{k,l}dz^l + a_{k,\bar{l}}d\bar{z}^l$. Similarly, we have $L_{a_i\partial_i}d\bar{z}^k = 0$.

Combinning this together with the identity

$$F_{0\bar{m}}A_{0\bar{0}} + F_{r\bar{m}}A_{0\bar{r}} = 0$$

and its differentiations with respect to z_s and $\bar{z}_s$, we finally obtain:

$$\begin{aligned} & L_X(F_{s\bar{m}}dz^s \wedge d\bar{z}^m)|_M = (\dot{F}_{s\bar{m}} + \frac{A_{0\bar{r}}}{A_{0\bar{0}}}F_{s\bar{m}r} \\ + & \quad F_{r\bar{m}}(\frac{A_{0\bar{r},s}}{A_{0\bar{0}}} - \frac{A_{0\bar{r}}A_{0\bar{0},s}}{A_{0\bar{0}}^2}))dz^s \wedge d\bar{z}^m + F_{r\bar{m}}(\frac{A_{0\bar{r}}}{A_{0\bar{0}}})_{\bar{s}}d\bar{z}^s \wedge d\bar{z}^m \\ = & \quad \frac{-A_{0\bar{0},s}}{A_{0\bar{0}}^2}(\dot{F}_{\bar{m}}A_{0\bar{0}} + F_{r\bar{m}}A_{0\bar{r}})dz^s \wedge d\bar{z}^m - (F_{r\bar{m}\bar{s}}\frac{A_{0\bar{r}}}{A_{0\bar{0}}} + \dot{F}_{\bar{m}\bar{s}})d\bar{z}^m \wedge d\bar{z}^s \\ = & \quad 0. \end{aligned}$$

as required.

Proposition 1. *W induces a family of Moser differential transformations $T(t)$ such that $T(\partial_M\bar{\partial}_M F|_M) = \partial_M\bar{\partial}_M F|_M$.*

By the parallel transformation formula in [16] and [5,7] we have:

Lemma 3. *A function h is a parallel transformation, regarding as a vector field in the moduli space of Kähler metrics, along a path $\omega_t = \omega_0 + \partial_M\bar{\partial}_M\varphi$ if and only if $\dot{h} - \frac{1}{2}(dh, d\dot{F}) = 0$.*

3 Existence of the Parallel Transformations

In general, let $\omega_t = (\omega_{t,i\bar{j}}) = \omega_0 + \partial\bar{\partial}\varphi_t$ be a curve in the space of Kähler metrics (might not be a geodesic curve), we see that

$$A_{0\bar{0}}(\dot{h} - \frac{1}{2}(dh, d\dot{F})) = \mathrm{Re}A_{0\bar{i}}h_i = A_{0\bar{0}}L_W h.$$

Therefore, we have:

Theorem 1. *h is parallel if and only if $L_W h = 0$. In particular, ω_t is a geodesic if and only if $L_W \dot{F} = 0$.*

Once we have a curve in the moduli space of Kähler metrics we have the orbits of the Moser vector field W. Let h_0 be any function on M, regarding as the initial vector field at ω_0, we can let h to be constant along the orbits. Then h is parallel along this given curve in the moduli space. In this way, we have an explicit construction of the vector field h from the initial value h_0. Therefore, we have:

Corollary 1. *Given any curve $C: \ \omega_t$ with $t \in [0, 1]$ in the moduli space of the Kähler metrics in a Kähler class and an initial vector h_0 at ω_0, there is a unique smooth parallel vector field along C.*

In [7], without knowing the first proof of Mabuchi in [16 p.234], we had used our Corollary 1 to find global parallel vector fields on the compact complex almost homogeneous Kähler manifolds with algebraic reductive groups and obtain parallel coordinates. Then by integrating the global parallel vector fields we shall obtain many smooth geodesics.

4 Poisson Bracket and the Curvature Symmetric Property of the Mabuchi Moduli Space of the Kähler Metrics in a Given Kähler Class

For a given Kähler structure we have a symplectic structure. Therefore, there is a Poisson structure $\{f, g\} = \omega^*(df, dg)$ or $df \wedge dg \wedge \omega^{n-1} = \{f, g\}\omega^n$. By identifying $\{f \in C^\infty(M)|_{\int_M f\omega^n = 0}\}$ with the tangent space at ω we have an infinite dimensional Lie algebra structure on each tangent space. It is actually an infinite dimensional Lie algebra with an invariant metric (more like a compact Lie algebra of the finite dimensional case, see [8]). Now we would like to see some more global pictures.

Let $\phi = \phi(t)$ be a curve in the Mabuchi moduli space, and φ_i be two parallel vector fields along ϕ for $i = 1, 2$.

Then in view of (Lemma 2 and) Proposition 1, identifying by the Moser transformation we can consider the family of Kähler structures as the same symplectic structure, and φ_i being regarded as functions of this symplectic manifold are independent of t by Theorem 1. The Poisson bracket $\{\varphi_1, \varphi_2\}$ also does not depend on t and therefore is parallel along the curve of the Kähler structures. This parallelism can be expressed as:

Theorem 2.

$$\frac{d}{dt}(\{\varphi_1, \varphi_2\}) = \frac{1}{2}(d\dot{\phi}, d\{\varphi_1, \varphi_2\}).$$

Therefore, by the formula of the curvature in [16] (see also [18], [3], [2 p.195 Theorem A]) that

$$R(f, g)h = -\frac{1}{4}\{\{f, g\}, h\}$$

we also have that the curvature is constant along the curve by the curvature formula.

Theorem 3. *On the moduli space* $\nabla R = 0$.

We call an infinite dimensional Riemannian manifold **curvature symmetric** if $\nabla R = 0$. Theorem 4 just says that the Mabuchi moduli space of the Kähler metrics in a given Kähler class is curvature symmetric.

Therefore, the moduli space has a quite good structure. However, we see in [6] that the moduli space, in general, is incomplete. This gives a negative answer to an expectation of Semmes [18] and Donaldson [3] that the moduli space is an infinite dimensional kind of (complete) symmetric space.

On the other hand, our result in this section does give some good properties similar to a convex region (see [2]) in a special symmetric space of nonpositive curvature (cf. [9]).

I personally do believe that the Mabuchi moduli space of the Kähler metrics is locally symmetric, i.e., at each point there is a local isometric map which fixes the given point and induces a map of -1 on the tangent space.

5 The Fourth Geodesic Principle and the Geodesic Stability Conjectures

When the curve is an infinite geodesic ray, we can obtain a good picture for the general geodesic stability provided that there is enough smoothness. Instead of using parallel geodesics, which works for the case with the moduli space being flat as in [6], we could use infinite geodesic rays with the same point at infinity. This was also pointed out in [11] Remark 4. It is just as in the case of finite dimensional symmetric spaces. For example, two maximal geodesic rays γ_1 and γ_2 with $d(\gamma_1(t), \gamma_2(t)) < C$ for some positive constant C go to the same point at infinity. One can use a finite dimensional nonpositively curved symmetric space as a model, e.g., the Poincare metric on the upper half plane $\mathrm{Im} z > 0$.

In the case with the Poincare metric, the geodesics are straight lines perpendicular to the x-axis and the half circles with the centers on the x-axis. Two geodesic rays go to the same point at infinity if their end points on the x-axis are the same.

Now let us go back to the infinite dimensional picture on the Mabuchi moduli space. Along each maximal geodesic ray, since $\nabla R = 0$, we can have a frame of vector fields such that the sectional curvatures are constant and nonpositive. The corresponding Jacobi field of family of geodesics going to the same point at infinity has the form

$$J = \sum_{i=0}^{+\infty} a_i e^{-b_i t} \varphi_i,$$

where φ_i are parallel along the geodesic, $b_0 = 0 < b_1 < \cdots < b_i < b_{i+1} < \cdots$ assuming that there exists an orthogonal basis with respect to both the metric and the curvature along the given geodesic (notice that this is not true in general). See [1 p.15] and notice a sign difference for the curvature. The above formula will be $J = \sum_{i=1}^{N} a_i e^{-b_i t} \varphi_i$ for the finite dimensional case and we just replace N by $+\infty$ here. We had dealt with some flat cases in [7] in which $a_i = 0$ for $i > 0$.

Let us have another way to understand this. Let $\mathcal{M}_K$ be the Mabuchi moduli space of the invariant Kähler metrics with respect to a maximal compact subgroup K of the complex automorphism group. Rescale it by a sequence of positive numbers $A_i \to 0$. Recall that on the tangent space at ω of the Mabuchi moduli space, the Riemannian metric is

$$g(\phi_1, \phi_2) = \int_M \phi_1 \phi_2 \omega^n.$$

We let $g_{A_i} = A_i g$ and fix a Kähler metric ω. We denote the corresponding marked Riemannian manifold by $A_i(\mathcal{M}_K, \omega)$. With assuming that we can have an exponential map, we identify $A_i(\mathcal{M}_K, \omega)$ as a subset of $(\mathcal{M}_K, \omega)$ by the differentiable map

$$j_{A_i} = exp_{(\mathcal{M}_K,\omega)} \cdot i_{A_i} \cdot exp^{-1}_{A_i(\mathcal{M}_K,\omega)}$$

where exp are the exponential maps at ω and i_{A_i} is the natural identitity map from the tangent space of $A_i(\mathcal{M}_K, \omega)$ to that of $(\mathcal{M}_K, \omega)$. Then for any point ω, the marked infinite dimensional Riemannian manifolds $A_i(\mathcal{M}_K, \omega)$

converge as subsets to a cone $\mathcal{C}$ (rather than a sequence of Riemannian manifolds) with ω as the vertex point. The structure of $\mathcal{C}$ depends on the choice of ω. Any other point ω_1 also converges to the vertex point. *The condition that an infinite geodesic ray γ_1 from ω_1 goes to the same point at infinity as that of a given infinite geodesic ray γ from ω is defined as* $\lim_{i\to+\infty}\gamma_1 = \lim_{i\to+\infty}\gamma$.

A general Jacobi field on $(\mathcal{M}_K,\omega)$ has the form

$$J = (a_0^1\varphi_0^1 + a_0^2 t\varphi_0^2) + \sum_{n=1}^{+\infty}(a_n^1 e^{-b_n t}\varphi_n^1 + a_n^2 e^{b_n t}\varphi_n^2),$$

where a_n^k are constants and φ_n^k are parallel functions for $k = 1, 2$ (or a_n^2 are zeros except finite of them).

For $A_i(\mathcal{M}_K,\omega)$, we have $t_i = A_i t$ and $A_i\varphi_{n,i}^k = \varphi_n^k$. Therefore,

$$J_i = (a_0^1 A_i\varphi_{0,i}^1 + a_0^2 t_i\varphi_{0,i}^2) + \sum_{n=1}^{+\infty}(A_i a_n^1 e^{-A_i^{-1}b_n t_i}\varphi_{n,i}^1 + A_i a_n^2 e^{A_i^{-1}b_n t_i}\varphi_{n,i}^2).$$

Therefore, formally we can let

$$J_\infty = a_0^2 t_\infty\varphi_{0,\infty} + \sum_{n=1}^{+\infty} a_n^2 e^{b_n t_\infty}\varphi_{n,\infty}.$$

Other terms tend to zero. Let $M(\omega_1,\omega_2)$ be the Mabuchi functional. For any maximal geodesic ray γ, we let

$$F(\gamma) = \lim_{t\to+\infty}\frac{dM}{dt}(\gamma(0),\gamma(t)).$$

We then have:

Conjecture 1. *Two infinite geodesic rays γ_1 and γ_2 have the same generalized Futaki invariant if* $\lim_{i\to+\infty}\gamma_1 = \lim_{i\to+\infty}\gamma_2$.

The structure of the Jacobi field strongly supports this conjecture. We already have three **geodesic stability principles** in [6], we might call Conjecture 1 the **geodesic stability principle 4**.

This is exactly what we have in the Remark 4 of [11]. But we have more detailed picture here.

Let $\mathcal{C}_1$ be the subcone of $\mathcal{C}$ of all incomplete maximal geodesic rays, i.e., those only parametrized on half lines, as in [6] we have:

Conjecture 2. *If we identify $\mathcal{C}_1$ as part of the tangent space at a given Kähler metric ω_0, Then there is an extremal metric if and only if the generalized Futaki invariants of maximal geodesic rays are positive and bounded from blow by a given seminorm.*

As in [5,6], by the convex property of the modified Mabuchi functional along the geodesics we see that the generalized Futaki invariant can not be $-\infty$. The generalized Futaki invariant of a maximal geodesic ray might be $+\infty$. We have:

Conjecture 3. *The subset $\mathcal{C}'$ of maximal geodesic ray $c \in \mathcal{C}_1$ with finite generalized Futaki invariant is a subcone and $F(c), c \in \mathcal{C}'$ is linear on $\mathcal{C}'$ with respect to J_∞.*

Then we have the following:

Conjecture 2'. *A Kähler class has an extremal metric if and only if $F(c) \geq |c| > 0, c \in \mathcal{C}'$ for a* **given** *seminorm $|\ |$.*

We remark here that we need a seminorm here, i.e., $F(c) > 0, c \in \mathcal{C}_1$ is not enough as we see in [6] and the Ding-Tian type generalized Futaki invariant therein (cf. [4]) might not come from an $F(c)$ as in [6]. We still need to understand what the *given seminorm* is. However, our picture of the existence of the extremal metrics is quite clear now.

Remark 1. To convince the readers that the geodesic stability is the right stability we should give a philosophical "proof" for the Kähler Einstein case. If we consider the Kähler Ricci flow, Perelman's estimate, also earlier by Cao and Koiso, shows that the change of the metric is bounded. If a subsequence of the metrics converges to a metric of a finite distance, even a singular metric, then by the first and the second geodesic stability principles in [6] (see also [15] for updated version), it has a finite Mabuchi functional and the sequence of the Kähler metrics converges to a Kähler Einstein metric. If there is not any finite limit, by the first stability principle there is a unique geodesic connecting g_0 and g_t. We call it γ_t. Extend the γ_t to be a maximal geodesic ray. Let t_i be a series of t such that γ_{t_i} converges to a possibly singular maximal geodesic ray γ. Then the stability conditon in the third geodesic stability principle (or our conjecture 2 here) would imply that the Mabuchi functional would increase along γ eventually, a contradiction.

6 Global Parallel Vector Fields

Moreover, by [5] the potential functions of the vector fields of the Lie algebra $\mathcal{K}$ of K is parallel on the moduli space of the invariant metrics. In particular, the vector potential functions of elements of the center $\mathcal{T}$ of the Lie algebra $\mathcal{K}$ of K also induce families of invariant Kähler metrics. Notice that no all the potential functions of Lie algebra in the tangent space of the moduli space of the *invariant* Kähler metrics—they are in the tangent space the bigger (nonnecessary invariant) moduli space. And it is natural for us to restrict our attention to the smaller moduli space. Therefore, these functions from the center are in the tangent space of the Mabuchi moduli space of the invariant Kähler metrics and are parallel. Thus, there might be some global parallel vector fields on $\mathcal{M}_K$. We denote the set of all the global parallel vector fields by $\mathcal{P}$. Then by our Theorem 1 and 2, we have:

Theorem 4. *Let $f_1, \cdots, f_k \in \mathcal{P}$, then for any analytic function $F(x_1, \cdots, x_k)$, the composition function $F(f_1, \cdots, f_k)$ is parallel. Moreover, the set of all these parallel vector fields is closed under the Poisson bracket.*

For these parallel vector fields we can obtain the geodesics by the method of section 3. However, as what happened for the toric varieties in [6], unlike those in $\mathcal{T}$ the maximal geodesic might be incomplete and might actually be finite.

Moreover, for any $f_1, f_2 \in \mathcal{P}$, starting from any invariant metric, one should get a 2-dimensional net of geodesics P_2. If we can prove this, then P_2 is a geodesic submanifold of $\mathcal{M}_K$. Since both f_1, f_2 are parallel on P_2, we expect that the curvature is zero. That is:

Theorem 5. *For any g, $\{g, f\} = 0$ if $f \in \mathcal{P}$. In particular, the Poisson structure on $\mathcal{P}$ is trivial, i.e., $\mathcal{P}$ is abelian.*

Proof: Let ω be any equivariant Kähler metric, and g be any function in the equivariant tangent space, $f \in \mathcal{P}$, $\omega(s,t)$ be a 2-parameter family of Kähler metrics such that $\omega(0,0) = \omega$ and $\omega(s,0)$ has tangent g, $\omega(0,t)$ has tangent f. Let $f \in \mathcal{P}$, then $R(g,f)f = (\nabla_g \nabla_f - \nabla_f \nabla_g) f = 0$. Since $R(g,f)f = -\frac{1}{4}\{\{g,f\},f\}$ (see [2 p. 195]) we get $\{\{g,f\},f\} = 0$ and hence $(R(g,f)f, g) = 0$, i.e., $\int \{g,f\}^2 \omega^n = 0$. Therefore, $\{g,f\} = 0$ for any tangent vector g.

In particular, if $g \in \mathcal{P}$ also, we have that $\mathcal{P}$ is abelian.

Now we see that $\mathcal{P}$ is in a way similar to $\mathcal{T}$. But some time $\mathcal{P}$ is bigger than the algebra of functions generated by $\mathcal{T}$. For the examples in [6,7], $\mathcal{T} = 0$ but $\mathcal{P}$ is generated by the function U there. Let $k = \sup_{m \in M} \dim d\mathcal{P}|_m$, then at a generic point $m \in M$, we can have k functions $f_i \in \mathcal{P}$ which are independent. We call $f_1, \cdots, f_k$ a parallel coordinate at m with regarding M as a symplectic manifold.

Theorem 6. *The L^2 closure of the infinite dimensional Lie algebra of equivariant functions $T_K = T_{\mathcal{M}_K}$ splits as an orthogonal direct sum of the closure of $\mathcal{P}$ and the closure of another infinite dimensional Lie subalgebra $\{T_K, T_K\}$.*

Proof: We want to prove that $f \in T_K$ is perpendicular to $\{T_K, T_K\}$ if and only if $f \in \mathcal{P}$.

$$\int_M f\{g,h\}\omega^n = 0$$

for any $g, h \in T_K$ if and only if $\int_M \{h, f\} g\omega^n = 0$ for any $g \in T_K$ with any given $h \in T_K$, that is, $\{h, f\} = 0$ for any $h \in T_K$. In the proof of our Theorem 6 we already see that if $f \in \mathcal{P}$ then $\{h, f\} = 0$. We can also prove the other direction. If $\{h, f\} = 0$ for any $h \in T_K$, then $R(g, h)f = 0$. We want to prove that f can be globally defined as a parallel vector field on $\mathcal{M}_K$. We notice that the Lie algebra $\{T_K, T_K\}$ is invariant under the parallel transformation and hence so is the function orthogonal to $\{T_K, T_K\}$. Let $\omega(s, t)$ be a family of curves in $\mathcal{M}_K$ and $\omega(s, 0) = \omega_0$, $\omega(s, 1) = \omega_1$. Let $f(s, t)$ be family of functions on M such that $f(s, 0) = f$ and $f_s(t)$ is parallel along the curves $C_s(t) = \omega(s, t)$. Then $f(s, t)$ are orthogonal to $\{T_K, T_K\}$. Therefore, $R(g, h)f(s, t) = 0$ for any $g, h \in T_K$. Now we want to prove that $f(s, 1)$ does not depend on s. Since f is parallel along each ω_s we have $\nabla_t f = 0$, then

$$\nabla_t \nabla_s f = (\nabla_t \nabla_s - \nabla_s \nabla_t) f = R(f_t, f_s) f = 0,$$

that is, $\nabla_s f$ is also parallel along each ω_t. But $\nabla_s f = 0$ at $t = 0$, we have $\nabla_s f = 0$ always. In particular, $\nabla_s f = 0$ at $t = 1$, i.e., $f(s, 1)$ is independent of s. Therefore, f can be extended as a global parallel vector field, i.e., $f \in \mathcal{P}$.

Now, for any $f \in T_K$, (f, g) $g \in \mathcal{P}$ definds a functional on the closure of $\mathcal{P}$. But the property of a Hilbert space we have a f_1 in the closure of $\mathcal{P}$ such that $(f_1, g) = (f, g)$ for any g in the closure of $\mathcal{P}$. Let $f_2 = f - f_1$, then f_2 is in the closure of $\{T_K, T_K\}$ and $f = f_1 + f_2$.

Q. E. D.

For those maximal geodesic rays generated by the functions in $\mathcal{P}$ we already applied our generalized Futaki invariants in [6]. If M is a compact spherical almost homogeneous manifold under an algebraic reductive group, i.e., the K invariant functions are generated by an abelian ideal, then $T_K = \mathcal{P}$, e.g., when M is cohomogeneity one. In that case, any tangent vector at a point can be extended to a global parallel vector field. In these cases, we can apply Mabuchi's definition to define the Futaki invariants for the maximal geodesic rays. The existance of the Kähler-Einstein metrics should imply the positivity of the Futaki invariant for those not coming from any holomorphic vector field. Even for the toric manifolds, this give new obstructions since the function might be a function of the potential functions of holomorphic vector fields but itself is not a potential function of any holomorphic vector field. That is exactly what we did in [6]. Also, for these manifolds, the invariant Mabuchi moduli space is flat. In general, even not all the tangent vectors in the tangent space of the invariant Mabuchi moduli space come from parallel vector fields, we can use the parallel ones to get new obstructions just like the original futaki obstruction. We call the related stability *parallel stability*, then we expect that for the cohomogeneity one and the spherical cases including the toric manifolds the existence is the same as the parallel stability.

We can actually expect that the direct sum can be realized as C^∞ functions.

Conjecture 4. $T_K = \mathcal{P} + \{T_K, T_K\}$.

While the conjectures 1, 2, 3 (2') might be very difficult to be proved, conjecture 4 would be easier. We should try to prove our conjecture 4 in a near future.

Remark 2. When the group G is Hamiltonian (This is true whenever the manifold is simply connected, e.g., when M is Fano), i.e., every element in the Lie algebra $\mathcal{G}$ of G corresponds to a given function, from (3) and (4) of [8 p.3362] one could easily to see that $\mathcal{P}$ is generated by the functions related to the Lie algebra of the torus which is the complement of H in J. We notice here that G is compact and J is locally a direct product of a torus and H. And the functions on the symplectic reduction produces the major part of $\{T_K, T_K\}$. To make the picture clearer to the reader, let $\Phi : M \to \mathcal{G}^*$ be the moment map with $\Phi(m)(g) = g(m)$. On each generic orbit G/H, we have the moment map $\Psi = \Phi|_{G/H} : G/H \to G/J$. The codimension of $G/J = \Psi(G/H)$ is the same as the dimension of the torus J/H. Therefore $\Psi(M)//G$ has a dimension of that of J/H. $\mathcal{P}$ is the pullback of the functions on $\Psi(M)//G$ and the symplectic reduction M_G gives the major part of $\{T_K, T_K\}$. That is, locally T_K is the functions on $\Psi(M)//G \times M_G$. This basically gives a proof for the Conjecture 4.

Remark 3. In the case of [6], one could easily calculate that θ is the square of the norm of $\Psi(m)$ under the standard product metric of $\mathbf{C}P^n \times \mathbf{C}P^n$. Therefore, the name *phase angle* (or square phase angle) is justified. Similarly, U is the square of the norm of $\Psi(m)$ for the corresponding metric.

Reference

[1] J. Cheeger & D. G. Ebin: Comparison Theorems in Riemannian Geometry, North-Holand Publishing Company 1975.

[2] X. X. Chen: *Space of Kähler metrics,* J. Diff. Geom. **56** (2000), 189–234.

[3] S. K. Donaldson: *Symmetric spaces, Kähler geometry and hamiltonian synamics,* AMS Transl. **196(2)** (1999), 13–33.

[4] W. Ding & G. Tian: *Kähler-Einstein metrics and the generalized Futaki invariants,* Invent. Math. **110** (1992), 315–335.

[5] D. Guan: *On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundles,* Math. Research Letters **6** (1999), 547–555.

[6] D. Guan: *Existence of extremal metrics on almost homogeneous manifolds of cohomogeneity one—III,* Intern'l J. of Math. **14** (2003), 259–287.

[7] D. Guan: *Jacobi fields and geodesic stability,* in preparation.

[8] D. Guan: *Compact symplectic manifolds with Lie group symmetries,* Transaction of AMS **257** (2005), 3358–3373.

[9] Z. Guan: *Curvature on Hermitian symmmetric spaces,* Acta Math. Sinica. New Series **4** (1988), 270–283.

[10] D. Guan: *Existence of extremal metrics on almost homogeneous manifolds of cohomogeneity one—II,* J. Geom. Anal. **12** (2002), 63–79.

[11] D. Guan: *Extremal-solitons and exponential C^∞ convergence of Calabi flow on certain $\mathbf{C}P^1$ bundles,* Pacific J. of Math. **233** (2007), 91–124.

[12] D. Guan: *Affine almost homogeneous manifolds of cohomogeneity one,* Central European J. of Math. **7** (2009), 84–123.

[13] D. Guan: *m-extreme metrics and m-Calabi flows,* in preparation 2012.

[14] D. Guan: *Positive lemmas, generalized extremal-solitons and second order linear differential equations,* Advancement and Developement in Mathematical science **1** (2012), 13–32.

[15] D. Guan: *Type I Compact Almost Homogeneous Manifolds of Cohomogeneity One—III,* Pacific J. Math. **261** (2013), 369–388.

[16] T. Mabuchi: *Some symplectic geometry on compact Kähler manifolds I,* Osaka J. Math. **24** (1987), 227–252.

[17] J. Moser: *On the volume elements on a manifold,* Trans. Amer. Math. Soc. **120** (1965), 286–294.

[18] S. Semmes: *Complex Monge-Ampére and symplectic manifolds,* Amer. J. Math. **114** (1991), 495–550.

Department of Mathematics
University of California at Riverside
Riverside, CA 92521 U. S. A.