Acta Mathematica Scientia, 2023, **43B**(1): 363–372
https://doi.org/10.1007/s10473-023-0120-9

Mathematica Acta Scientia
数学物理学报
http://actams.apm.ac.cn

MAXWELL-EINSTEIN METRICS ON COMPLETIONS OF CERTAIN C* BUNDLES*

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Abstract In this paper, we prove that for some completions of certain fiber bundles there is a Maxwell-Einstein metric conformally related to any given Kähler class.

Key words Hermitian metrics; Maxwell-Einstein metrics; complex manifolds; scalar curvature; fiber bundle; almost homogeneous manifolds

2010 MR Subject Classification 53C10; 53C21; 53C26; 53C55; 32L05; 32M12; 32Q20

1 Introduction

In every Kähler class of a compact almost homogeneous manifold with two ends, we found a unique Calabi extremal metric in [15, 16]. Moreover, we found a unique extremal metric in a given Kähler class on some completion of certain $\mathbf{C}^*$ bundle if the function Φ there is positive. In [19] we observed that this is equivalent to the geodesic stability of the Kähler class.

In this paper, we shall prove that for these kinds of manifolds, there is another natural Kähler metric in the given Kähler class.

Definition For any given Kähler class, there is a Maxwell-Einstein metric conformally related to the Kähler class if $h = u^{-2}g$ is a Hermitian metric with a constant scalar curvature such that u is the Hamiltonian function of a holomorphic vector field related to a Kähler metric g in the given Kähler class.

Theorem For any Kähler class on a compact almost homogeneous manifold with two ends, there is at least one Maxwell-Einstein metric in the given Kähler class.

We notice that when the complex dimension of the manifold M is 2, some extremal metrics, e.g., on $\mathbf{C}P^2$ blowing up a point, are actually the same as the Maxwell-Einstein metrics. Moreover, in this case, the corresponding Hermitian metrics are, as Riemannian metrics, actually Hermitian-Einstein.

Therefore, Maxwell-Einstein metrics should be as standard as Calabi extremal metrics, but Maxwell-Einstein metrics are not, in general, Einstein metrics, just as quaternion Kähler are not in general Kähler.

About twenty years ago, motivated by [20, 21] and [7], I developed this type of metric and obtained some partial results for this Theorem. I told Professor Kobayashi about it. However,

*Received May 5, 2021; revised June 29, 2022. This research was supported by NSFC (12171140).

somehow, first we did not get the further Hermitian-Einstein metric in the Riemannian sense. Also, by the proof of the Yamabe conjecture, every Kähler metric is conformally related to a Hermitian metric with a constant scalar curvature, just as every Hermitian metric has a smooth Riemannian scalar curvature, so we did not pay much attention to these metrics.

Recently, however, after the publication of two papers [28, 29] by LeBrun, it seems that the Maxwell-Einstein metrics have become a hot topic.

It came to our attention when we were finishing the completion of this paper, that in [2], the authors proved that on any admissible Kähler class there is an extremal-Maxwell-Einstein metric with a given number $a > 1$; that is, the scalar curvature is a potential function of a holomorphic vector field (see, for example, Theorem 1 therein). Our result is a more general form of their conjecture 1 in 4.2, but their result is more like a soliton version of Maxwell-Einstein metrics. Hence, their result does not imply our result. Also, the admissible metrics are very restricted and do not include all of the compact almost homogeneous Kähler manifolds with two ends.

In my original exchange with Professor Kobayashi, there was a conclusion that the Maxwell-Einstein Hermitian metric is a Kähler metric if and only if the classical Futaki invariant is zero. Although this could easily be seen to be true, the original simple argument, which came directly out of our calculation, was not able to be recalled.

We also remark that these metrics are usually referred to as Einstein-Maxwell metrics. These metrics originally came from physics, and some of them are actually Hermitian-Einstein in the Riemannian sense. However, our metrics are generally quite different. To emphasise this difference, we call them Maxwell-Einstein metrics instead; they are more like some kind of pseudo-Einstein metric. The same method has been used for a cohomogeneity one version of a Yau's conjecture in [9]. which was also dealt with in [6] and the references mentioned therein.

2 Certain Completion of Line Bundles

Our results can be regarded as a continuation of those in [25–27] and [15, 16, 18, 19]; in what follows, we state without detail proof lemmas similar to some that are found in those papers. Readers can refer to [16, 18] for more details and most of the relevant Lemmas can be actually found in [16].

Let $p:\ L \to M$ be a holomorphic line bundle over a compact complex Kähler manifold M and let h be a hermitian metric of L. Denote by L^0 the open subset $L - \{0\text{-section}\}$ of L and let $s \in C^\infty(L^0)_{\mathbf{R}}$ be defined by $s(l) = \log |l|_h$ $(l \in L^0)$, where $|\ |_h$ is the norm defined by h. Now we consider a function $\tau = \tau(s) \in C^\infty(L^0)_{\mathbf{R}}$ which is only dependant on s and is monotone-increasing with respect to s.

Let $\tilde{J}$ be the complex structure of L and let J be the complex structure of M. Now we consider a Riemannian metric on L^0 of the form

$$\tilde{g} = \mathrm{d}\tau^2 + (\mathrm{d}\tau \circ \tilde{J})^2 + g, \tag{2.1}$$

where $g(l) = p^* g_{\tau(s(l))}(m)$ with $m = p(l) \in M$, and where g_τ is a one parameter family of Riemannian metrics on M. This form of the metrics is general, and is achieved by using the function τ as the length of the geodesics perpendicular to the generic orbits. Define a function

u on L^0 depending only on τ by $u(\tau)^2 = \tilde{g}(H,\ H)$, where H is the real vector field on L^0 corresponding to the $\mathbf{R}^*$ action on L^0.

Lemma 2.1 (cf [26, 27], [16, p. 2257]) Suppose that the range of τ contains 0. Then $\tilde{g}$ is Kähler if and only if g_0 is Kähler and $g_\tau = g_0 - UB$, where B is the curvature of L with respect to h, $U = \int_0^\tau u(\tau)\mathrm{d}\tau$.

Throughout this paper, we assume that

(1) $\hat{L}$ is a compactification of L^0 and that $\tilde{g}$ is the restriction of a Kähler metric of $\hat{L}$ to L^0;

(2) the range of τ contains 0;

(3) the eigenvalues of B with respect to g_τ are constants on M;

(4) the traces of the Ricci curvature r of g on each eigenvector space of B are constant.

The condition (4) here is much more general than that in [15, 16], in that we have

(4)' the eigenvalues of r are constants.

Our results cover some others that have appeared in recent years. For example, for if g has a constant scalar curvature and B has only one eigenvalue. This is the case of admissible metrics mentioned in [2]. However, in general, for a rational homogeneous space, B only has constant eigenvalues, but these might be distinct. Therefore, a general Kähler metric in this paper is most likely not admissible at all.

By abuse of language, we call the constants in (4) trace eigenvalues.

Let $(z^1, \cdots, z^n)$ be a system of holomorphic local coordinates on M. $n = \dim_{\mathbf{C}} M$. Using a trivialization of L^0, we take a system of holomorphic local coordinates $(z^0, \cdots, z^n)$ on L^0 such that $\partial/\partial z^0 = H - \sqrt{-1}\tilde{J}H$.

Remark 2.2 Here we notice that z^0 corresponds to w_1 in [18, p. 552]. s can be regarded as $\mathrm{Re} z^0$ near the considered point, so s is the x_1 in [18, p. 552]. As in [16], we let $\varphi = u^2$ as a function of U and we let F be the Kähler potential as in [18, p. 552]. Then by comparing [16, Lemma 2] (or Lemma 2.4 below) with [18, p. 552], we immediately have

Lemma 2.3 $4\varphi = \frac{\partial^2 F}{\partial s^2}$.

From $(\frac{\mathrm{d}\tau}{\mathrm{d}s})^2 = \varphi$, we obtain that $\frac{\mathrm{d}\tau}{\mathrm{d}s} = u$. $U = \int_0^\tau u\mathrm{d}\tau = \int_{s(0)}^s u^2 \mathrm{d}s = \int_{s(0)}^s 4^{-1}\frac{\partial^2 F}{\partial s^2}\mathrm{d}s$ is $\frac{\partial F}{\partial s} = y_1$ up to a constant in [18, p. 552]; i.e., we have

Lemma 2.4 U is the Legendre transformation of s.

Here we use the Legendre transformation in [17] instead of the moment map in [16]. This remark is only for readers who might be interested in deeper understanding of our construction, and is not really needed in this paper.

Let $\hat{X}_i,\ \hat{X}_{\bar{i}}$ $(0 \le i \le n)$ be the partial differentiations $\partial/\partial z^i,\ \partial/\partial \bar{z}^i$ on L^0 and let $X_i,\ X_{\bar{i}}$ $(1 \le i \le n)$ be the partial differentiations $\partial/\partial z^i,\ \partial/\partial \bar{z}^i$ on M.

Lemma 2.5 (cf [26, 27], [15, Lemma 2]) We have

$$\tilde{g}_{0\bar{0}} = 2u^2, \quad \tilde{g}_{0\bar{i}} = 2u\hat{X}_{\bar{i}}\tau, \quad \tilde{g}_{i\bar{j}} = g_{i\bar{j}} + 2\hat{X}_i\tau \cdot \hat{X}_{\bar{j}}\tau, \tag{2.2}$$

where $1 \le i, j \le n$. At the point where $P \in L^0$ considered, we can choose a local coordinate system around $m = p(P) \in M$ such that $(\partial/\partial z^i)\tau = 0$ at m, so $\hat{X}_i\tau = \hat{X}_{\bar{j}}\tau = 0$ at the point we are considering, and if f is a function on L^0 depending only on τ, we have that

$$\hat{X}_0\hat{X}_{\bar{0}}f = u\frac{\mathrm{d}}{\mathrm{d}\tau}\left(u\frac{\mathrm{d}f}{\mathrm{d}\tau}\right) \quad \hat{X}_i\hat{X}_{\bar{0}}f = 0,$$

$$\hat{X}_i \hat{X}_{\bar{j}} f = -\frac{1}{2} u B_{i\bar{j}} \frac{\mathrm{d}f}{\mathrm{d}\tau}. \tag{2.3}$$

The Ricci curvature at this point is

$$\tilde{r}_{0\bar{0}} = -u \frac{\mathrm{d}}{\mathrm{d}\tau}\left(u \frac{\mathrm{d}}{\mathrm{d}\tau} \log(u^2 Q)\right) \quad \tilde{r}_{0\bar{i}} = 0,$$

$$\tilde{r}_{i\bar{j}} = p^* r_{0\ i\bar{j}} + \frac{1}{2} u \frac{\mathrm{d}}{\mathrm{d}\tau} \log(u^2 Q) \cdot B_{i\bar{j}}, \tag{2.4}$$

where $Q = \det(g_0^{-1} \cdot g_\tau)$. In particular, we have the scalar curvature

$$\tilde{R} = \frac{\Delta}{Q} - \frac{1}{2Q} \frac{\mathrm{d}}{\mathrm{d}U}\left(\frac{\mathrm{d}}{\mathrm{d}U} Q\varphi\right), \tag{2.5}$$

where $\varphi = u^2$ is regarding as a function of U and $\Delta(U) = Q \sum_{i,j} r_{0\ i\bar{j}} g^{i\bar{j}}_{\tau(U)}$. We also have $\varphi'(\min U) = 2$, $\varphi'(\max U) = -2$.

Lemma 2.6 (cf. [11, 30], [16, Lemma 3]) We can also regard U as a moment map corresponding to $(\tilde{g},\ \tilde{J}H)$, and g_τ is just the symplectic reduction of $\tilde{g}$ at $U(\tau)$. $\tilde{g}$ is extremal if and only if $\tilde{R} = aU + b$ for some $a, b \in \mathbf{R}$.

Let $M_0 = U^{-1}(\min U)$ and let $M_\infty = U^{-1}(\max U)$. They are complex sub-manifolds, since they are components of the fixed point set of $H - \sqrt{-1}\tilde{J}H$, which is semisimple. Let D_0 be the codimension of M_0 in $\hat{L}$, and let D_∞ be the codimension of M_∞ in $\hat{L}$.

Lemma 2.7 (cf. [16, Lemma 4]) Suppose that there is another Kähler metric $\tilde{g}^\vee$ on $\hat{L}$ in the same Kähler class which is of form (1) on L^0. Let $\tau^\vee$, $g^\vee$, $U^\vee$, $Q^\vee$, $\Delta^\vee$, $\varphi^\vee$, $u^\vee$ be the corresponding metric and the corresponding functions of s. Then there is a unique corresponding $\tau^\vee$ such that $g_0^\vee = g_0$. In this case, $\min U^\vee = \min U$ (or $\max U^\vee = \max U$) and $Q^\vee = Q$, $\Delta^\vee = \Delta$ hold. Thus we may write $D = \max U$ and $-d = \min U$. Then

$$Q(U) = (1 + \frac{U}{d})^{D_0 - 1} Q_{-d}$$

$$(\text{or } = (1 - \frac{U}{D})^{D_\infty - 1} Q_D), \tag{2.6}$$

where Q_{-d} (or Q_D) is a polynomial of U such that $Q_{-d}(-d) \neq 0$ (or $Q_D(D) \neq 0$) and

$$\Delta(U) = D_0(D_0 - 1)\frac{1}{d}(1 + \frac{U}{d})^{D_0 - 2} Q_{-d} \pmod{(1 + \frac{U}{d})^{D_0 - 1}}$$

$$(\text{or } = D_\infty(D_\infty - 1)\frac{1}{D}(1 - \frac{U}{D})^{D_\infty - 2} Q_D \pmod{(1 - \frac{U}{D})^{D_\infty - 1}}). \tag{2.7}$$

Proof Let $\tilde{g} - \tilde{g}^\vee = \hat{\partial}\bar{\hat{\partial}}\phi$. Then

$$\tilde{g}^\vee_{i\bar{j}} = \tilde{g}_{i\bar{j}} + \frac{1}{2} u \frac{\mathrm{d}\phi}{\mathrm{d}\tau} B_{i\bar{j}} = (g_0)_{i\bar{j}} - (U - \frac{1}{2} u \frac{\mathrm{d}\phi}{\mathrm{d}\tau}) B_{i\bar{j}} \tag{2.8}$$

for $1 \le i, j \le n$, so at $\min U$ (or $\max U$) $\tilde{g}_{i\bar{j}} = \tilde{g}^\vee_{i\bar{j}}$. There is a τ_0 such that $g^\vee_{\tau^\vee(\tau_0)} = g_0$. By choosing $\tau^\vee$ such that $\tau^\vee(\tau_0) = 0$, one sees that $\min U^\vee = \min U$, $\max U^\vee = \max U$, as desired.

The last statement follows from the fact that the scalar curvature R is finite on both M_0 and M_∞. □

We need normalization in this paper. By rescaling, we can choose $U^\vee = a_1^2 U + a_2$ for any $a_1 > 0$ and $a_2 \in \mathbf{R}$, allowing us to assume that $\max U - \min U = 2$ and $\min U = -1$, so $\max U = 1$.

3 Existence of Maxwell-Einstein Metrics

We recall our definition of the Maxwell-Einstein metrics: for any given Kähler class, there is a Maxwell-Einstein metric conformally related to the Kähler class if $h = u^{-2}g$ is an Hermitian metric with a constant scalar curvature such that u is the Hamiltonian function of a holomorphic vector field related to a Kähler metric g in the given Kähler class.

From Lemma 2.6, it can be seen that if $\tilde{g}$ is a Maxwell-Einstein metric, then $u = aU + b$ for some $a, b \in \mathbf{R}$.

The following formula is well-known for the conformal geometry. It can be found in [8, p. 119, (6.1)], or [3, p. 126, (1)]:

$$S_h = -2\frac{2n-1}{n-1}v^{-\frac{n+1}{n-1}}\Delta v + Sv^{-\frac{2}{n-1}} = 2(2n-1)(u\Delta u - n|Du|^2) + Su^2. \tag{3.1}$$

Here $v = u^{-n+1}$ and $S = \tilde{R}$ for the scalar curvature of our Kähler metric in Lemma 2.5. Notice that here we have a different sign for the Laplacian and that n is the complex dimension of the big manifold, but not the one in the last section, i.e., $n = \dim_{\mathbf{C}} M + 1$.

Letting f be a function of U, by Lemma 2.5, we have that

$$\begin{aligned}
\tilde{\Delta} f &= \tilde{g}^{\bar{\alpha}\beta}\hat{X}_{\bar{\alpha}}\hat{X}_{\beta} f \\
&= \tilde{g}^{\bar{0}0}\hat{X}_{\bar{0}}\hat{X}_0 f + \tilde{g}^{\bar{a}0}\hat{X}_{\bar{a}}\hat{X}_0 f + \tilde{g}^{\bar{0}a}\hat{X}_{\bar{0}}\hat{X}_a f + \tilde{g}^{\bar{a}b}\hat{X}_{\bar{a}}\hat{X}_b f \\
&= \frac{1}{2u^2}(\hat{X}_{\bar{0}}\hat{X}_0 f) + 0 + 0 + \tilde{g}^{\bar{a}b}(\hat{X}_{\bar{a}}\hat{X}_b f) \\
&= \frac{1}{2u^2}u\frac{\mathrm{d}}{\mathrm{d}\tau}(u\frac{\mathrm{d}}{\mathrm{d}\tau}f) + g_\tau^{\bar{a}b}(-2^{-1}u\frac{\mathrm{d}f}{\mathrm{d}\tau}B_{b\bar{a}}) \\
&= 2^{-1}\frac{\mathrm{d}}{\mathrm{d}U}(\varphi(U)\frac{\mathrm{d}}{\mathrm{d}U}f) - 2^{-1}\varphi(U)(\frac{\mathrm{d}}{\mathrm{d}U}f)g_t^{\bar{a}b}B_{b\bar{a}} \\
&= 2^{-1}\frac{\mathrm{d}}{\mathrm{d}U}(\varphi\frac{\mathrm{d}}{\mathrm{d}U}f) + 2^{-1}\varphi(\frac{\mathrm{d}}{\mathrm{d}U}f)\frac{1}{Q}\frac{\mathrm{d}}{\mathrm{d}U}Q \\
&= \frac{1}{2Q}\frac{\mathrm{d}}{\mathrm{d}U}(\varphi Q\frac{\mathrm{d}}{\mathrm{d}U}f).
\end{aligned} \tag{3.2}$$

We then have that $\Delta u = \frac{a}{2Q}(\varphi Q)'$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}U}((aU+b)^{-2n+1}Q\varphi) = -2\int_{-1}^{U} c(ax+b)^{-2n-1}Q(x)\mathrm{d}x \\
+2\int_{-1}^{U}(ax+b)^{-2n+1}\Delta(x)\mathrm{d}x + 2(b-a)^{-2n+1}Q(-1).
\end{aligned} \tag{3.3}$$

Notice that it is easy to check that this is consistent, even if $Q(-1) = 0$ or $Q(1) = 0$, with our Lemma 2.7.

We also have that

$$\begin{aligned}
&c\int_{-1}^{1}(ax+b)^{-2n-1}Q(x)\mathrm{d}x \\
&= (a+b)^{-2n+1}Q(1) + (b-a)^{-2n+1}Q(-1) + \int_{-1}^{1}(ax+b)^{-2n+1}\Delta(x)\mathrm{d}x.
\end{aligned} \tag{3.4}$$

Then

$$(aU+b)^{-2n+1}Q\varphi = -2\int_{-1}^{U}\Big[\int_{-1}^{y}[c(ax+b)^{-2n-1}Q(x) - (ax+b)^{-2n+1}\Delta(x)]\mathrm{d}x$$

$$
\begin{aligned}
&-(b-a)^{-2n+1}Q(-1)\Big]\mathrm{d}y\\
=&-2\Big[\int_{-1}^{U}(U-x)[c(ax+b)^{-2n-1}Q(x)-(ax+b)^{-2n+1}\Delta(x)]\mathrm{d}x\\
&-(b-a)^{-2n+1}(U+1)Q(-1)\Big].
\end{aligned}
$$

We denote the right side by $\Phi(U)$.

We also have that

$$
\begin{aligned}
&c\int_{-1}^{1}x(ax+b)^{-2n-1}Q(x)\mathrm{d}x\\
=&\int_{-1}^{1}x(ax+b)^{-2n+1}\Delta(x)\mathrm{d}x+(a+b)^{-2n+1}Q(1)-(b-a)^{-2n+1}Q(-1).
\end{aligned}
\tag{3.5}
$$

Now, we have that $aU+b>0$. In particular, $b>0$. We might assume that $b=1$, so $-1<a<1$.

(3.4) and (3.5) imply that

$$
\begin{aligned}
&\int_{-1}^{1}(ax+1)^{-2n-1}Q(x)\mathrm{d}x\Big[\int_{-1}^{1}(ax+1)^{-2n+2}\Delta(x)\mathrm{d}x\\
&+(a+1)^{-2n+2}Q(1)+(1-a)^{-2n+2}Q(-1)\Big]\\
=&\int_{-1}^{1}(ax+1)^{-2n}Q(x)\mathrm{d}x\Big[\int_{-1}^{1}(ax+1)^{-2n+1}\Delta(x)\mathrm{d}x\\
&+(a+1)^{-2n+1}Q(1)+(1-a)^{-2n+1}Q(-1)\Big].
\end{aligned}
\tag{3.6}
$$

This actually comes from (3.4) and $a(3.4)+(3.5)$. If we write (3.4) as $cA=B$ and (3.5) as $cD=E$, then, by $A>0$, we have that $c=B/A$. We get that $(B/A)D=E$, i.e., that $BD=AE$. What we have in (3.6) is actually that $aBD=aAE$. Note that there is a trivial solution of $a=0$ in (3.6), which does not come from (3.4) and (3.5).

Lemma 3.1 When a is near 1, the right side of (3.6) is bigger than the left side. When a is near -1, the right side (after dividing a) is smaller than the left side (after dividing a).

Proof When $a\to 1$, the major part of $\int_{-1}^{1}(ax+1)^{-k}Q(x)\mathrm{d}x$ comes from $\int_{-1}^{1}(ax+1)^{-k}Q(-1)\mathrm{d}x$, which is $\frac{1}{(-k+1)a}[(1+a)^{-k+1}-(1-a)^{-k+1}]Q(-1)$ if $Q(-1)\neq 0$. Further, the major part of the left side is $\frac{1}{2n}(1-a)^{-4n+2}Q^2(-1)$, if $Q(-1)\neq 0$. Similarly, in this case, the major part of the right side is $\frac{1}{2n-1}(1-a)^{-4n+2}Q^2(-1)$. More work is needed on the case in which $Q(-1)=0$.

On the other hand, when $a\to -1$, the major part $\int_{-1}^{1}(ax+1)^{-k}Q(x)\mathrm{d}x$ comes from $\int_{-1}^{1}(ax+1)^{-k}Q(1)\mathrm{d}x$, which is $\frac{1}{(-k+1)a}[(1+a)^{-k+1}-(1-a)^{-k+1}]Q(1)$ if $Q(1)\neq 0$. Similarly to the above, the major part of the left side is $\frac{1}{2n}(1+a)^{-4n+2}Q^2(1)$ if $Q(1)\neq 0$. In this case, the major part of the right side is $\frac{1}{2n-1}(1+a)^{-4n+2}Q^2(1)$, similarly to the above. Also, more work is needed for the case in which $Q(1)=0$.

All of this implies that, both when $a\to 1$ or -1, the right side is bigger than the left side. However, after dividing by a, which is an extra factor, when $a\to 1$ (or -1), the left side is smaller (or bigger) than the right side. Therefore, there is actually at least one $a \in (-1,1)$ such that both identities (3.4) and (3.5) hold with a corresponding number c.

When $Q(-1)=0$, a similar result holds.

Let $L_{k,l} = \int_{-1}^{1}(ax+1)^{-k}(1+x)^l \mathrm{d}x$ with $k > l+1$. Then

$$\begin{aligned} L_{k,l+1} &= \int_{-1}^{1}(ax+1)^{-k}(1+x)^{l+1}\mathrm{d}x \\ &= \int_{-1}^{1}(ax+1)^{-k}a^{-1}((ax+1)-(1-a))(1+x)^l\mathrm{d}x \\ &= a^{-1}[L_{k-1,l}-(1-a)L_{k,l}]. \end{aligned}$$

Now, $L_{k,0} = \frac{1}{(k-1)a}[(1-a)^{-k+1}-(1+a)^{-k+1}]$ with $k>1$, which is equivalent to $\frac{1}{(k-1)(1-a)^{k-1}}$ when a turns to 1. We have that

$$L_{k,1} = a^{-1}[L_{k-1,0}-(1-a)L_{k,0}],$$

which is equivalent to $\frac{1!(k-1-2)!}{(k-1)!(1-a)^{k-1-1}}$ when a turns to 1.

Therefore, by our induction formula, we can prove that $L_{k,l}$ is equivalent to

$$\frac{l!(k-l-2)!}{(k-1)!(1-a)^{k-l-1}} \tag{3.7}$$

when a turns to 1.

The major part of the difference between the two sides of equation (3.6) comes from

$$\begin{aligned} &\int_{-1}^{1}(ax+1)^{-2n-1}Q\mathrm{d}x\int_{-1}^{1}(ax+1)^{-2n+2}\Delta\mathrm{d}x \\ &-\int_{-1}^{1}(ax+1)^{-2n}Q\mathrm{d}x\int_{-1}^{1}(ax+1)^{-2n+1}\Delta\mathrm{d}x. \end{aligned}$$

By the formula of Δ near -1 in (2.7) of Lemma 2.7, we only need to check that

$$L_{2n+1,D_0-1}L_{2n-2,D_0-2}-L_{2n,D_0-1}L_{2n-1,D_0-2} \tag{3.8}$$

has a negative major part. By (3.7) it is proportional to

$$\frac{(D_0-1)!(D_0-2)!}{(1-a)^{4n-2D_0}}$$
$$\Big[\frac{(2n+1-D_0+1-2)!(2n-2-D_0+2-2)!}{(2n)!(2n-3)!}-\frac{(2n-D_0+1-2)!(2n-1-D_0+2-2)!}{(2n-1)!(2n-2)!}\Big],$$

and is determined by the sign of

$$\frac{1}{2n(2n-D_0-1)}-\frac{1}{(2n-D_0)(2n-2)};$$

this is the sign of D_0-n, which is negative if $n>D_0$.

This takes care of the cases with $D_0 \neq n$.

When $D_0=n$, then $Q=(1+x)^{n-1}$, and Δ is a proportion of $(1+x)^{n-2}$. In this case, one might easily see that M is $\mathbf{C}P^{n-1}$, by blowing up, and L^0 is a line bundle over $\mathbf{C}P^{n-1}$. The line bundle over $\mathbf{C}P^{n-1}$ can be classified by the Chern class. There is only one line bundle over $\mathbf{C}P^{n-1}$ such that the zero section can be contracted, and the contracted line bundle is just $\mathbf{C}^n$. Therefore, the big manifold is just $\mathbf{C}P^n$. Our Lemma holds.

Here, we give an elementary proof. We only need to take care of the major part of

$$L_{2n+1,n-1}[L_{2n-2,n-2}+A]-L_{2n,n-1}[L_{2n-1,n-2}+B],$$

here, A and B are the constants from the $Q(1)$ terms.

As above, we know that the power of $1-a$, which comes from the major part of $L_{k,l}$, cancels out. Therefore, we also need to take care of the second major part of those $L_{k,l}$'s.

From the induction formula, we see that

$$L_{k,l}=a^{-l}\sum_{s=0}^{l}C_l^s(-1)^s(1-a)^sL_{k-l+s,0}.$$

This can also be obtained by expanding $(1+x)^l$ as a function of $ax+1$.

Each $L_{k-l+s,0}$ term contributes an $(1-a)^s$ term other than the major term. Therefore, the second major term only comes from the constant term in the first term $L_{k-l,0}$.

Therefore, the second major term in $L_{k,l}$ is the second term of

$$a^{-l}L_{k-l,0}=\frac{1}{(k-l-1)a^{l+1}}[(1-a)^{-k+l+1}-(1+a)^{-k+l+1}].$$

Now, we see that the second major part of the difference of the two sides of (3.6) actually comes from the major part of $L_{2n+1,n-1}$, and the sign only depends on the sign of the constant term in

$$\int_{-1}^{1}(ax+1)^{-2n+2}\Delta\mathrm{d}x+(1+a)^{-2n+2}Q(1).$$

Now, $Q=(1+x)^{n-1}$. By Lemma 6, $\Delta=n(n-1)(1+x)^{n-2}$. Therefore, the constant term from $\int_{-1}^{1}(ax+1)^{-2n+2}\Delta\mathrm{d}x$ is

$$-a^{-n+2}n(n-1)(1+a)^{-2n+2+n-2+1}/(n-1)=-a^{-n+2}n(1+a)^{-n+1}.$$

However, $(1+a)^{-2n+2}Q(1)=(1+a)^{-2n+2}2^{n-1}$. Therefore, the sign of the major part is the sign of $-n2^{-n+1}+2^{-n+1}$, which is the sign of $-n+1<0$, since $n>1$, otherwise, the $\mathbf{C}^*$ bundle is trivial and M is homogeneous.

The same argument also works for $a\to-1$ by the symmetric argument. $\square$

We then have

Lemma 3.2 There is a nontrivial solution a for equation (3.6).

Let

$$L(U)=\int_{-1}^{U}2(ax+1)^{-2n-1}[(ax+1)^2\Delta(x)-cQ(x)]\mathrm{d}x+2(1-a)^{-2n+1}Q(-1),\tag{3.9}$$

so $(aU+1)^{-2n+1}Q\varphi=\int_{-1}^{U}L(y)\mathrm{d}y$.

Theorem 3.3 ([cf. [26], [16, Lemma 6]) There is a Maxwell-Einstein metric in the same Kähler class of $\tilde{g}$ if $\phi_0=\Phi/Q$ is positive on $(-1,\ 1)$.

Lemma 3.4 (cf. [15–17]) If r has nonnegative trace eigenvalues, then for a given a, Φ as above is always positive on $(-1,\ 1)$.

Proof Since the derivative of $Q\varphi(aU+1)^{-2n+1}$ is $L(U)$ and is $p(U)(aU+1)^{-2n}$ with $p(U)$ being a polynomial, we have that

$$(aU+1)^{2n+1}\frac{\mathrm{d}}{\mathrm{d}U}(\frac{\mathrm{d}}{\mathrm{d}U}(Q\varphi(aU+1)^{-2n+1}))=2[(aU+1)^2\Delta(U)-cQ(U)].\tag{3.10}$$

Diagonalizing B, we see that Q is a product of polynomials of degree 1. Letting

$$-a_1^{-1} < \cdots < -a_p^{-1} < b_1^{-1} < \cdots < b_q^{-1},$$

denote the distinct roots of Q for which some corresponding Ricci curvature $r_{i\bar{i}}$ is nonzero, where a_i and b_j are positive. When $Ric(g_0) \geq 0$, $\Delta(x) \geq 0$ on $[-1, 1]$. By (3.4), $c > 0$. Let

$$S(U) = U \prod_{i=1}^{p} (1 + a_i U) \prod_{j=1}^{q} (1 - b_j U),$$

$$P(U) = UQ(U)/S(U)$$

and

$$\Psi(U) = \left((aU + 1)^{2n+1} \frac{\mathrm{d}}{\mathrm{d}U} (\frac{\mathrm{d}}{\mathrm{d}U} [(Q\varphi)(U)(aU + 1)^{-2n+1}]) \right) / P(U).$$

Then Ψ is a polynomial of degree $p + q + 1$, and $\Psi(w) = -k_w S'(w)$ for w being a root of $S(U)/U$ if there is no a root w such that $1 + aw = 0$ (that is, $-1/a$ is one of the roots), where $k_w \in \mathbf{R}^+$, since r is nonnegative. We can see that $S'(w) \neq 0$ and that > 0 (or < 0) if and only if $S' < 0$ (or > 0) for the root before w and after w (if such exists). Even if $w = -1/a$ for a w above, we then have that $\Psi(w) = 0$. In that case, $\Psi(U)$ either has a value opposite to the values of those two other roots next to it at a point near w and therefore has two zeros between those two roots, or w is a double root for Ψ. Our proof still works. Because $S'(0) > 0$, we have that $S'(-a_p^{-1}) < 0$ and that $S'(b_1^{-1}) < 0$; that is, $\Psi(-a_p^{-1}) > 0$ and $\Psi(b_1^{-1}) > 0$. Now there are $p - 1$ (or $q - 1$) zero points of Ψ in $(-a_1^{-1}, -a_p^{-1})$ (or in (b_1^{-1}, b_q^{-1})) if p and q are not zero (one may also check the case where $q = 0$ or $p = 0$). If φ has some non-positive points in $(-1, 1)$, then in $(-1, 1)$, $Q\varphi$ has at least two maximal points and one minimal point since $\varphi(-1) = \varphi(1) = 0,\ \varphi(-1 + \epsilon) > 0,\ \varphi(1 - \epsilon) > 0$ for ϵ small enough. Thus, we get that there are at least 4 zero points of Ψ in $(-a_p^{-1}, b_1^{-1})$. Ψ has at least $(p - 1) + (q - 1) + 4 = p + q + 2$ zero points, i.e., $\Psi(U) = 0,\ Q\varphi = (c_1 + c_2 U)(aU + 1)^{2n-1}$. However, $\varphi(-1) = \varphi(1) = 0$, so we have that $Q\varphi = 0$, which is a contradiction. Thus we have the Lemma. □

This, in particular, concludes our Theorem.

Acknowledgements I would like to thank Professor Feng and the School of Mathematics and Statistics, Henan University for their support. I thank Professor Peter Li for his support with [16]. I also thank Professor D. Chen for telling me of Futaki's work [12, 13], which led to this article. Some thanks go to the referees for pointing out the reference [1] to me. After adding [1] to the references, we realized that the authors in [2] are in the same group with those in [1]. The definition of the Einstein-Maxwell metrics in [1] and [2] are basically from [28] and [29], and is quite different from ours. This implies that ours here have a quite different origin. We believe though, that the definitions are equivalent. Recently, we have make some progress on new Hermitian-Einstein metrics which are not conformally Kähler.

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