Type II Compact Almost Homogeneous Manifolds of Cohomogeneity One—II

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Abstract: In this paper we start the program of the existence of the smooth equivariant geodesics in the equivariant Mabuchi moduli space of Kähler metrics on type II cohomogeneity one compact Kähler manifold. In this paper, we deal with the manifolds M_n obtained by blowing up the diagonal of the product of two copies of a $\mathbf{C}P^n$.

1 Introduction

Motivated with the Hilbert scheme construction in [Gu4], we consider the manifolds $N_n = M_n/S_2$ in [Gu8], where M_n is constructed by blowing up the diagonal of the product of two copies of a complex projective space and S_2 is the symmetry group of two elements.

The manifolds M_n are Fano. In [GC], we proved the existence on M_n of the Kähler-Einstein metric in the Ricci class by considering the symmetric Ricci curvature equation and in [Gu7] we dealt with the general Kähler classes on M_n by the symmetric scalar curvature equations as in [Gu2].

In [Gu8] we deal with a similar situation for N_n. We adapt the method in [Gu5] to our situation to solve the uniqueness and then to obtain a clearer picture of the existence. We proved that the existence of extremal metric is the same as the negativity of certain integral, i.e., the positivity of the generalized Futaki invariants (see also [Gu6]). This is a demonstration of the relation between the existence and the stability.

The relation between the existence and the stability can also be seen in [Gu2] where we obtain a solution, up to the *positivity* of φ, for *any Kähler class* on the projective line bundle with the conditions therein (except the positivity of the Ricci curvature). The positivity of φ is depended on the negativity of the partial integrals $\int_a^D \Phi'(U)dU$ with $D > a > -d$ (we call the Kähler class being stable if this is true by comparing with the condition (9) in [Gu8]), or equivalently, the negativity of $\int_a^{-d} \Phi'(U)dU$. Hence, we studied the stability of [Gu2] together with the situation of N_n. We were able to prove the equivalence between the stability and the *geodesic stability* in [Gu8].

To test the geodesic stability further, we need first to find the smooth geodesics.

We shall deal with this for M_n in this paper and for general type II of almost homogeneous manifolds of cohomogeneity one in [Gu9].

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2 Preliminary

Let M_n be the blow-up of $\mathbf{C}P^n \times \mathbf{C}P^n$ along the diagonal. Here we recall some formulas in [GC] and [Gu7] on the calculation of Kähler metrics on M_n.

To calculate the Kähler metric ω on M_n, we consider the pushdown of the metric by the map $p: \ M_n \to P_n = \mathbf{C}P^n \times \mathbf{C}P^n$. Then

$$p_*\omega = a\omega_1 + b\omega_2 + \partial\bar{\partial}F, \tag{1}$$

where ω_i $(i = 1, 2)$ are the standard metrics on the first and the second copy of $\mathbf{C}P^n$ and F is a function with some singularities on the diagonal.

Now the automorphism group of M_n is $PSL(n+1, \mathbf{C})$, and the maximal connected compact subgroup $K = PSU(n+1)$ has real hypersurface orbits. All K invariant functions are functions of $\theta = \frac{|(z,w)|^2}{|zw|^2}$ where z and w are the homogeneous coordinates of the $\mathbf{C}P^n$'s and $|zw| = |z||w|$. The diagonal corresponds to $\theta = 1$. The orbit with $\theta = 0$ is a very special orbit. If the metric is K invariant, then F is a function of θ.

Recall that $\omega_1 = \partial\bar{\partial}\log|z|^2$ and $\omega_2 = \partial\bar{\partial}\log|w|^2$. We also have that $\partial\bar{\partial}F = \partial(F'\bar{\partial}\theta) = \partial(\theta F'\bar{\partial}\log\theta)$. If we let $f = \theta F'$, then $f(0) = 0$ and

$$p_*\omega = a\omega_1 + b\omega_2 + \theta f'\partial\log\theta \wedge \bar{\partial}\log\theta + f\partial\bar{\partial}\log\theta. \tag{2}$$

Because of the action of the isometric group we only need to calculate $p_*\omega$ at points with $z_0 = 1,\ w_0 = 1$ and $z_i = w_j = 0$ for $i \neq 0, 1,\ j \neq 0$. This choice covers a dense set of the values of θ. By calculation we obtain:

$$\partial\bar{\partial}\log|z|^2 = \theta(dz \wedge d\bar{z} - (1-\theta)dz_1 \wedge d\bar{z}_1)$$

$$\partial\bar{\partial}\log|w|^2 = dw \wedge d\bar{w}$$

$$\partial_z\partial_{\bar{w}}\log\theta = dz \wedge d\bar{w} \quad \partial_w\partial_{\bar{z}}\log\theta = dw \wedge d\bar{z}$$

$$\partial\log\theta = \bar{z}_1(dw_1 - \theta dz_1)$$

$$\begin{aligned}
p_*\omega &= (a - f(\theta))\partial\bar{\partial}\log|z|^2 + (b - f(\theta))\partial\bar{\partial}\log|w|^2 \\
&+ f(\theta)(\partial_z\partial_{\bar{w}} + \partial_w\partial_{\bar{z}})\log\theta + \theta f'(\theta)\partial\log\theta \wedge \bar{\partial}\log\theta \\
&= (a - f(\theta))\theta(\theta dz_1 \wedge d\bar{z}_1 + \sum_{i>1} dz_i \wedge d\bar{z}_i) + (b - f(\theta))dw \wedge d\bar{w} \\
&+ f(\theta)(dz \wedge d\bar{w} + dw \wedge d\bar{z}) \\
&+ (1-\theta)f'(\theta)(\theta dz_1 - dw_1) \wedge (\theta d\bar{z}_1 - d\bar{w}_1).
\end{aligned}$$

We observe that the complex 2-dimensional subspaces V_i generated by $\frac{\partial}{\partial z_i}$, $\frac{\partial}{\partial w_i}$ are orthogonal to each other for different i with $1 \leq i \leq n$. If we regard the tangent space as the complex vector space generated by the vector fields corresponding to the elements of the Lie algebra of K, then the semisimple part of the centralizer of the isotropy group has these V_i's as invariant subspaces of the tangent space. To calculate the volume form, we only need to calculate the determinant τ_i of the restricted metric for each V_i and compare them with the corresponding determinants on the standard Kähler-Einstein metric on $\mathbf{C}P^n \times \mathbf{C}P^n$.

We notice that $\tau_i,\ i \geq 2$ are all equal to

$$\begin{vmatrix} \theta(a - f(\theta)) & f(\theta) \\ f(\theta) & b - f(\theta) \end{vmatrix} = \theta(a - f(\theta))(b - f(\theta)) - f^2(\theta).$$

If ω comes from the pullback of the standard metric, then $a = b = n+1$ and the determinants of the standard metric on each V_i are $\tau_i^0 = \frac{(n+1)^2}{|zw|^2}$ for $i > 1$. So $\frac{\tau_i}{\tau_i^0} = \frac{|zw|^2}{(n+1)^2}(\theta(a - f(\theta))(b - f(\theta)) - f^2(\theta))$ must be a function of θ, we have

$$\tau_i = \frac{1}{|zw|^2} A$$

with $i > 1$, where $A = (a - f(\theta))(b - f(\theta)) - \theta^{-1} f^2(\theta)$.

For $i = 1$ we have:

$$\begin{aligned}\tau_1 = &\begin{vmatrix} \theta^2(a - f(\theta) + (1-\theta)f'(\theta)) & f(\theta) - \theta(1-\theta)f'(\theta) \\ f(\theta) - \theta(1-\theta)f'(\theta) & b - f(\theta) + (1-\theta)f'(\theta) \end{vmatrix} \\ &= \theta^2[(a - f(\theta) + (1-\theta)f'(\theta))(b - f(\theta) + (1-\theta)f'(\theta)) \\ &-(\theta^{-1}f(\theta) - (1-\theta)f'(\theta))^2].\end{aligned}$$

In the same way, we observe that $\tau_1^0 = \frac{(n+1)^2}{|zw|^4}$ and hence

$$\tau_1 = \frac{1}{|zw|^4} B$$

with

$$\begin{aligned} B &= (a - f(\theta) + (1-\theta)f'(\theta))(b - f(\theta) + (1-\theta)f'(\theta)) \\ &- \quad (\theta^{-1}f(\theta) - (1-\theta)f'(\theta))^2. \end{aligned}$$

We have the following result in [GC]:

Proposition 1. *The volume form is*

$$\frac{1}{|zw|^{2n+2}} A^{n-1} B dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \wedge dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n.$$

Now we try to describe the conditions for $f(\theta)$ such that the 2-form defined by $f(\theta)$ is a Kähler form at any point outside the diagonal. We have the following (see [GC]):

Proposition 2. $(A(1-\theta))' = -B$.

We let $C = A(1-\theta)$ and

$$dV = dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \wedge dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n.$$

Then the volume form is

$$\frac{C^{n-1}BdV}{|zw|^{2n+2}(1-\theta)^{n-1}} = \Phi dV.$$

We will see more geometrical meaning of this formula later on.

If ω is positive, then $\tau_1 > 0$. That is, $B > 0$. Proposition 2 says that $A(1-\theta)$ is decreasing. We observe that A is positive if and only if $\lim_{\theta\to 1} A(1-\theta) \geq 0$. Let $\theta = 0$. We observe that $a, b > 0$ since ω is positive on V_i with $i > 1$. On the other hand, if $a, b > 0$ and $A > 0$, then ω is positive on V_i with $i > 1$ at $\theta = 0$; and ω is always positive on V_i $(i > 1)$ by the continuity (otherwise ω has zero direction on some V_i $(i > 1)$, but this contradicts $A > 0$). In the same way, we observe that ω is positive on V_1 if and only if $a + f'(0), b + f'(0) > 0$ and $B > 0$. We have (see [GC]):

Proposition 3. *ω is a Kähler metric outside the diagonal of M_n if and only if* (1) $B > 0$, (2) $\lim_{\theta\to 1} C \geq 0$ *and* $\min(a, b, a + f'(0), b + f'(0)) \geq 0$.

We consider how the Kähler metric ω in Proposition 3 extend to a metric on the diagonal. As above, we only consider the points at which $z_0 = w_0 = 1$ and $z_i = w_j = 0$ for $i > 1, j > 0$.

Proposition 4. *Let ω be a Kähler metric as in Proposition* 3*. Then it is a metric on the diagonal of M_n (resp. N_n) if and only if* $\lim_{\theta\to 1} f(\theta)(1-\theta) < 0$, $\lim_{\theta\to 1} B > 0$ *(resp.* $= 0$*), and* $\lim_{\theta\to 1} C > 0$.

To calculate the Ricci curvature of ω on M_n, we notice that $C(0) = A(0) = ab$. We let

$$\begin{aligned}
U = ab - C & \\
= & \; ab - (1-\theta)((a - f(\theta))(b - f(\theta)) - \theta^{-1} f^2(\theta)) \\
= & \; \theta ab + (a+b) f(\theta)(1-\theta) + \theta^{-1} f^2(\theta)(1-\theta)^2 \\
= & \; \theta[ab - \frac{(a+b)^2}{4} + (\frac{a+b}{2} + \theta^{-1} f(\theta)(1-\theta))^2] \\
= & \; \theta[-\frac{(a-b)^2}{4} + (\frac{a+b}{2} + \theta^{-1} f(\theta)(1-\theta))^2] \\
= & \; -\frac{(a-b)^2}{4}\theta + g^2,
\end{aligned}$$

where $g = \theta^{\frac{1}{2}}(\frac{a+b}{2} + \theta^{-1}(1-\theta) f)$, then $\Phi = \frac{(ab-U)^{n-1}U'}{|zw|^{2n+2}(1-\theta)^{n-1}}$ and $U(0) = 0$. Therefore, the Ricci curvature is $-\partial\bar{\partial} \log \Phi$. We also let

$$V = \theta(\log \Phi |zw|^{2n+2})',$$

then $V(0)=0$ and

$$\text{Ricci}(\omega)=(n+1)\omega_1+(n+1)\omega_2-\theta V'\partial\log\theta\wedge\bar\partial\log\theta+V\partial\bar\partial\log\theta. \qquad (3)$$

Again, we only need to calculate the Ricci curvature at points with $z_0=1,\ w_0=1$ and $z_i=w_j=0$ for $i\neq 0,1,\ j\neq 0$. By calculation we obtain:

$$\begin{aligned}
\text{Ricci}(\omega) &= ((n+1)+V)\partial\bar\partial\log|z|^2+((n+1)+V)\partial\bar\partial\log|w|^2\\
&- V(\partial_z\partial_{\bar w}+\partial_w\partial_{\bar z})\log\theta-\theta V'\partial\log\theta\wedge\bar\partial\log\theta\\
&= (n+1+V)(\frac{dz\wedge d\bar z}{|z|^2}-\frac{|z_1|^2dz_1\wedge d\bar z_1}{|z|^4})\\
&+ (n+1+V)\frac{dw\wedge d\bar w}{|w|^2}-V(\partial_z(\frac{zd\bar w}{(z,w)})+\partial_w(\frac{wd\bar z}{(w,z)}))\\
&- \theta V'(-\frac{\bar z_1dz_1}{|z|^2}+\frac{\bar z_1dw_1}{(w,z)})\wedge(-\frac{z_1d\bar z_1}{|z|^2}+\frac{z_1d\bar w_1}{(z,w)})\\
&= (n+1+V)(\frac{dz_1\wedge d\bar z_1}{|z|^4}+\sum_{i>1}\frac{dz_i\wedge d\bar z_i}{|z|^2})\\
&+ (n+1+V)dw\wedge d\bar w-V(dz\wedge d\bar w+dw\wedge d\bar z)\\
&- \theta V'|z_1|^2(\frac{dz_1}{|z|^2}-dw_1)\wedge(\frac{d\bar z_1}{|z|^2}-d\bar w_1).
\end{aligned}$$

We observe that the complex 2-dimensional subspaces V_i generated by $\frac{\partial}{\partial z_i},\ \frac{\partial}{\partial w_i}$ are orthogonal to each other for different i with $1\le i\le n$ with respecting both the metric and the Ricci curvature. To calculate the scalar curvature, we only need to calculate the restricted $\sigma_i=\frac{1}{2}\text{Ricci}(\omega)\wedge\omega|_{V_i}$ for each V_i and compare them with the corresponding items on the standard Kähler-Einstein metric on $\mathbf{C}P^n\times\mathbf{C}P^n$.

We notice that $\sigma_i,\ i\geq 2$ are all equal to

$$\begin{vmatrix} \frac{\frac{a+b}{2}-f(\theta)}{|z|^2} & f(\theta)\\ -V & n+1+V\end{vmatrix}=\frac{1}{|z|^2}(\frac{a+b}{2}-f(\theta))(n+1+V)+f(\theta)V.$$

If ω comes from the pullback of the standard metric, then $a=b=n+1$ and $\sigma_i^0=\frac{(n+1)^2}{|zw|^2}$ for $i>1$. So $\frac{\sigma_i}{\sigma_i^0}=\frac{|zw|^2}{(n+1)^2}(\frac{1}{|z|^2}(\frac{a+b}{2}-f(\theta))(n+1+V)+f(\theta)V)$ must be a function of θ, and we have

$$\sigma_i=\frac{1}{|zw|^2}A_1$$

with $i > 1$, where $A_1 = (\frac{a+b}{2} - f(\theta))(n+1+V) + \theta^{-1} f(\theta) V$.

For $i = 1$ we have:

$$\sigma_1 = \begin{vmatrix} \frac{\frac{a+b}{2} - f(\theta) + |z_1|^2 \theta f'(\theta)}{|z|^4} & \frac{f(\theta)|z|^2 - |z_1|^2 \theta f'(\theta)}{|z|^2} \\ \frac{-V|z|^2 + |z_1|^2 \theta V'}{|z|^2} & n+1+V-\theta V'|z_1|^2 \end{vmatrix}$$
$$= \frac{1}{|z|^4}((\frac{a+b}{2} - f(\theta) + (1-\theta) f'(\theta))(n+1+V-(1-\theta)V')$$
$$+(\theta^{-1} f(\theta) - (1-\theta) f'(\theta))(\theta^{-1} V - (1-\theta) V')).$$

In the same way, we observe that $\sigma_1^0 = \frac{(n+1)^2}{|zw|^4}$ and hence

$$\sigma_1 = \frac{1}{|zw|^4} B_1$$

with $B_1 = (\frac{a+b}{2} - f(\theta) + (1-\theta) f'(\theta))(n+1+V-(1-\theta)V') + (\theta^{-1} f(\theta) - (1-\theta) f'(\theta))(\theta^{-1} V - (1-\theta) V')$.

We have the following result which is similar to Proposition 2 (see [Gu7]):

Proposition 5. *$(A_1(1-\theta))' = -B_1$.*

Proof: $A_1' = -\theta^{-2} f(\theta) V + V'(\frac{a+b}{2} - f(\theta) + \theta^{-1} f(\theta)) - f'(\theta)(n+1+V-\theta^{-1} V)$. Therefore, we have

$$\begin{aligned} (A_1(1-\theta))' &= A_1'(1-\theta) - A_1 \\ &= -\theta^{-2} f(\theta) V - (\frac{a+b}{2} - f(\theta))(n+1+V) \\ &- (1-\theta)(-V'(\frac{a+b}{2} - f(\theta) + \theta^{-1} f(\theta)) + f'(\theta)(n+1+V-\theta^{-1} V)) \\ &= -B_1. \end{aligned}$$

Q. E. D.

We let $C_1 = A_1(1-\theta)$. Then we have (see [Gu7]):

Proposition 6. *The scalar curvature is $2n \frac{(C_1 C^{n-1})'}{(C^n)'}$.*

We shall determine the equation for metrics with constant scalar curvatures. By Proposition 6, this is the same as

$$\frac{(C_1 C^{n-1})'}{(C^n)'} = R_0.$$

Let

$$\lim_{\theta \to 1} (1-\theta) f(\theta) = -c,$$

then by Proposition 4 we have $c > 0$.

$$\begin{aligned}
U(1) &= ab - C(1) \\
&= ab - \lim_{\theta \to 1}(-a-b)f(\theta) - \theta^{-1}f^2(\theta)(1-\theta))(1-\theta) \\
&= ab - (a+b)c + c^2 = (a-c)(b-c) > 0
\end{aligned}$$

we have $a, b > c$ since if we increase only a without changing other quantities we should still get a metric. We have:

Proposition 8. *The Kähler classes on M_n (resp. N_n) are one to one to with the elements in the set $\Delta = \{(a,b,c)|_{a,b>c>0}\}$ (resp. $(a,b)|_{a>c>0}$). For any Kähler metric on N_n, we have $U' = (1-\theta)u(\theta)$ with $u(1) \neq 0$.*

We can regard M_n as a K-equivariant fiber bundle over $\mathbf{C}P^n$. To calculate the equivariant integrals we only need to calculate them on an open dense set of the fiber. Therefore, the total volume is

$$\begin{aligned}
\int_{M_n} \Phi dV &= -C(n) \int_0^{+\infty} \frac{((ab-U)^n)'}{n(1+r^2)^{n+1}(1-\theta)^{n-1}} \frac{r^{2n-1}}{(2n-1)!} S(2n-1)dr \\
&= A(n) \int_1^0 ((ab-U)^n)' d\theta = A(n)((ab)^n - (ab-(a-c)(b-c))^n) \\
&= A(n)((ab)^n - (c(a+b-c))^n,
\end{aligned}$$

where $C(n)$ (resp. $S(2n-1)$) is the volume of $\mathbf{C}P^n$ (resp. the sphere S_{2n-1}) and

$$\theta = \frac{1}{1+r^2}, \ A(n) = \frac{C(n)S(2n-1)}{(2n)!}.$$

In the same way, we have that the total scalar curvature on M_n is

$$\begin{aligned}
&A(n)(C_1(ab-U)^{n-1})|_1^0 \\
&= A(n)(\frac{n+1}{2}(a+b)(ab)^{n-1} - C_1(1)(c(a+b-c))^{n-1}).
\end{aligned}$$

3 Some general results on almost homogeneous manifolds

Theorem 1. *Let K be a compact Lie subgroup of the automorphism group of a compact Kähler manifold M such that $K^{\mathbf{C}}$ has an open orbit. Then the Mabuchi moduli space of the K equivariant Kähler metrics is flat.*

The flatness comes from the fact that the curvature of the Mabuchi moduli space is determined by the Poisson brackets of two functions. Since the the complexification Lie group $K^{\mathbf{C}}$ has an open orbit, say U, at each poit in U the vector fields related to K generates the whole holomorphic tangent space, the $\mathcal{K}^{\perp}$ to the metric is perpendicular to itself with respect to the symplectic structure.

Theorem 2. *Assume that the Mabuchi moduli space is flat. Let $\varphi(s,t)$ be a two parameters family of the Kähler metrics in the Mabuchi moduli space such that $\varphi(s,0)=\varphi_1, \varphi(s,1)=\varphi_2$, ϕ_0 be a function on M. Then the parallel transformation $\phi_1(s)$ of ϕ_0 along the curve $\varphi_s=\varphi(s,t)$ are independent of s.*

Proof: Along each curve φ_s, ϕ is parallel, therefore, $\nabla_t \phi = 0$. We have $\nabla_t \nabla_s \phi = \nabla_s \nabla_t \phi = 0$, i.e., $\nabla_s \phi$ is also parallel. But $\nabla_s \phi = 0$ at $t=0$, we have $\nabla_s \phi = 0$ at $t=1$ also.

Theorem 3. *Let M be a Kähler manifold, $\varphi(t)$ be a path in the Mabuchi moduli space of the Kähler metrics. If ϕ_0 is a function on M, we can regard ϕ_0 as a vector at φ_0, then the parallel transformation of ϕ_0 along $\varphi(t)$ exists.*

Proof: We need to solve the equation

$$\dot{\phi} - \frac{1}{2}(d\phi, d\dot{\varphi})_{\varphi(t)} = 0,$$

which is $\dot{\phi} - g^{i\bar{j}}\dot{\varphi}_i \phi_{\bar{j}} = 0$.

Here we apply a method from [Ga p.18]. Regarding the graph of $v = \phi$ as a hypersurface we have the normal vector $(-\dot{\phi}, -\phi_{\bar{j}}, 1)$. If $\alpha(s) = (t(s), z_j(s), v(s))$ is a curve such that

$$\alpha'(s) \cdot (-\dot{\phi}, -\phi_{\bar{j}}, 1) = 0,$$

then $\alpha(s)$ is a curve on the graph if and only if the initial point is on the graph.

In particular, if $\alpha'(s) = (1, -g_t^{i\bar{j}}\varphi_i, 0)$, $\alpha(s)$ is a curve on the graph if $\alpha(0)$ is. We call the curves with this condition the characteristic curves of this equation. In this case we have $v'(s)=0, t'(s)=1, \frac{d\bar{z}_j}{ds} = -g^{i\bar{j}}\dot{\varphi}_i$.

Therefore, ϕ is constant along these characteristic curves, $t=s$, $\frac{d\bar{z}_j}{dt} = -g^{i\bar{j}}\dot{\varphi}_i$.

Now the ordinary differential equations have good enough conditions such that these curves exist and unique. Therefore, the parallel transformation exists.

Q. E. D.

4 The Existence of the Geodesics

In [GC] we proved that there is a unique Kähler-Einstein metric in the Ricci class on M_n. By the result of [LS], we see in [Gu7] that there is an open subset Ω_0 such that for any Kähler class in Ω_0 there is a unique Kähler metric of constant scalar curvature on M_n. Although this argument does not work for N_n with a general n, we applied a modified method of [Gu5] and proved the existence of the geodesics in the Mabuchi moduli space of Kähler metrics. We can not apply our method of Legendre transformation as in [Gu5,6] to our case when $a \neq b$. Here we will apply our Theorem 2 and 3 in section 3 and then integrate the parallel vector field to get the geodesics.

The parallel transformation equation is

$$\begin{aligned}\dot{\phi} &= \frac{1}{2}(d\phi, d\dot{F}) \\ &= \frac{\dot{f}}{2\theta}(d\phi, d\theta) \\ &= \theta \dot{f}\phi'(\partial \log\theta, \bar{\partial}\log\theta) \\ &= 2\dot{f}\phi'(1-\theta)\theta^{-\frac{1}{2}} g B^{-1} \\ &= 2g\dot{g}\phi' B^{-1} = \dot{U}\phi'(U')^{-1}.\end{aligned}$$

But if we regard ϕ as a function of U and t, then

$$\dot{\phi}(\theta, t) = \dot{\phi}(U, t) + \frac{\partial\phi}{\partial U}\dot{U} = \dot{\phi}(U,t) + \phi'\frac{\dot{U}}{U'}.$$

Therefore, we have $\dot{\phi}(U,t) = 0$, i.e., ϕ only depends on U.

The equation for the geodesic is:

$$\dot{F} = \phi(U)$$

then $\phi_s = \dot{g}$. where $g = \theta^{\frac{1}{2}}(\frac{a+b}{2} + \theta^{-1}(1-\theta)f) = (U + \frac{(a-b)^2}{4}\theta)^{\frac{1}{2}}$ and $s = \ln\frac{1+\theta^{\frac{1}{2}}}{1-\theta^{\frac{1}{2}}}$. In particular,

$$\dot{U} = 2g\dot{g} = 2(U + \frac{(a-b)^2}{4}\theta)^{\frac{1}{2}}\phi_U U_s.$$

Regard $z = U(s,t)$ as a graph, $(-U_s, -\dot{U}, 1)$ is the normal vector. The characteristic curve is $U =$constant, $t = t$,

$$\frac{ds}{dt} = -2(U + \frac{(a-b)^2}{4}\theta)^{\frac{1}{2}}\phi_U.$$

By integration we get that if

$$D = \sqrt{\frac{(a-b)^2}{4}\theta + U} + \sqrt{\theta(\frac{(a-b)^2}{4} + U)}$$

then

$$D^2 = C_0 U(1-\theta)\exp(-2t\phi_U\sqrt{\frac{(a-b)^2}{4} + U}).$$

Let $t = 0$ we obtain

$$C_0 = (U(1-\theta_0))^{-1}(\sqrt{\frac{(a-b)^2}{4}\theta_0 + U} + \sqrt{\theta_0(\frac{(a-b)^2}{4} + U)})^2 = \frac{D^2(\theta_0, U)}{U(1-\theta_0)}$$

if $\theta_0 \neq 1$. That is $\theta = \frac{(x-U)^2}{(x+U)^2+(a-b)^2 x}$ where $x = C_0 U e^{kt}$,

$$k = -2\phi_U\sqrt{\frac{(a-b)^2}{4} + U}.$$

We notice that

$$C_0 > \frac{(1+\theta_0^{\frac{1}{2}})^2}{1-\theta_0} = \frac{1+\theta_0^{\frac{1}{2}}}{1-\theta_0^{\frac{1}{2}}} > 1.$$

Since both x and $U > 0$, it is not difficult to see that $0 \leq \theta < 1$ always hold if $\theta_0 < 1$. For $\theta_0 = 0$, we have

$$C_0(0) = \frac{(\sqrt{\frac{(a-b)^2}{4} + U'(0)} + \frac{|a-b|}{2})^2}{U'(0)}.$$

If $a = b$. then $D = U^{\frac{1}{2}}(1+\theta_0^{\frac{1}{2}})$, $D^2 = C_0 U(1-\theta) exp(-2t\phi_U U^{\frac{1}{2}})$. $C_0 = \frac{1+\theta_0^{\frac{1}{2}}}{1-\theta_0^{\frac{1}{2}}}$, $x = \frac{U(1+\theta_0^{\frac{1}{2}})}{1-\theta_0^{\frac{1}{2}}} exp(-2\phi_U U^{\frac{1}{2}} t)$. Let $y = \frac{x}{U}$, then when $t = 0$, we have $y = \frac{1+\theta_0^{\frac{1}{2}}}{1-\theta_0^{\frac{1}{2}}} > 1$. Therefore, by $\theta = \left(\frac{y-1}{y+1}\right)^2$, we have $\theta^{\frac{1}{2}} = \frac{y-1}{y+1}$. That is $\ln y = s$.

Therefore, in general, we might use $s_1 = \ln y$ in the place of s. But in general, s_1 depends on U and can not be a function of θ only.

Now we try to connect two metrics by a geodesic. Regard x as a function of t. Let θ_1 be the value of θ of the second metric which corresponds to $U(\theta_0)$.

We notice that the characteristic curve has U as constant, we can then solve the equation

$$\theta_1 = \frac{(x(1)-U)^2}{(x(1)+U)^2+(a-b)^2x(1)}.$$

we obtain that

$$C_0Ue^k = \frac{2U(1+\theta_1)+(a-b)^2\theta_1+\sqrt{(4U+(a-b)^2\theta_1)(4U\theta_1+(a-b)^2\theta_1)}}{2(1-\theta_1)}.$$

We can also have

$$\frac{(1-\theta_0)D^2(\theta_1,U)}{(1-\theta_1)D^2(\theta_0,U)} = \exp(-2\phi_U\sqrt{U+\frac{(a-b)^2}{4}}) = e^k.$$

From this we get k and hence ϕ. To get $\phi_U(1)$, we apply

$$\lim_{U\to(a-c)(b-c)} \frac{1-\theta_0(U)}{1-\theta_1(U)} = \lim_{U\to(a-c)(b-c)} \frac{\theta_{0,U}}{\theta_{1,U}}.$$

To prove this does give a path of metric we only need to prove that $\theta_U = [\frac{(x-U)^2}{(x+U)^2+(a-b)^2x}]_U > 0$ always.

$$\theta_U = \frac{x-U}{((x+U)^2+(a-b)^2x)^2}[x'(4U+(a-b)^2)(x+U)-2x(2(x+U)+(a-b)^2)]$$

for $0 < U < U(1)$. For $U = 0$, we have

$$\theta_U(0) = \frac{(y-1)^2}{(a-b)^2y}$$

with $y = x/U$.

For $U = U(1)$, we only need to check $\lim_{U\to U(1)} m(t)/x > 0$, i.e., $\lim_{U\to U(1)} \frac{y'}{y^2} > 0$. That is same as $\lim_{U\to U(1)} C_0^{-1}(\log C_0)' > 0$. This is always true.

Therefore, $\theta_U > 0$ if and only if

$$m(t) = \frac{x'}{x}[4U+(a-b)^2] - 4 - \frac{2(a-b)^2}{x+U} > 0$$

for $U \neq 0$ and $y > 0$ at $U = 0$.

If this is true for the two metrices at the end of the geodesic, i.e., for $x(0)$ and $x(1)$, then by $\ln x = t \ln x(0) + (1-t)\ln x(1)$, we have

$$\begin{aligned}\frac{x'}{x}[4U+(a-b)^2]-4 &= (t\frac{x'(0)}{x(0)}+(1-t)\frac{x'(1)}{x(1)})[4U+(a-b)^2]-4 \\ &> 2(a-b)^2[\frac{t}{x(0)+U}+\frac{1-t}{x(1)+U}] \\ &\geq \frac{2(a-b)^2}{x+U}.\end{aligned}$$

The last inequality comes from the fact that the function $h(t) = (e^t+1)^{-1}$ is concave ($h'' > 0$) for $t > 0$ by noticing that $y(0), y(1)$ hence $y > 1$.

We notice that if $b = a$, we only need $\frac{x'}{x}U - 1 > 0$ or $(\ln y)' > 0$.

Theorem 4. *On M_n, there is always a smooth geodesic curve between two metrics in the same Kähler class.*

Therefore, we have:

Theorem 5. *There is an unique Kähler metric of constant scalar curvature up to the automorphism group in each Kähler class on M_n if there is one.*

We notice that the existence problem for the compact type II cohomogeneity one Kähler manifolds were solved in [Gu6].

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