A New Proof Of A Conjecture On Nonpositive Ricci Curved Compact Kähler Einstein Surfaces

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In [Gu] we gave a proof of the conjecture of the pinching of the bisectional curvature mentioned in [HGY] and [CHY]. Moreover, we proved that any compact Kähler-Einstein surface M is a quotient of the complex two dimensional unit ball or the complex two dimensional plane if (1) M has nonpositive Einstein constant and (2) at each point, the average holomorphic sectional curvature is closer to the minimal than to the maximal. Following [SY], [HGY] and [CHY], we used a minimal holomorphic sectional curvature direction argument, which was easier for the experts in this direction to understand our proof. In this note, we should use a maximal holomorphic sectional curvature direction argument, which is shorter and easier for the readers who are new in this direction.

1 Introduction

In [SY] the authors conjectured that any compact Kähler-Einstein surface with negative bisectional curvature is a quotient of the complex two dimensional unit ball. They proved that there is a number $a \in (1/3, 2/3)$ such that if at every point P, $K_{av} - K_{min} \leq a[K_{max} - K_{min}]$, then M is a quotient of the complex ball. Here, K_{min} (K_{max}, K_{av}) is the minimal (maximal, average) of the holomorphic sectional curvature. The number a they obtained is $a < \frac{2}{3[1+\sqrt{6/11}]}$ (almost 0.38, see [P2] page 398). In [HGY], Yi Hong[1] pointed out that this is also true if $a \leq \frac{2}{3[1+\sqrt{1/6}]} < 0.476$. We also observed in Theorem 2 that if $a \leq \frac{1}{2}$, then there is a ball-like point P.

Key Words and Phrases: Kähler-Einstein metrics, compact complex surfaces, pinching of the curvatures.

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[1] For this part, it is due to Professor Hong. Notice that he was the first author there.

That is, at P, $K_{max} = K_{min}$. We notice that $\sqrt{1/6} > 1/3$. Therefore, we conjectured in [HGY] that M is a quotient of the complex ball if $a = \frac{1}{2}$. In general, we believe that we might not get a quotient of the complex ball if $a > \frac{1}{2}$. In [P1, P2], the author used a different method and proved that a can be $(3 + \frac{4\sqrt{3}}{3})/11$ (almost 0.48 according to [CHY] page 2628 right before Theorem 1.2), see [P1] page 669, or [P2] page 398. In [CHY], the authors improved the constant to $a < \frac{1}{2}$ that gave a proof of a weaker version of the conjecture.

In [Gu], we proved:

PROPOSITION. Let M be a connected compact Kähler-Einstein surface with nonpositive scalar curvature, if we have

$$K_{av} - K_{min} \leq \frac{1}{2}[K_{max} - K_{min}]$$

at every point, then M is a compact quotient of either the complex two dimensional unit ball or the complex two dimensional complex plane.

For important mathematics work, there is a common practice that people gave different and (possibly) simpler proofs (to certain experts and readers). For examples, see [Bo], [Hu], [CF], [Gu1], [Ti], [DW], [PS], [ACGT], etc..

This note is for the experts who are new in this direction. In the second section, we shall review the basic material from [SY] with an emphasis on the maximal holomorphic sectional direction instead of the minimal holomorphic sectional direction in [SY], [HGY], [CHY], [Gu]. We shall prove the existence of the ball-like points as we did in the second section in [Gu] by using a different but similar function. In the third section, we again use the Hong-Cang Yang's function and a different but similar calculation with respect to the maximal direction instead of the minimal direction. We put some detail calculation the Appendix as the last section of this paper.

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2 Existence of Ball-like Points

Here, we repeat the argument in the proof of our Proposition 1 given in [Gu] by using a different but similar argument:

Proposition 1(Cf. [HGY] p.597–599, [Gu] Proposition 1) *Suppose that*

$$K_{av} - K_{min} \leq \frac{1}{2}[K_{max} - K_{min}]$$

for every point on the compact Kähler Einstein surface with nonpositive Ricci curvatures. There is at least one ball-like point.

Proof of Proposition 1: Throughout this section, as in [SY] and [CHY], [Gu], we assume that $\{e_1, e_2\}$ be an unitary basis at a given point P with

$$R_{1\bar{1}1\bar{1}} = R_{2\bar{2}2\bar{2}} = K_{min},\ R_{1\bar{1}1\bar{2}} = R_{2\bar{2}2\bar{1}} = 0$$

$$A = 2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}} \geq 0,\ B = |R_{1\bar{2}1\bar{2}}|$$

As in [SY], we always have that $A \geq |B|$ and we assume that $B \geq 0$. This also implies if the sectional curvatures have a 1/4 pinching, i.e., the section curvature is inside an interval $[-\frac{1}{4}a(P), -a(P)]$ at every point P for a nonnegative function $a(P)$, then M is covered by a ball. This was pointed out in [CHY]. This is because if we let $a(P) = -R_{1\bar{1}1\bar{1}}$, $e_i = X_i + \sqrt{-1}Y_i$, then at least one of $R(X_1, X_2, X_1, X_2)$ and $R(X_1, Y_2, X_1, Y_2)$ is bigger or equal to $-\frac{1}{4}a(P)$. Same argument works for the higher dimension case. Our PROPOSITION is a kind of the generalization of the 1/4 pinching.

If P is not a ball-like point, according to [SY], we can do as above for a neighborhood $U(P)$ of P whenever $A > B$ (Case 1 in [SY] page 475). In [Gu] we took a lot of efforts in handling the case in which $A = B$. We write

$$\alpha = e_1 = \sum a_i \partial_i,\ \beta = e_2 = \sum b_i \partial_i$$

and

$$S_{1\bar{1}1\bar{1}} = R(e_1, \bar{e}_1, e_1, \bar{e}_1) = \sum R_{i\bar{j}k\bar{l}} a_i \bar{a}_j a_k \bar{a}_l$$

and so on. In particular, we have

$$S_{1\bar{1}1\bar{1}} = S_{2\bar{2}2\bar{2}} = K_{min},\ S_{1\bar{1}1\bar{2}} = S_{2\bar{2}2\bar{1}} = 0$$

According to [SY], we have

$$K_{max} = K_{min} + \frac{1}{2}(A + B),\ K_{av} = K_{min} + \frac{1}{3}A$$

$$\frac{1}{3}[K_{max} - K_{min}] \leq K_{av} - K_{min} \leq \frac{2}{3}[K_{max} - K_{min}]$$

This also shows that A and B is independent of the choice of e_1 and e_2. Also, our condition in Proposition 1 is therefore the same as $A \leq 3B$.

In this section, we denote the maximal direction by e_{1^*} and use * in the notation of the corresponding terms minimal direction case. Assume that P is not a ball-like point. Under our assumption, $B > 0$. According to [SY] page 474 the e_{1^*} could be $\frac{1}{\sqrt{2}}(e_1 + e_2)$. We could pick up $e_{2^*} = \frac{i}{\sqrt{2}}(e_1 - e_2)$. We have:

$$A^* = 2R_{1^*\bar{1^*}2^*\bar{2^*}} - R_{1^*\bar{1^*}1^*\bar{1^*}} = -\frac{1}{2}(A + 3B)$$

$$B^* = R_{1^*\bar{2^*}1^*\bar{2^*}} = \frac{1}{2}(A - B)$$

In our case, we have $A^* + 3B^* = A - 3B \leq 0$, i. e., $-A^* \geq 3B^*$. Moreover, from both the arguments in [SY] page 474 and 475, the choices of the directions of e_{1^*} are isolated on the projective holomorphic tangent space. Those two cases are: Case 1: $A > B$; and Case 2: $A = B$. In the case 1, there is only one direction for the minimal holomorphic sectional curvature and there is only one direction for maximal holomorphic sectional curvature since by our assumption $A \leq 3B$, that is, B is not zero at a nonball-like point. The second statement also follows from the argument in [SY] by applying it to the maximal direction instead of the minimal direction. In the case 2, there is a circle for the minimal direction but there is an unique maximal direction. That is, one could always have a smooth frame of e_{1^*}. This might make the proof simpler. However, near the points with $B^* = A - B = 0$, we might still have a difficulty to get a smooth frame nearby such that $B^* \geq 0$. Therefore, we only assume that $B^* \geq 0$ at P but not necessary true near by if $B^*(P) = 0$.

In [Gu] and [HGY], we let $\Phi_1 = \frac{|B|^2}{A^2} = \tau^2$. Here, we let $\Phi_1^* = \frac{|B^*|^2}{(A^*)^2} = (\tau^*)^2$ and $\tau^* = -\frac{|B^*|}{A^*} \geq 0$.

Our condition is same as $\tau^* \leq 1/3$. If there is no ball-like point, there is a maximal point.

Now, $\tau^* = \frac{A-B}{A+3B} = \frac{1}{3}(-1 + \frac{4}{1+3\tau})$. The maximal of τ^* is just the minimal of τ.

The calculation of the Laplace of Φ_1 at a minimal point, which is not a ball-like point and $A \neq B$ in [Gu] showed that $B^* = 0$.

A similar calculation of the Laplacian of Φ_1^* with $B^* \neq 0$ shows that

$$\Delta\Phi_1^* = 6A^*(\tau^*)^2((\tau^*)^2 - 1) + h^* \qquad (1)$$

Here $\Delta\Phi_1^*$ has two general terms, just as the formula for the $\Delta\Phi_1$ in [Gu]. See the Appendix at the end of this paper. The first term is always nonnegative since $\tau^* \leq \frac{1}{3} \leq 1$. The second term is a hermitian form h^* to y^*. We can separate y^* into two groups: y_{2j}^* in one group and y_{1j}^* in the other. These two groups of variables are orthogonal to each other with respect to this hermitian form. That is, $h^* = h_1^* + h_2^*$ with h_1^* (or h_2^*) only depends on the first (second) group of variables.

We need to check the nonnegativity for each of them.

For y_{11}^*, y_{12}^*, the corresponding matrix of h_2^* is:

$$\begin{bmatrix} 2(9(\tau^*)^2 - 1)((\tau^*)^2 - 1) & 0 \\ 0 & 0 \end{bmatrix}$$

And the matrix for h_1^* of y_{21}^*, y_{22}^* is:

$$\begin{bmatrix} 0 & 0 \\ 0 & 2(9(\tau^*)^2 - 1)((\tau^*)^2 - 1) \end{bmatrix}$$

When P is a critical point of Φ_1^*, the matrices on y^* is clearly semi positive. Therefore, if there is no ball-like point, then we have that at the maximal point of Φ_1^*, $\tau^* = 0$ or $A^* = 0$ since $\tau^* \leq \frac{1}{3}$.

If $A^* = 0$, then we have a ball-like point. And we are done.

On the other hand, if $\tau^* = 0$, we have $B^* = 0$ at P. Since P is a maximal point for τ^*, this implies that $B^* = 0$ on the whole manifold. In this case, we could always assume that $B^* \geq 0$.

According to [SY] page 475 case 2, i. e., when $A = B$, we have a smooth coordinates with $K_{max} = R_{1\bar{1}1\bar{1}}$ (this works fortunately when $A = B$ always. In general, the original argument might not always work since one might not have $A = B$ always nearby. However, as [SY] case 1 also works for the maximal direction instead of the minimal direction, this implies that under our condition the directions for K_{max} are always isolated. Therefore, it might be better one chose K_{max} instead of K_{min} from the very beginning). Using this new coordinate, we can define the similar function A^* and B^*. In general, $B^* = \frac{1}{2}(A - B)$ and $A^* = -\frac{1}{2}(A + 3B)$. In our case, $B^* = 0$ and $A^* = -2A$. Using this new coordinate, one can do the calculation for any of the functions in [SY], [P1], [P2] (or [CHY], see the next section) that the set of ball-like points is the whole manifold. If one does not like Polombo's function Φ_α ([P2] page 418) with $\alpha = -\frac{8}{7}$ (e.g., [P2] page 417 Lemma), then one might simply use the function with $\alpha = -1$ (in [P1, P2], not the vector we mentioned in this paper earlier), i.e., the new function is proportional to $\Phi_2 = (3B - A)A$. In our case, this is just $2A^2$. We can apply

$\Phi_2^{\frac{1}{3}}$. This is relatively easy that we just leave it to the readers (or see (4) in the generalization). One can also use the function in [SY] page 477

$$3\gamma_2 - \gamma_1^2 = \frac{1}{2}(A^2 + 3B^2).$$

We can also still use the argument in [SY] case 1, in which the minimal vectors are not isolated any more but they are points in a smooth circle bundle over the manifold that we could just choose a smooth section instead.

Also, this paragraph is not needed in the following Corollary 1 and Lemma 1 since in those two propositions, we already have $A = 3B$. With $A = B$, one could readily get that $A = B = 0$.

If $A = 0$, $K_{max} = K_{min}$ and P is a ball-like point. We have a contradiction. Therefore, the set of ball-like points is not empty.

Q. E. D.

Observe that if $A = 3B$ at P, then Φ_1 achieves the minimal value at P and $A \neq B$ unless P is a ball-like point. That is the first part of the proof of Proposition 1 goes through. That is, P must be a ball-like point.

Corollary 1. *Assume the above, if $K_{av} - K_{min} = \frac{1}{2}[K_{max} - K_{min}]$ at P, then P is a ball-like point.*

Therefore, we have:

Lemma 1. *If $K_{av} - K_{min} \leq \frac{1}{2}[K_{max} - K_{min}]$ on M, then we have $K_{av} - K_{min} < \frac{1}{2}[K_{max} - K_{min}]$ on $M - N$, where N is the subset of all the ball-like points.*

Therefore, we can apply the argument of [CHY]. To do that one need following Proposition 4 in [SY]:

Propositon 2.(Cf. [SY], also [HGY Theorem 3]) *If $N \neq M$, then N is a real analytic subvariety and* $\operatorname{codim} N \geq 2$.

As in [SY], Proposition 2 give us a way to the conjecture by finding a superharmonic function on M which was obtained by Hong Cang Yang around 1992. In [SY] and [HGY], the authors used $\Phi = 6B^2 - A^2$. In [P2], Polombo used $(11A - 3B)(B - A) + 16AB$, see [P2] page 417 Lemma. One might ask why do we need another function but do not use our Φ_1. The answer is that by a power of Φ_1, we can only correct the Laplace by $|\nabla \Phi_1|^2$. But that could only change the upper left coefficients of our matrices as it only provides $|x|^2$ terms. In the case of Φ_1, it does not work since $\frac{\tau}{A} \neq 0$ but the coefficients of $|y_{12}|^2, |y_{21}|^2$ are zeros.

[2] The paragraph is not needed for the proofs of Corollary 1 and Lemma 1. Also, in this special case, the original frame in [SY] actually work. So, one could simply apply [SY]

Therefore, we need another function, which was provided by Hong Cang Yang.

Remark 1. Whenever there is a bounded continuous nonnegative function f on M such that (1) $f(N) = 0$, (2) f is real analytic on $M - N$ and (3) $\Delta f \leq 0$ on $M - N$, then $f = 0$. *Here N could be just a codimension two subset*. This is in general true for extending continuous superharmonic functions over a codimensional two subset. See [SY], [HGY] and [CHY]. Here, we would like to give our own reasons why this is true in these special cases. If we define $M_s = \{x \in M|_{\mathrm{dist}(x,N)\geq s}\}$ and $h_s = \partial M_s$, then the measure of h_s is smaller than $O(s)$ when s tends to zero. Therefore,

$$0 \geq \ln 2 \int_{M_{2\delta}} \Delta f \omega^n \geq \int_{\delta}^{2\delta} [\int_{M_s} \Delta f \omega^n] s^{-1} ds = \int_{\delta}^{2\delta} [\int_{h_s} \frac{\partial f}{\partial n} d\tau] s^{-1} ds.$$

But by applying an integration by parts to the single variable integral, the last term is about $(\delta)^{-1} \int_{h_{2\delta}} (f - g) d\tau \to 0$ since f is bounded and $f - g$ tends to 0 near N, where g is the f value of the corresponding point on h_δ. For example, if $f = r^a$ with $a > 0$, then

$$\frac{\partial f}{\partial n} = a r^{a-1} = a s^{a-1}$$

and

$$\int_{h_s} \frac{\partial f}{\partial n} d\tau = O(s^a) \to 0.$$

Therefore, $\Delta f = 0$ on $M - N$. Therefore f extends over N as a harmonic function. This implies that $f = 0$ on M.

Now, let $f = (3B - A)^a$, this is natural after the proof of Proposition 1, we will show in the next section that $\Delta f \leq 0$ for $a \leq \frac{1}{3}$ (see also a proof in [CHY]). Therefore, $A = 3B$ always. By the Corollary 1, we have $A = B = 0$. This function is also related to the functions in [P2] page 417 with $a_1 = a_3 = 0$. In [P2] Polombo had to pick up functions with $a_1 = a_2$ to avoid a complication of the singularities. See [P2] page 406 and the first paragraph in page 418 (see also [P1], the last paragraph of page 668). While we shall completely resolve the difficulty in the next section.

3 Generalized Hong Cang Yang's Function

Let $\Psi = 3B - A = -A^* - 3B^*$. About 1992, Hong Cang Yang considered $f = \Psi^{\frac{1}{3}}$. In [CHY], they had a formula for the Laplacian of Ψ. To apply the method to the maximal direction, we notice that same formula holds. Moreover, if we let $\Psi_k = 3B + kA$ then $\Psi = \Psi_{-1}$ and

Lemma 2. (Cf. [CHY] p.2630 (13)) *We have:* $A = \frac{3B^*-A^*}{2}$,

$$B = -\frac{A^* + B^*}{2},$$

$$\begin{aligned}\Delta(3B + kA) &= 3[\Psi_k R_{1\bar{1}2\bar{2}} - B(3A + kB)] \\ &+ \frac{3}{B}|\nabla(\mathrm{Im}R_{1\bar{2}1\bar{2}})|^2 + 6[(B + kA)\sum|y|^2 + 2(A + kB)\mathrm{Re}\sum y_{i1}\bar{y}_{i2}]\end{aligned}$$

In particular, we have:

$$\begin{aligned}\Delta\Psi &= 3[\Psi R_{1^*\bar{1}^*2^*\bar{2}^*} + B^*(3A^* + B^*)] \\ &- \frac{3}{B^*}|\nabla(\mathrm{Im}R_{1^*\bar{2}^*1^*\bar{2}^*})|^2 - 6(A^* + B^*)\sum|y^*_{i1} + y^*_{i2}|^2.\end{aligned}$$

It is obvious that in the case of the maximal direction, we have to assume $B^* \neq 0$. That is, we still need to deal with the case in which $A = B$. This is because, in general, one could not calculate the second derivatives of B^* even if we could get a smooth frame near the considered point. Therefore, we still need deal with the singularities as we did in our earlier paper.

To make everything easier for us, in the rest of this section and the next section (except Remark 2), we use the notation without * *for the maximal direction instead of the minimal direction if there is no confusion.*

Let $z_i = \nabla_i\Psi$. Then

$$-z_1 = \nabla_1(3B + A) = \frac{3}{2}\nabla_1(R_{1\bar{2}1\bar{2}} + R_{2\bar{1}2\bar{1}} - 2R_{1\bar{1}1\bar{1}})$$

$$\begin{aligned}\sqrt{-1}\nabla_1(\mathrm{Im}R_{1\bar{2}1\bar{2}}) &= \frac{1}{2}\nabla_1(R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}) \\ &= -\frac{1}{3}z_1 - \nabla_1 R_{2\bar{1}2\bar{1}} + \nabla_1 R_{1\bar{1}1\bar{1}} \\ &= -\frac{1}{3}z_1 - \nabla_2\bar{R}_{1\bar{1}1\bar{2}} - \nabla_2 R_{1\bar{1}1\bar{2}} \\ &= -\frac{1}{3}z_1 + (A + B)y_{22} + (B + A)y_{21}\end{aligned}$$

$$-z_2 = \nabla_2(3B + A) = \frac{3}{2}\nabla_2(R_{2\bar{1}2\bar{1}} + R_{1\bar{2}1\bar{2}} - 2R_{1\bar{1}1\bar{1}})$$

$$
\begin{aligned}
\sqrt{-1}\nabla_2(\mathrm{Im}R_{1\bar{2}1\bar{2}}) &= \frac{1}{2}\nabla_2(R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}) \\
&= \frac{1}{3}z_2 - \nabla_2 R_{1\bar{1}1\bar{1}} + \nabla_2 R_{1\bar{2}1\bar{2}} \\
&= \frac{1}{3}z_2 - \nabla_1 R_{2\bar{1}1\bar{1}} - \nabla_1 R_{1\bar{1}1\bar{2}} \\
&= \frac{1}{3}z_2 + (B+A)y_{12} + (A+B)y_{11}
\end{aligned}
$$

we can write the formula in the Lemma 2 as:

$$
\begin{aligned}
\Delta\Psi &= 3[\Psi R_{1\bar{1}2\bar{2}} + B(B+3A)] \\
&+ \quad 3\frac{A+B}{B}\Psi\sum|y_{i1}+y_{i2}|^2 \\
&- \quad 2\frac{A+B}{B}\mathrm{Re}[(y_{12}+y_{11})\bar{z}_2 - (y_{22}+y_{21})\bar{z}_1] - \sum\frac{1}{3B}|z|^2
\end{aligned} \tag{2}
$$

Similar to what we have in the last section, we have two general terms, the first is negative as the constant term of z and y. The second is a hermitian form on z and y. We can actually let $w_i = y_{i^*1} - y_{i^*2}$ with $i^* \neq i$. Then the second term is a sum of two hermitian forms. One of them is on w_1, z_1 and the other is on w_2, z_2. We notice that the second term is also nonpositive on y (or nonpositive on w, if we assume that $z = 0$). We can modify the coefficient of $|z|^2$ (only) by taking the power of Ψ. More precisely, if we let $g = \Psi^a$, to make sure that $\Delta g < 0$, after taking out a factor $3\frac{A+B}{B}$ we need

$$
\begin{vmatrix} \Psi & 1/3 \\ 1/3 & -\frac{1+3\Psi^{-1}(1-a)B}{9(A+B)} \end{vmatrix} \geq 0
$$

That is,

$$
A + 3B - 3(1-a)B - A - B = (3a-1)B \leq 0.
$$

We have $1 - 3a \geq 0$. So, $a \leq 1/3$.

Therefore, we have:

Lemma 3. *$\Delta g < 0$ for $a \leq 1/3$ on $M - N$.*

This is exactly the same as what they had in [CHY]. Actually, the number $1/6$ was already in [SY], [HGY], [P1, 2] for those quadratic functions.

So, finally we have:

Theorem 1. *If $K_{av} - K_{min} \leq \frac{1}{2}[K_{max} - K_{min}]$, then M has a constant holomorphic sectional curvature.*

Remark 2. The reason we did not get this earlier was that there was a difficulty when $A = B$. In that case, the argument in [SY] page 475 case 2 seems not working. Polombo resolved the problem by using a function which is symmetric to $\lambda_1 = -\frac{A}{3}$ and $\lambda_2 = \frac{A-3B}{6}$ (see [P2] page 418 first paragraph and the end of page 397). However, Hong Cang Yang's function Ψ is only $-6\lambda_2$ and therefore is not symmetric after all. To overcome the difficulty, we let $\Omega = \{x \in M|_{A=B}\}$. Then according to [SY], all our calculation are good on $M - \Omega$ since $N \subset \Omega$. In [CHY] page 2632, there was a suggestion to prove that codim $\Omega \leq 2$, although it was not very well explained. Then everything went through. The relation was that if we use the argument in [SY] page 475 case 2, using the maximal instead of the minimal, we let $B_1 = |R_{1\bar{2}1\bar{2}}|$ then $2B_1 = A - B$. That is $\Omega = \{x \in M|_{B_1=0}\}$. The argument goes as follows:

Case 1: If Ω is a closed region, we have:

$$\begin{aligned} 0 \geq \int_{M-\Omega} \Delta g \\ &= a \int_{-\partial\Omega} \Psi^{a-1} \frac{\partial(-A_1 - 3B_1)}{\partial n} \\ &\geq a \int_{-\partial\Omega} (2A)^{a-1} \frac{\partial(-A_1)}{\partial n} \\ &= -\int_{\Omega} \Delta F_1 \geq 0 \end{aligned}$$

where F_1 can be chosen from one of the functions in [P2] which satisfies the symmetric condition on M, e.g., a power of Φ_2 in the proof of Proposition 1, or one of our functions with a calculation using the new smooth coordinate in [SY] page 475 with $R_{1\bar{1}1\bar{1}} = K_{max}$ (e.g., see (4) in the next section). Actually, A_1 itself is proportional to the λ_2 in [P2] and is symmetric in the sense of Polombo. On Ω, F_1 is just our g since $B_1 = 0$. We notice that there is a sign difference for the Laplace operator in [P2]. Again, on Ω, since $A = B$ on a neighborhood, the set of minimal directions is a S^1 bundle over Ω, therefore, one might choose a smooth section of it locally that the calculation of [SY] still works in our case. That is, one could simply choose F_1 to be g.

Case 2: If Ω is a hypersurface. Same argument went through except that $\int_{\partial(M-\Omega)} (A)^{a-1} \frac{\partial A}{\partial n} = 0$ since $A \neq 0$ outside a codimension one subset and on $\Omega_1 = \{x \in \Omega|_{A \neq 0}\}$ the integral is integrated from both sides.

Therefore, Ω is a subset of codimension two and we can apply Remark 1. By the calculation in Remark 1, we see that g is harmonic on $M - \Omega$.

Now, by Lemma 2, that implies that $B(B-3A)=0$ and hence $A=B=0$ by our assumptions.

4 The Generalization

Actually, in the first section of [SY], the authors did not require any negativity. We also see that in our second section, we do not really need any negativity except when we applied the formula in the Lemma 2 in the third section.

In the first section of [SY], they also considered the coordinate in which $R_{1\bar{1}1\bar{1}}$ achieves the maximal instead of the minimal. With using the maximal direction, it is much easier to see that the constant term in the Laplacian is negative. We only need to check:

$$\begin{aligned} C = R_{1\bar{1}2\bar{2}} & \\ = \quad & k - R_{1\bar{1}1\bar{1}} \\ = \quad & k/2 - (K_{\max} - k/2) \\ = \quad & k/2 - (K_{\max} - K_{min}) \\ = \quad & k/2 + A/3 \leq 0. \end{aligned} \tag{3}$$

One might compare this with [Gu] to see the advantage of this new method.

Now, with $C \leq 0$, we could also easily cover the arguments in both at the end of the proof of Proposition 1 and in Remark 2 in the case of $B=0$ (using the maximal direction). Similar to the calculation in section 2 we obtain:

$$\Delta R_{1\bar{1}1\bar{1}} = -AC + B^2 = -AC \leq 0.$$

See also [MZ] page 27 for a good calculation for this Laplacian at a maximal direction for any complex dimension.

We also have:

$$\nabla R_{1\bar{1}1\bar{2}} = -A\nabla a_2 - B\nabla \bar{a}_2 = -A\nabla a_2,$$

$$\Delta S_{1\bar{1}1\bar{1}} = -2A\sum |y|^2 - AC.$$

$$\nabla_1 A = -3\nabla S_{1\bar{1}1\bar{1}} = -3Ay_{21},$$

$$\nabla_2 A = 3Ay_{12}.$$

$$\nabla_{\bar{1}} R_{1\bar{2}1\bar{2}} = -A\bar{y}_{22} = 0,$$

$$\nabla_2 R_{1\bar{2}1\bar{2}} = Ay_{11} = 0.$$

$$
\begin{aligned}
\Delta(|A|^a) = 3a|A|^{a-1}\Delta S_{1\bar{1}1\bar{1}} + a(a-1)|A|^{a-2}|\nabla A|^2 & \\
&= 3a \times (-A)^{a-1}(-2A\sum|y|^2 - AC) \\
&+ 9a(a-1)(-A)^a \sum |y|^2 \qquad (4) \\
&= 3a(-A)^a[(2-3(a-1))\sum|y|^2 + C]
\end{aligned}
$$

is nonpositive when $a \leq 1/3$. This is same as in the Lemma 3 and that in [CHY].

Therefore, we concluded the general case. One might conjecture that our Theorem is also true in the higher dimensional cases.

Remark 3. Notice that this generalization basically covers the results in [P1] and [P2] for the Kähler-Einstein case (see [P2] page 398 Corollary). See also [De] page 415 Proposition 2 for the W^+ for a Kähler surface. One might ask whether our result could be generalized to the Riemannian manifolds with closed half Weyl curvature tensors. This is out of the scope of this paper although a similar result is true, i.e., if $\lambda_2 \leq 0$ at every point. To make the relation between this paper and [P1], [P2] clearer to the readers, we just mention that any one of the half Weyl tensors is harmonic if and only if it is closed since the tensor is dual to either itself or the negative of itself. The Remark (i) in [P2] page 397 says that if M is Riemannian-Einstein, the second Bianchi identity says that the half Weyl tensors are closed (see also [De] page 408 formula (9) and page 411 remark 1).

5 Appendix

Here, we repeat the argument in the proof of the Proposition 1 in [Gu] by using a different but similar argument:

Throughout this Appendix, as in [SY] and [CHY], [Gu], we assume that $\{e_1, e_2\}$ be an unitary basis at a given point P with

$$R_{1\bar{1}1\bar{1}} = R_{2\bar{2}2\bar{2}} = K_{min},$$

or K_{max}.

$$R_{1\bar{1}1\bar{2}} = R_{2\bar{2}2\bar{1}} = 0$$

$$A = 2R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{1}} \geq 0,$$

or ≤ 0 in the maximal direction case.

$$B = |R_{1\bar{2}1\bar{2}}|$$

As in [SY], we always have that $A \geq |B|$ or $-A \geq |B|$. And we assume that $B \geq 0$.

If P is not a ball like point, We write

$$\alpha = e_1 = \sum a_i \partial_i, \ \beta = e_2 = \sum b_i \partial_i$$

and

$$S_{1\bar{1}1\bar{1}} = R(e_1, \bar{e}_1, e_1, \bar{e}_1) = \sum R_{i\bar{j}k\bar{l}} a_i \bar{a}_j a_k \bar{a}_l$$

and so on.

In particular, we have

$$S_{1\bar{1}1\bar{1}} = S_{2\bar{2}2\bar{2}}, \ S_{1\bar{1}1\bar{2}} = S_{2\bar{2}2\bar{1}} = 0$$

We calculate the Laplace of $\Phi_1 = \tau^2 = \frac{|B|^2}{A^2}$ at a critical point.

We let

$$x_i = \nabla_i \Phi_1 = 2\frac{\tau}{A}[\mathrm{Re}\nabla_i S_{1\bar{2}1\bar{2}} + 3\tau \nabla_i S_{1\bar{1}1\bar{1}}]$$

As in [SY]. [HGY], [CHY], we have:

$$\Delta R_{1\bar{1}1\bar{1}} = -AR_{1\bar{1}2\bar{2}} + B^2$$

$$\Delta R_{1\bar{2}1\bar{2}} = 3(R_{1\bar{1}2\bar{2}} - A)B.$$

At P we have $a_1 = b_2 = 1$ and $a_2 = b_1 = 0$, $\nabla a_1 = \nabla b_2 = 0$, $\nabla a_2 + \nabla \bar{b}_1 = 0$. Therefore, we write $y_{i1} = \nabla_i a_2$ and $y_{i2} = \nabla_i \bar{a}_2$. We also have:

$$\Delta(a_1 + \bar{a}_1) = -|\nabla a_2|^2, \Delta(a_2 + \bar{b}_2) = 0$$

$$\nabla_i R_{1\bar{1}1\bar{2}} = -Ay_{i1} - By_{i2}$$

since

$$0 = \nabla S_{1\bar{1}1\bar{2}} = \nabla R_{1\bar{1}1\bar{2}} + 2R_{2\bar{1}1\bar{2}} \nabla a_2 + B\nabla \bar{a}_2 + R_{1\bar{1}1\bar{1}} \nabla \bar{b}_1,$$

i.e.,

$$\nabla R_{1\bar{1}1\bar{2}} = -A\nabla a_2 - B\nabla \bar{a}_2.$$

This also gives a similar formula for $\nabla_{\bar{i}} R_{1\bar{1}1\bar{2}}$. Similarly,

$$\nabla S_{1\bar{1}1\bar{1}} = \nabla R_{1\bar{1}1\bar{1}}$$

$$\nabla S_{1\bar{2}1\bar{2}} = \nabla R_{1\bar{2}1\bar{2}}$$

$$\Delta S_{1\bar{1}1\bar{1}} = -2A\sum |y|^2 - 4B\mathrm{Re}\sum y_{i1}\bar{y}_{i2} - AR_{1\bar{1}2\bar{2}} + B^2$$

$$\mathrm{Re}\Delta S_{1\bar{2}1\bar{2}} = 4A\sum \mathrm{Re} y_{i1}\bar{y}_{i2} + 2B\sum |y|^2 + 3(R_{1\bar{1}2\bar{2}} - A)B.$$
$$\nabla_{\bar{1}} S_{1\bar{2}1\bar{2}} = -A\bar{y}_{22} - B\bar{y}_{21}$$
$$\nabla_2 S_{1\bar{2}1\bar{2}} = Ay_{11} + By_{12}$$
$$\nabla_1 S_{1\bar{2}1\bar{2}} = -A(6\tau^2-1)y_{22} - 5A\tau y_{21} + x_1$$
$$\nabla_{\bar{2}} S_{1\bar{2}1\bar{2}} = 5A\tau\bar{y}_{12} + A(6\tau^2-1)\bar{y}_{11} + \bar{x}_2$$

As in [HGY] p. 598, at P we have:

$$\begin{aligned}
\Delta\Phi_1 &= \frac{2\tau\Delta B}{A} + \frac{6\tau^2}{A}\Delta S_{1\bar{1}1\bar{1}} \\
&+ \frac{1}{A^2}\sum(|\nabla S_{1\bar{2}1\bar{2}}|^2 + |\bar{\nabla} S_{1\bar{2}1\bar{2}}|^2) + \frac{54\tau^2}{A^2}\sum|\nabla S_{1\bar{1}\bar{1}}|^2 \\
&+ \frac{12\tau}{A^2}\sum \mathrm{Re}(\nabla_i S_{1\bar{1}1\bar{1}}(\nabla_{\bar{i}}(S_{1\bar{2}1\bar{2}} + S_{2\bar{1}2\bar{1}})) \qquad (5)\\
&= 2\tau[3A\tau(\tau^2-1) - 4\tau\sum|y|^2 + 4(1-3\tau^2)\sum\mathrm{Re}(y_{i1}\bar{y}_{i2})] \\
&+ |y_{22}+\tau y_{21}|^2 + |y_{11}+\tau y_{12}|^2 \\
&+ \frac{1}{A^2}[|x_1 + A[(1-6\tau^2)y_{22} - 5\tau y_{21}]|^2 + |x_2 + A[(6\tau^2-1)y_{11} + 5\tau y_{12}]|^2 \\
&- 18\tau^2[y_{12}+\tau y_{11}|^2 + |y_{21}+\tau y_{22}|^2] \\
&+ \frac{12\tau}{A}[\mathrm{Re}[(y_{21}+\tau y_{22})\bar{x}_1] - \mathrm{Re}[(y_{21}+\tau y_{11})\bar{x}_2]]
\end{aligned}$$

Here we notice that $\Delta\Phi_1$ has two general terms. The first term has nothing to do with x and y, and therefore can be regarded as constant term to them. That term is always nonpositive since $\frac{1}{3} \le \tau \le 1$.

The second term can be regarded as a hermitian form h to x and y. We can separate x and y into two groups: x_1, y_{2j} in one group and x_2, y_{1j} in the other. These two groups of variables are orthogonal to each other with respect to this hermitian form. That is, $h = h_1 + h_2$ with h_1 (or h_2) only depends on the first (second) group of variables.

We need to check the nonpositivity for each of them.

For x_2, y_{11}, y_{12}, the corresponding matrix of h_2 is:

$$\begin{bmatrix} \frac{1}{A^2} & -\frac{1}{A} & -\frac{\tau}{A} \\ -\frac{1}{A} & 2(9\tau^2-1)(\tau^2-1) & 0 \\ -\frac{\tau}{A} & 0 & 0 \end{bmatrix}$$

And the matrix for h_1 of x_1, y_{21}, y_{22} is:

$$\begin{bmatrix} \frac{1}{A^2} & \frac{\tau}{A} & \frac{1}{A} \\ \frac{\tau}{A} & 0 & 0 \\ \frac{1}{A} & 0 & 2(9\tau^2-1)(\tau^2-1) \end{bmatrix}$$

When P is a critical point of Φ_1, then $x_1 = x_2 = 0$. The matrices on y is clearly semi definite.

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