Extremal-solitons and Exponential $C^\infty$
Convergence of the Modified Calabi Flow on
Certain $\mathbb{C}P^1$ Bundles

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We explain in this paper that, for certain completions of $\mathbb{C}^*$ bundles, the existence of the Calabi extremal metrics are the same as the geodesic stability and prove the exponential $C^\infty$ convergence of the modified Calabi flow whenever the extremal metric exists, i.e., when the Kähler class is geodesically stable, for those cases with hypersurface ends. In particular, we solved the problem of the convergence of the modified Calabi flow on those almost homogeneous manifolds with two hypersurface ends in [Gu2]. As a byproduct, we also found a family of Kähler metrics called extremal-soliton metrics which interpolates the extremal metrics and the generalized quasi-einstein metrics. We also proved the existence of these metrics on compact almost homogeneous manifolds of two ends. For the completions of the $\mathbb{C}^*$ bundles we considered in this paper, we found a functional which we call the generalized Mabuchi functional. The existence of extremal-soliton metrics on these manifolds is again equivalent to the geodesic stability of the Kähler class with respect to this functional.

1 Introduction

In every Kähler class of a compact almost homogeneous manifold with two ends we found a unique Calabi extremal metric in [Gu1,2]. Moreover, we

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found a unique extremal metric in a given Kähler class on a certain completion of $\mathbb{C}^*$ bundle, if the function $\Phi$ of the Kähler class therein is positive. We realized recently in [Gu5] that this is equivalent to the geodesic stability of the Kähler class. A natural question is:

If the Kähler class is geodesically stable, does the modified Calabi flow converge to the extremal metric pointwise?

The answer is yes. We shall deal in this paper with these manifolds with hypersurface ends. We shall treat the case of higher codimension ends later on.

In particular we have:

**THEOREM A.** For any positive integer $k$ the modified Calabi flow exponentially converges in the $C^k$ norm to a unique extremal metric on any compact almost homogeneous manifold with two hypersurface ends.

We shall remove the assumption of hypersurface ends later on in another paper, since to obtain such a result we need more technical tools.

In this paper, we shall explain how the positivity of $\Phi$ in [Gu2] is the same as the geodesic stability, and in this case how natural is the convergence of the modified Calabi flow.

Similar questions for Riemann surfaces has been solved by Chrusciel in [Cr], then later on reproved in [Ce] and [St]. See also [Ca1] and [Ca2] (the second paper dealt with a conformal version of the usual surface Calabi flow, which in general is not the same as the usual Calabi flow), as well as the references therein for related works. Our manifolds are the first examples that are not related to conformal geometry and of higher dimensions.

In the second section, we shall define certain completions of $\mathbb{C}^*$ bundles, and we shall consider and prove the existence of the extremal metrics on them. To interpolate the extremal metrics and those quasi-einstein metrics in [Gu3], which are a kind of Kähler-soliton metrics as a generalization of Ricci-soliton metrics, we define extremal-solitons. A Kähler metric is **extremal** if

$$R - HR = \phi$$

where $R$ is the scalar curvature, $HR$ is the average of the scalar curvature and $\phi$ a potential function of a holomorphic vector field. A Kähler metric is a quasi-einstein metric or a **Kähler-soliton** if

$$R - HR = \Delta \phi$$
where $\Delta \phi$ is the Laplacian of a potential function of a holomorphic vector field. When the Kähler class is the Ricci class or the negative Ricci class we have exactly the Ricci-soliton. In that direction, there are some quite interesting results, see [Ki2], [Gu3], [TZ], [WZ].

A Kähler metric is an **extremal-soliton** if we have

$$R - HR = \phi_1 + \Delta \phi_2$$

with two potential functions $\phi_1, \phi_2$ of holomorphic vector fields.

In section 2 we consider the existence of extremal-solitons, which is a generalization of the results in [Gu2,3].

In particular we have that:

**THEOREM B.** There is a family of extremal-soliton metrics in every Kähler class on a compact almost homogeneous manifolds with two ends, which interpolates the extremal metric and the generalized quasi-einstein metric we obtained before.

Even for extremal metrics and quasi-einstein metrics the results are more general than those in [Gu2] and are relatively new, but the methods are the same and we just use a more general setting. In section 2 the manifolds might not be a $\mathbb{C}P^1$ bundle.

In the third section we shall explain the equivalence between the existence and the geodesic stability. Some detailed calculations could also be found in [Gu5 p.279–280]. We also calculate the modified Mabuchi functional for these manifolds.

Moreover, we defined the generalized Mabuchi functional on the manifolds under consideration and obtained:

**THEOREM C.** There is an extremal-soliton metric in a given Kähler class on our manifold under consideration with respect to two given holomorphic vector fields if and only if the generalized Mabuchi functional is geodesically stable.

In the fourth section we shall deal with the short time existence of the modified Calabi flow for the compact Kähler manifold, we shall apply a linearization method there.

It turns out that there are two kind of curvature flow equations for extremal-soliton metrics. The first one which we used around 1993 is the **modified Ricci flow**

$$\frac{\partial}{\partial t} \log \det g = -R + HR + \phi_1 + \Delta \phi_2$$
where $g$ are the Kähler metrics. This is a quasi-second-order fourth order heat equation. It has fourth order derivatives of the potential function of the Kähler metric, just as those in the equation of the metrics with constant scalar curvatures. However, one might regard the major terms as a heat equation of $\log \frac{\det g}{\det g_0}$, which was the motivation for considering this equation in 1992 (see [Gu3]). However, I could only solve this equation for metrics with a condition, which I was not able to check even now\footnote{I also gave a talk on this matter in 1996 invited by Professor Paul Yang (see also [Gu3]). This flow was recently used by Simanca [Sm] to the extremal metrics. However, there are some very serious mistakes in his paper. For example, Proposition 3.10 therein is not correct. See the end of our last section for the counterexamples on the simplest manifold $CP^1$.}. Although the condition was satisfied for many concrete cases by direct checking, I could not prove it for all of our cases (see section 11). Later on I started look at the modified Calabi flow (see below). Although the modified Ricci flow has a fourth order equation, it behaves more like a second order heat equation. One might apply the maximal principle. However, the Calabi functional could not be decreasing under this flow, e.g., an extremal metric with a nonzero $\phi$ in the Ricci class could achieve the minimal of the Calabi functional while it could not be a stationary metric of the Ricci flow even up to an action of a one parameter group of holomorphic automorphisms (see also the end of our last section).

The second equation is the \textbf{modified Calabi flow}, which is also called the Calabi-Robinson-Trautment equation

$$\frac{\partial}{\partial t} F = R - HR - \phi_1 - \Delta \phi_2,$$

where $F$ are the Kähler potentials. This is a fourth order heat flow equation.

In sections 5 through 10, we shall show the convergence of the modified Calabi flow to the extremal metrics for the special case in the second section for those hypersurface ends. Because of our setting for the problem we have to deal with weighted Sobolev inequalities, which seems a little more complicated than those in [Cr]. We believe that this is the first step toward the similar problem for the toric manifolds that we dealt with in [Gu4]. From our argument one sees that the convergence is very natural, comparing to the more complicated situation for the modified Ricci curvature flow in [Ki1,2] and also [Gu3,6]. We apply our formula of modified Mabuchi functional where the geodesic stability is hidden.

We also use a family of \textit{higher order Calabi functionals} which are essential for our higher order estimates.
In the last section we compare the modified Calabi flow with the modified Ricci flow and explain why we think that the modified Calabi flow is more natural.

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2 Existence of the Extremal-solitons on Certain Completions of Line Bundles

Our results can be regarded as a continuation of [KS1,2], [Ki2], [Gu1,2,3,4,5], which we suppose the readers should get familiar with. We state without detail proof in this section the Lemmas and the Theorem 1 similar to those in these papers as following. The readers might take [Gu2,3,4] as the standard references. Most Lemmas and Theorem 1 can be actually found in [Gu2]. Lemma 2 and 3 could be found in [Gu4].

Let $p: L \to M$ be a holomorphic line bundle over a compact complex Kähler manifold $M$ and $h$ a hermitian metric of $L$. Denote by $L^0$ the open subset $L - \{0\}$ of $L$ and let $s \in C^\infty(L^0)_R$ be defined by $s(l) = \log |l|_h (l \in L^0)$, where $|.|_h$ is the norm defined by $h$. Now we consider a function $\tau = \tau(s) \in C^\infty(L^0)_R$ which depends only on $s$ and is monotone-increasing with respect to $s$.

Let $\tilde{J}$ be the complex structure of $L$ and $J$ be the complex structure of $M$. Now we consider a Riemannian metric on $L^0$ of the form

$$\tilde{g} = d\tau^2 + (d\tau \circ \tilde{J})^2 + g$$

where $g(l) = p^* g_{\tau(s(l))}(m)$ with $m = p(l) \in M$, $g_\tau$ is a one parameter family of Riemannian metrics on $M$. Define a positive function $u$ on $L^0$ depending only on $\tau$ by $u(\tau)^2 = \tilde{g}(H, H)$, where $H$ be the real vector field on $L^0$ corresponding to the $\mathbb{R}^*$ action on $L^0$.

**Lemma 1.** (Cf [KS1,2], [Gu2 p.2257]) Suppose that the range of $\tau$ contains 0. Then $\tilde{g}$ is Kähler if and only if $g_0$ is Kähler and $g_\tau = g_0 - UB$, where $B$ is the curvature of $L$ with respect to $h$, $U = \int_0^\tau u(\tau)d\tau$. 


Throughout this paper, we assume that

(1) \( \hat{L} \) is a compactification of \( L^0 \) and \( \tilde{g} \) is the restriction of a Kähler metric of \( \hat{L} \) to \( L^0 \).

(2) the range of \( \tau \) contains 0 and

(3) the eigenvalues of \( B \) with respect to \( g_\tau \) are constants on \( M \).

(4) the traces of the Ricci curvature \( r \) of \( g \) on each eigenvector space of \( B \) are constant.

The condition (4) here is much more general than that in [Gu1,2] in which we have:

(4)’ the eigenvalues of \( r \) are constants.

Our results cover some results which appeared in recent years. For example, if \( g \) has a constant scalar curvature and \( B \) has only one eigenvalue.

By abusing the language, we call the constants in (4) the \textit{trace eigenvalues}.

Let \( (z^1, \ldots, z^n) \) be a system of holomorphic local coordinates on \( M \). \( n = \dim_{\mathbb{C}} M \). Using a trivialization of \( L^0 \), we take a system of holomorphic local coordinates \( (z^0, \ldots, z^n) \) on \( L^0 \) such that \( \frac{\partial}{\partial z^0} = H^p \uparrow \mathcal{J}H \).

Here we notice that \( z^0 \) is corresponding to \( w_1 \) in [Gu4 p.552]. \( s \) can be regarded as \( \text{Re}(z^0) \) near the considered point. So \( s \) is the \( x_1 \) in [Gu4 p.552]. As in [Gu2] we let \( \varphi = u^2 \) as a function of \( U \) and we let \( F \) be the Kähler potential as in [Gu4 p.552], then by comparing [Gu2 Lemma 2] (or the Lemma 4 below) with [Gu4 p.552] we immediately have\(^3\)

\[
\frac{\partial^2 F}{\partial s^2} = \tilde{g}_{00} = 2\varphi.
\]

**Lemma 2.** \( 2\varphi = \frac{\partial^2 F}{\partial s^2} \).

From \( H = 2^{-1}\frac{\partial}{\partial s}, \frac{1}{4}(\frac{\partial}{\partial s})^2 = \varphi \), we obtain \( \frac{\partial}{\partial s} = 2u \).

\[
U = \int_0^\tau u\,d\tau = \int_{s(0)}^s 2u^2\,ds = \int_{s(0)}^s \frac{\partial^2 F}{\partial s^2}ds
\]

is \( \frac{\partial F}{\partial s} = y_1 \) up to a constant in [Gu4 p.552], i.e.,

**Lemma 3.** \( U \) is the Legendre transformation of \( s \).

Here we use the Legendre transformation in [Gu4] instead of the moment map in [Gu2] since we need the new insight in the later sections.

\(^3\)The \( F \) we used in [Gu4] is the \( \frac{1}{4} \) of the usual potential function in the Kähler geometry. The difference might cause a constant factor in the calculations, e.g., for Lemma 2 and the Calabi flow equation, but does not affect our conclusions.
Remark 1. We shall see in [Gu7, 8, 9] that the function $U$ here, the Legendre transformation in [Gu4] and the miracle function $U$ in [GC, Gu5, Gu8] are special cases of the parallel coordinates along the curves in the Mabuchi moduli space of Kähler metrics on compact almost homogeneous manifolds with actions of reductive groups.

Let $X_i, X_i^0 (0 \leq i \leq n)$ be the partial derivatives $\partial/\partial z^i, \partial/\partial \bar{z}^i$ on $L^0$ and $X_i, X_i^1 (1 \leq i \leq n)$ be the partial derivatives $\partial/\partial z^i, \partial/\partial \bar{z}^i$ on $M$.

Lemma 4. (Cf [KS1,2], [Gu2 Lemma 2]) We have
\[ g_{00} = 2u^2, \quad g_{0i} = 2uX_i^0, \quad g_{ij} = g_{ij} + 2X_i \cdot X_j \tau \] (2)
where $1 \leq i, j \leq n$. At the point $P \in L^0$ considered, we can choose a local coordinate system around $m = p(P) \in M$ such that $(\partial/\partial \tau) \tau = 0$ at $m$, so $X_i \tau = X_j \tau = 0$ at $P$, then if $f$ is a function on $L^0$ depending only on $\tau$, we have
\[ \dot{X}_0 \dot{X}_0 f = u \frac{df}{d\tau} \quad \dot{X}_i \dot{X}_0 f = 0 \]
\[ \dot{X}_i \dot{X}_j f = -\frac{1}{2} u B_{ij} \frac{df}{d\tau} \] (3)
The Ricci curvature at this point is
\[ \ddot{\delta}_{00} = -u \frac{d}{d\tau} \left( u \frac{d}{d\tau} \log(u^2 Q) \right) \quad \ddot{\delta}_{0i} = 0 \]
\[ \ddot{\delta}_{ij} = p^r r_{0i} i_j + \frac{1}{2} u \frac{d}{d\tau} \log(u^2 Q) \cdot B_{ij} \] (4)
where $Q = \det(g_{0i}^{-1} \cdot g_{r})$. In particular, we have the scalar curvature
\[ \dddot{R} = \frac{\Delta}{Q} - \frac{1}{4Q} \frac{d}{d\tau} \left( \frac{d}{d\tau} Q \varphi \right) \] (5)
where $\varphi = u^2$ as a function of $U$ and $\Delta(U) = Q \sum_{i,j} r_{0i} i_j g_{r}(U)$. We also have $\varphi'(\min U) = 2, \varphi'(\max U) = -2$.

Lemma 5. (Cf. [FMS], [Mb], [Gu2 Lemma 3]) We can also regard $U$ as a moment map corresponding to $(\tilde{g}, \tilde{JH})$ and $g_T$ just be the symplectic reduction of $\tilde{g}$ at $T(\tau)$. $\tilde{g}$ is extremal if and only if $\dddot{R} = a + bU$ for some $a, b \in \mathbb{R}$.

Let $M_0 = U^{-1}(\min U)$ and $M_\infty = U^{-1}(\max U)$, they are complex submanifolds, since they are components of the fixed point set of $H - \sqrt{-1}JH$. 

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which is semisimple. Let $D_0$ be the codimension of $M_0$ in $\hat{L}$, $D_\infty$ be the codimension of $M_\infty$ in $\hat{L}$.

**Lemma 6.** (Cf. [Gu2 Lemma 4]) Suppose that there is another Kähler metrics $\tilde{g}^\vee$ on $\hat{L}$ in the same Kähler class which is of form (1) on $L^0$. Let $\tau^\vee$, $g^\vee$, $U^\vee$, $Q^\vee$, $\varphi^\vee$, $u^\vee$ be the corresponding metric and the corresponding functions of $s$. Then there is a unique corresponding $\tau^\vee$ such that $g_0^\vee = g_0$. In this case, $\min U^\vee = \min U$ (or $\max U^\vee = \max U$) and $Q^\vee = Q$, $\Delta^\vee = \Delta$ hold. So we may write $D = \max U$ and $-d = \min U$. Then

$$Q(U) = (1 + \frac{L}{d})D_0^{-1}Q_{-d}$$

(or $Q(U) = (1 - \frac{L}{D})D_\infty^{-1}Q_D$),

where $Q_{-d}$ (or $Q_D$) is a polynomial of $U$ such that $Q_{-d}(-d) \neq 0$ (or $Q_D(D) \neq 0$) and

$$\Delta(U) = D_0(D_0 - 1)\frac{1}{2}(1 + \frac{L}{d})D_0^{-2}Q_{-d} \pmod{(1 + \frac{L}{d})D_0^{-1}}$$

(or $\Delta(U) = D_\infty(D_\infty - 1)\frac{1}{2}(1 - \frac{L}{D})D_\infty^{-2}Q_D \pmod{(1 - \frac{L}{D})D_\infty^{-1}}$). \hfill (6)

Proof: Let $\tilde{g} - \tilde{g}^\vee = i\bar{\partial}\partial\phi$, then

$$\tilde{g}_{ij}^\vee = \tilde{g}_{ij} + \frac{1}{2}u\frac{d\phi}{d\tau}B_{ij} = (g_0)_{ij} - (U - \frac{1}{2}u\frac{d\phi}{d\tau})B_{ij}$$

for $1 \leq i, j \leq n$, so at $\min U$ (or $\max U$) $\tilde{g}_{ij} = \tilde{g}_{ij}^\vee$, therefore there is a $\tau_0$ such that $g_{\tau_0}^\vee = g_0$. By choosing $\tau$ such that $\tau^\vee(\tau_0) = 0$, one sees that $\min U^\vee = \min U$, $\max U^\vee = \max U$ as desired.

The last statement follows from the fact that the scalar curvature $\tilde{R}$ is finite on both $M_0$ and $M_\infty$. \quad Q. E. D.

We need normalization in this paper. By rescaling we have:

**Lemma 7.** (Cf. [Gu3 Lemma 5]) For any given $a_1 \in \mathbb{R}$ $\tilde{g}$ is extremal-soliton if and only if $\tilde{g}^\vee = a_1^2\tilde{g}$ is, and we can choose $U^\vee = a_1^2U + a_2$ for any $a_2 \in \mathbb{R}$. So we may assume $\max U - \min U = 2$ and $\min U = -1$, then $\max U = 1$. 

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For example, if $\hat{L} = C\mathbb{P}^{n+1}$, then $M_0$ is a point, $M_{\infty} = M = C\mathbb{P}^n$. In this case $\hat{L}$ is the one point completion (compactification) of the hyperplane line bundle $L$ over $M$ with $M_{\infty}$ as the zero section. The anticanonical line bundle is $(n + 1)L$. Therefore $r_{0,ii} = n + 1$ and $Q = (1 + U)^n$. We have that the Kähler metric at $U = 0$ is the curvature of $L$ and therefore $\Delta = n(n + 1)(1 + U)^{n-1}$.

From Lemma 5, it can be seen that if $\tilde{g}$ is an extremal-soliton metric, then

\[ \tilde{R} = a + bU + c\Delta U \]  \hspace{1cm} (9)

for some $a, b, c \in \mathbb{R}$ with $\phi_2 = cU + d$.

By Lemma 4 we have

\[ \tilde{\Delta} f = \tilde{g}^{\alpha\beta} \hat{X}_\alpha \hat{X}_\beta f \]
\[ = \tilde{g}^{00} \hat{X}_0 \hat{X}_0 f + \tilde{g}^{\alpha0} \hat{X}_\alpha \hat{X}_0 f + \tilde{g}^{0\alpha} \hat{X}_0 \hat{X}_\alpha f + \tilde{g}^{\alpha\beta} \hat{X}_\alpha \hat{X}_\beta f \]
\[ = \frac{1}{2u^2} (\hat{X}_0 \hat{X}_0 f) + 0 + 0 + g^{\alpha\beta} (\hat{X}_\alpha \hat{X}_\beta f) \]  \hspace{1cm} (10)
\[ = \frac{1}{2u^2} u \frac{d}{dt} (u \frac{d}{dt} f) + g^{\alpha\beta} (-2^{-1} u \frac{df}{dt} b\alpha) \]
\[ = 2^{-1} \frac{d}{dU} (\varphi(U) \frac{d}{dU} f) - 2^{-1} \varphi(U) \frac{d}{dU} f) g^{\alpha\beta} b\alpha \]
\[ = 2^{-1} \frac{d}{dU} (\varphi \frac{d}{dU} f) + 2^{-1} \varphi \frac{d}{dU} f) \frac{1}{Q} \frac{d}{dU} \]
\[ = \frac{1}{2Q} \frac{d}{dU} (\varphi Q \frac{d}{dU} f) \]

we get

\[ \tilde{\Delta} \phi_2 = \frac{1}{2Q} \frac{d}{dU} (\varphi Q \frac{d}{dU} (cU + d)) = \frac{c}{2Q} \frac{d}{dU} (\varphi Q) \]
\[ = \tilde{R} - (\int_{-1}^{1} \tilde{R} Q dU) / (\int_{-1}^{1} Q dU) \]
\[ = \frac{\Delta}{Q} - \frac{1}{2Q} \frac{d}{dU} (\frac{d}{dU} Q \varphi) - (a + bU) \]  \hspace{1cm} (11)

Let $m = \int_{-1}^{1} \tilde{R} Q dU / \int_{-1}^{1} Q dU$, $\alpha = \int_{-1}^{1} Q dU$, $\beta = \int_{-1}^{1} U Q dU$. We have

\[ \int_{-1}^{1} \tilde{R} Q dU = \int_{-1}^{1} [\Delta - 2^{-1} \frac{d}{dU} (\frac{d}{dU} Q \varphi)] dU \]
\[ \begin{align*}
&= \delta - 2^{-1} \frac{d}{dU}(Q\varphi)|_{-1}^{1} \\
&= \delta - 2^{-1}[Q \frac{d}{dU} \varphi + \varphi \frac{d}{dU} Q]|_{-1}^{1} \\
&= \delta - 2^{-1}[Q(1) \cdot (-2) - Q(-1) \cdot 2] \\
&= \delta + Q(1) + Q(-1)
\end{align*} \]  

(12)

where \( \delta = \int_{-1}^{1} \Delta dU \). Therefore, \( m = (\delta + Q(-1) + Q(1))/\alpha \).

We obtain

\[ c\varphi Q = -\frac{d}{dU}(Q\varphi) - 2 \int_{-1}^{U} (a + bx)Q(x)dx + 2 \int_{-1}^{U} \Delta(x)dx + c_1. \]  

(13)

Let \( U = -1 \), we have \( 0 = -2Q(-1) - 0 - 0 + c_1 \), i.e., \( c_1 = 2Q(-1) \).

Let \( U = 1 \) we have \( 0 = 2Q(1) - 2a\alpha - 2b\beta + 2\delta + 2Q(-1) \). Therefore, \( a = \frac{m - b\beta}{\alpha} \).

Then

\[ Q\varphi = e^{-cU} \left[ \int_{-1}^{U} [\int_{-1}^{U} [-2(a + bx)Q(x) + 2\Delta(x)]dx + 2Q(-1)]e^{cy}dy + f \right] \]  

(14)

with a constant \( f \). We denote the right side by \( \Phi(U) \).

Let \( U = -1 \), we have \( 0 = e^{\alpha}[0 + f] \), i.e., \( f = 0 \). And at \( U = 1 \) we have the equation

\[ 0 = \int_{-1}^{1} [\int_{-1}^{U} [-2(a + bx)Q(x) + 2\Delta(x)]dx + 2Q(-1)]e^{cy}dy \]  

(15)

Let

\[ p(U) = \int_{-1}^{U} 2[\Delta(x) - (a + bx)Q(x)]dx + 2Q(-1), \]  

(16)

then

\[ p(U) = \int_{U}^{1} 2[(a + bx)Q(x) - \Delta(x)]dx - 2Q(1) \]  

(17)

from (13). By the last statement of Lemma 6 we know that \( p(U) \) is nonnegative near \(-1\) and nonpositive near \(1\). So the right side of the above equation (15) turns to \(-\infty\) (or \(+\infty\)) when \( c \) turns to \(+\infty\) (or \(-\infty\), there is at least one solution \( c \). We pick up the smallest one, and have

**Lemma 8.** For any \( b \), there is a solution \( c \) for (15).
Theorem 1. (Cf. [KS1], [Gu2 Lemma 6]) There is an extremal-soliton metric in the Kähler class of $\tilde{g}$ for a given $b$ if $\varphi^0 = \Phi/(Qe^{cU})$ is positive on $(-1, 1)$.

If we let $b = 0$ we have the (generalized) quasi-einstein metric as in [Gu3].

To obtain an extremal metric we just let $c = 0$ and solve (15) to find $a$ and $b$ as we did in [Gu2 p.2259] (see Lemma 5 there). Let $c = 0$, $\delta_1 = \int_{-1}^{1} x\Delta dx$, $\gamma = \int_{-1}^{1} x^2 Qdx$, then (15) becomes

$$0 = \int_{-1}^{1} \int_{-1}^{1} ((a + bx)Q(x) - \Delta(x))dxdy - Q(-1)$$

$$= \int_{-1}^{1} \int_{-1}^{1} ((a + bx)Q(x) - \Delta(x))dydx - Q(-1)$$

$$= \int_{-1}^{1} (1 - x)((a + bx)Q - \Delta)dx - Q(-1)$$

$$= a\alpha + b\beta - a\beta - b\gamma - \delta + \delta_1 - Q(-1)$$

$$= m\alpha + \delta_1 - Q(-1) - a\beta - b\gamma - \delta$$

$$= m\alpha + \delta_1 - Q(-1) - m\beta + \frac{b}{\alpha}(\beta^2 - \alpha\gamma) - \delta.$$ 

We notice that the coefficient of $b$ can not be zero since

$$\alpha t^2 + 2\beta t + \gamma = \int_{-1}^{1} (t + U)^2 QdU > 0$$

for any $t$. Therefore, there is a unique solution of $b$.

Lemma 9. (Cf. [Gu1,2,3]) if $r$ has nonnegative trace eigenvalues, then for a given $b$ we have: (1) $\Phi$ above is always positive on $(-1, 1)$. (2) the solution $c$ in Lemma 8 is unique.

Proof: We first consider the case in which $r$ has no negative trace eigenvalue and consider a more relax condition in Corollary 2. Since the derivative of $Q\varphi e^{cU}$ is $p(U)e^{cU}$, we have

$$\frac{d}{dU}(e^{-cU} \frac{d}{dU}(Q\varphi e^{cU})) = 2\Delta(U) - 2(a + bU)Q(U).$$

(18)

Diagonalizing $B$, we see that $Q$ is a product of polynomials of degree 1. Let

$$-a_1^{-1} < ... < -a_p^{-1} < b_1^{-1} < ... < b_q^{-1}$$
denote the distinct roots of $Q$ for which some corresponding Ricci curvature $r_{ij}$ is nonzero, where $a_i, b_j$ are positive. Let

$$S(U) = U \prod_{i=1}^{p} (1 + a_i U) \prod_{j=1}^{q} (1 - b_j U),$$

$$P(U) = UQ(U)/S(U)$$

and

$$\Psi(U) = \left( \frac{d}{dU} (e^{-ct} \frac{d}{dU} [(Q \varphi(U)e^{ct})]) \right) / P(U).$$

Then $\Psi$ is a polynomial of degree $p + q$ and $\Psi(a) = -k_a S'(a)$ for $a$ a root of $S(U)/U$, where $k_a \in \mathbb{R}^+$ since $r$ is nonnegative. We can see that $S'(a) \neq 0$ and $> 0$ (or $< 0$) if and only if $S' < 0$ (or $> 0$) for the root before $a$ and after $a$ (if there exists). And $S'(0) > 0$, so $S'(-a_p^{-1}) < 0$, $S'(b_1^{-1}) < 0$, that is $\Psi(-a_p^{-1}) > 0$, $\Psi(b_1^{-1}) > 0$. Now there are $p - 1$ (or $q - 1$) zero points of $\Psi$ in $(-a_1^{-1}, -a_p^{-1})$ (or in $(b_1^{-1}, b_q^{-1})$) if $p, q$ are not zero (one may also check the case of $q = 0$ or $p = 0$). If $\varphi$ has some nonpositive points in $(-1, 1)$, then in $(-1, 1)$, $Q \varphi$ has at least two maximal and one minimal points since $\varphi(-1) = \varphi(1) = 0$, $\varphi(-1 + \epsilon) > 0$, $\varphi(1 - \epsilon) > 0$ for $\epsilon$ small enough. So we get that there are at least 4 zero points of $\Psi$ in $(-a_p^{-1}, b_1^{-1})$. $\Psi$ has at least $(p-1)+(q-1)+4 = p+q+2$ zero points, i. e., $\Psi(U) = 0$, $Q \varphi = c_1 + c_2 e^{-ct}$. But $\varphi(-1) = \varphi(1) = 0$, we have $Q \varphi = 0$, a contradiction, we have (1).

For (2) we only need to prove that the function $p(U)$ in (16) only has one zero point in $(-1, 1)$. If $p(U)$ has at least two zero points in $(-1, 1)$, then $p(U)$ has at least three zero points since it is nonnegative near $-1$ and nonpositive near $1$. So $\Psi$ has at least 4 zero points in $(-a_p^{-1}, b_1^{-1})$, a contradiction.

Q. E. D.

Corollary 1. (Cf. [Gu2,3]) For every Kähler class of a compact almost homogeneous manifold with two ends, there exists an extremal-soliton metric for any given $b$. In particular, there is always an extremal metric and a (generalized) quasi-Einstein metric.

Proof: Since every compact Kähler almost homogeneous space is a completion of a $\mathbf{C}^*$ bundle over a product of a torus $A$ and a $\mathbf{C}$-space $N$ with two homogeneous Kähler spaces as two ends ([IHS] Theorem 3.2), a maximal compact subgroup of the identity component of the automorphism group of this manifold is $G = A \times S \times S^1$, where $A$ is also the Albanese torus, $S$ is a maximal compact subgroup of the identity component of the automorphism
group of $N$. For any Kähler metric $g$, $g_G = \int_{h \in G} h^* g dm$ is a Kähler metric of form (1), where $m$ is the Haar measure on $G$; it is invariant under $G$. And also the Ricci curvature of $A \times N$ is nonnegative, the condition in our assumption follows from the well-known property of the invariant cohomology (1,1) classes for these manifolds (see [DG] p. 326 proof of the Proposition, for example).

Q. E. D.

In the case of $b = 0$ we can have a little bit more. We say that the trace eigenvalues (see three lines under the condition (4)') is nonnegative at one side if the trace eigenvalues are nonnegative for all $-a_i^{-1}$ or for all $b_j^{-1}$ in the proof of the Lemma 9. We have:

**Corollary 2.** (Cf. [Gu2,3]) *If the trace eigenvalues only change sign once and nonnegative at one side, then there is a (generalized) quasi-einstein metric. In particular, the completion of the Hodge line bundle over a Hodge manifold with a constant scalar curvature admits (generalized) quasi-einstein metric.*

Proof: In the proof the Lemma 9, if $b = 0$ the polynomial $p'$ is one degree lower. Therefore by ignore the root at which the trace eigenvalue is negative and change sign, the proof still goes through. By the argument in the proof of the Theorem 5.4 in [KS1] p. 177 we have our Corollary.

Q. E. D.

### 3 Geodesic Stability

For any Kähler manifold $X$ with a given Kähler form $\omega_0$, any Kähler form in the same Kähler class has the form

$$\omega = \omega_0 + i\partial\bar\partial f.$$ 

Therefore, a curve of Kähler forms corresponds to a family of functions $f_t$. The tangent corresponds to $\dot{f}$ which is also a function on $X$. The Mabuchi metric is

$$g_K(f_1, f_2) = \int_X f_1 f_2 \omega^n.$$ (19)

The Mabuchi metric to be considered as an infinite version of Riemann metric, in [Gu5 p.279–280] we found that on the manifolds we considered in last section the existence of the extremal metric is equivalent to certain stability, which we called the *geodesic stability*, of the Kähler class.
In [Gu4], we found that the geodesics of the Mabuchi metric come from the linear pathes of the Legendre transformation of the pair \((F, s)\). The Legendre transformation of \((F, s)\) is \((G, U)\) where \(G(s) = sF_s - F\) which can be considered as a function of \(U\).

For a family of Kähler metrics in a given Kähler class, we consider \(G\) as a function of \(U\) and a time \(t\). Let \(\dot{G}(t, U)\) be the partial derivative to \(t\) and \(\ddot{G}(t, U)\) be the second partial derivative to \(T\), then:

**Lemma 10.** (Cf. [Gu4 p.552]) The geodesic equation is \(\ddot{G}(t, U) = 0\).

Under the Legendre transformation, we always have \(\frac{\partial F}{\partial t}(t, s) = -\dot{G}(t, U)\). Since we have \(2\varphi = \frac{\partial^2 F}{\partial x^2} = (\frac{\partial^2 G}{\partial U^2})^{-1}\) (see Lemma 2), on the geodesic \((\varphi)^{-1} = 2\frac{\partial^2 G}{\partial U^2}\) is linear.

In [GC p.819] we see that the modified Calabi flow is the gradient flow of the modified Mabuchi functional

\[
M(\omega_0, \omega_1) = -\int_a^b \int_X \left( R - HR - \phi_E \right) \omega^n dt,
\]

where \(\phi_E\) is the function corresponding to the extremal vector field \(E\) in [FM]. In our case \(\phi_E = a + bU - HR\) for the \(a, b\) in (9) with \(c = 0\).

For a Ricci-soliton metric, Tian and Zhu obtain a modified Futaki invariant

\[
F_E(Y) = \int_X Y(h - \phi_E) e^{\phi_E} \omega^n
\]

where \(h\) satisfies \(\text{Ricci}(\omega) - \omega = \partial \bar{\partial} h\). This actually comes from a generalized Mabuchi functional

\[
M_E(\omega_0, \omega_1) = \int_0^1 \int_X (\nabla \hat{f}, \nabla (h - \phi_E)) e^{\phi_E} \omega^n dt.
\]

In this paper we shall use a generalized Mabuchi functional with two given potential functions \(\phi_1, \phi_2\) of holomorphic vector fields \(E_1, E_2\):

\[
M_{E_1, E_2}(\omega_0, \omega_1) = \int_0^1 \int_X (\nabla \hat{f}, \nabla (\gamma - \phi_2)) e^{\phi_2} \omega^n dt
\]

where \(\gamma\) satisfies \(R - HR - \phi_1 = \Delta \gamma\).

For the case of an extremal metric, in [Gu5 p.280] we already calculate the derivative of the modified Mabuchi functional along a geodesic. Therefore we shall have:
Lemma 11. The generalized Mabuchi functional is independent of the path which we choose and is convex along the geodesics. Moreover, if we formally let

$$\frac{1}{2} M_{E_1, E_2}(\varphi) = \int_{-1}^{1} (G - 1 - \ln G) e^{\phi_2} Q dU,$$

(21)

where $G = \frac{\psi^0}{\varphi}$ (see Theorem 1 for the definition of $\varphi^0$), then

$$M_{E_1, E_2}(\omega_0, \omega_1) = M_{E_1, E_2}(\varphi_1) - M_{E_1, E_2}(\varphi_0).$$

Proof: First we assume the existence of the extremal-soliton and deal with the extremal metric case first. Since the derivative of the modified Mabuchi functional along the geodesic is (see [Gu5 p. 280])

$$\int_{-1}^{1} h'' Q (\varphi^0 - \varphi) dU$$

where $\varphi = \frac{\psi^0}{(\varphi^0)^{-1} + th''}$. Integrating by $t$, we obtain our formula.

Now in general if there is an extremal-soliton, by

$$R - HR - \phi_1 = \frac{\Delta}{Q} - \frac{1}{2Q} (\varphi Q)'' - a - bU$$

we have

$$\int_X (\nabla \tilde{J}, \nabla (\gamma - \phi_2)) e^{\phi_2} \omega^n$$

$$= \int (\tilde{J}')(\varphi Q)(\frac{1}{\varphi Q}(\int_{-1}^{U}(2\Delta - 2(a + bU)Q)dU - (\varphi Q)' - c\varphi Q)e^{cU}dU$$

$$= \int (\tilde{G})'(\varphi Q e^{cU})'(\varphi Q e^{(cU)'}dU$$

$$= \int (\tilde{G})''((\varphi Q e^{cU})' - (\varphi^0 Q e^{cU})')dU$$

$$= -\int (\tilde{G})''(\varphi - \varphi^0) Q e^{cU}dU$$

$$= \frac{1}{2} \int \frac{\partial}{\partial t}(\varphi^{-1})(\varphi - \varphi^0) Q e^{cU}dU$$

$$= \frac{1}{2} \int \varphi^{-2}(\varphi - \varphi^0) Q e^{cU}dU.$$

Now we integrate with $t$ and have our formula.
The formula shows that it is independent on the path which we choose. And the second derivative of the generalized Mabuchi functional is
\[ \int \left( \frac{\partial}{\partial t} (\varphi^{-1}) \right)^2 \varphi^2 Q e^{\varphi} dU > 0. \]
We have our lemma.

Q. E. D.

A consequence of above Lemma is:

**Corollary 3.** For any given \( E_1, E_2 \) there is at most one extremal-soliton metric of the given form.

For the case of extremal metric, we notice that the formula of [Gu5 p. 280] is true even if the \( \varphi^0 \) in the Theorem 1 is not positive everywhere on \((-1, 1)\). In general we have a similar formula

\[ \int_{-1}^{1} h''(\varphi^0 - \varphi) Q e^{\varphi} dU \]

for the slope of the generalized Mabuchi functional from our above calculation by applying that \( h = \hat{G} \) and \( \hat{G}'' = h'' \). If we fix a metric with a given function \( \varphi_0 \), along the geodesic connecting \( \varphi_0 \) and \( \varphi \) we have \( \varphi = ((\varphi_0)^{-1} + th'')^{-1} \). The geodesic ray can not be infinite if \( h'' < 0 \) at some point and the limit of the slope is positive infinity.

If \( h'' \geq 0 \) then we have an infinite geodesic ray with \( \int_{-1}^{1} h'' \varphi^0 Q e^{\varphi} dU \) as the limit of the slope, which is finite. We call it the generalized Futaki invariant \( F(h) \) along this geodesic ray. Regarding it as a functional of \( h'' \) we see that \( \varphi^0 \) is positive on \((-1, 1)\) if and only if \( F(h) \geq C \int_{-1}^{1} h'' |\varphi_0 Q dU \) for some positive number \( C \) and all \( h \) with nonnegative \( h'' \). We call the last condition the **geodesic stability**. Therefore, the existence of the extremal metrics is the same as the geodesic stability. Similar results are also true for extremal solitons.

Here we like to explain the stability a little bit more. If the solution \( \varphi^0 \) exist, then the norm \( \int_{-1}^{1} |h''|\varphi^0 Q dU \) is equivalent to the norm \( \int_{-1}^{1} |h''|(1 + U)^{D_0}(1 - U)^{D_1} dU \).

Therefore, we have:

**Theorem 2.** (Cf. [Gu5 p. 280]) For the manifolds in the last section, a Kähler class has an extremal-soliton metric with two given holomorphic vector fields if and only if the given Kähler class has the geodesic stability.

**Remark 2.** With the formula in [Gu5 pp. 279] we can also obtain a formula for the modified Mabuchi functional without refering to \( \varphi^0 \) by integrating as we did in the proof of our Lemma 11. By a direct method of
finding the minimal of the functional we can get another method of finding \( \varphi^0 \).

Remark 3. Combining Theorem 2 with the series of papers [Gu8, 9] we proved that a Kähler class on a compact almost homogeneous manifold of cohomogeneity one admits a unique extremal-soliton metric with two given holomorphic vector fields if and only if the given Kähler class is geodesic stable.

4 General Result on Short Time Existence

The modified Calabi flow is (Cf. [GC p.820])

\[
\dot{f} = -\Delta \log \det(g) - HR - \phi_1 - \Delta \phi_2,
\]

where \( \phi_1, \phi_2 \) are the functions corresponding to holomorphic vector fields \( E_1, E_2 \). After changing the Kähler metric by a function \( v \), the function \( \phi_1, \phi_2 \) are changed by \( E_1(v), E_2(v) \) (Cf. [FM]). Since we always consider the case in which the metrics are invariant under a maximal compact group of the holomorphic automorphism group and \( JE_1, JE_2 \) are Killing vector fields, the potentials of \( E_1 \) and \( E_2 \) are always real. We have \( E_k(v) = \frac{1}{2}(E_k(v) + E_k(v)) \), \( k = 1, 2 \). The linearization is

\[
\dot{v} = -\Delta^2 v - v_{ij}(R^{ji} - \phi_2^{ji}) - \frac{1}{2}(E_1^i v_i + E_1^i v_i) - DE_2(v),
\]

where the indices \( i \) corresponds to \( z_i \) in the local holomorphic coordinates \((z_1, \cdots, z_i, \cdots, z_n)\). Multiply by \( v \) and observe that the functions \( E_1^i, E_2^i, R^{ij}, \phi_2^{ij} \) as well as the volume are smooth we obtain:

\[
\frac{d}{dt} \int v^2 dV \leq -\int (\Delta v)^2 dV + C_1 \int v^2 dV + C_2 \int |v v_{ij}| dV + C_3 \int |v_i| dV + C_4 \int \Delta v |v_i| dV
\]

\[
\leq -(1 - \epsilon) \int (\Delta v)^2 dV + C_5 \int v^2 dV + C_6(\int (\Delta v)^2 dV)^{\frac{1}{2}} \left( \int v^2 dV \right)^{\frac{1}{2}}
\]

\[
\leq -(1 - \epsilon) \int (\Delta v)^2 dV + C_5 \int v^2 dV + C_6 \left( \int (\Delta v)^2 dV \right)^{\frac{1}{2}} \left( \int v^2 dV \right)^{\frac{1}{2}}
\]

\[
\leq -(1 - 3\epsilon) \int (\Delta v)^2 dV + C \int v^2 dV \leq C \int v^2 dV
\]
for some constants $C_1, C_2, C_3, C_4, C_5, C_6, C$ and $\epsilon$ which can be chosen as small as we want. Here we apply [Ab p.93 Theorem 3.69] to the last two terms and the formula

$$\int |v_{ij}|^2 = -\int v_i v_{ij\bar{j}}$$

$$= -\int v_i v_{j\bar{j}} = -\int \bar{v}_i v_{j\bar{j}}$$

$$= \int \bar{v}_i v_{j\bar{j}} = \int (\Delta v)^2$$

to the third term. We also apply repeatedly the Young’s inequality $ab \leq ca^2 + \frac{b^2}{c}$ by choosing suitable $c$’s which are related to $\epsilon$.

Let $A = \int v^2 dV$, then $\dot{A} \leq CA$. And hence $\frac{d}{dt} (e^{-Ct} A) \leq 0$. Therefore, if $A(0) = 0$ then $A = 0$. This will allow us to use the argument in [Kb p.212]. To do that we only need to replace the Theorem 5.9 in [Kb p.212] by [HP Theorem 7.9] (see also Theorem 7.14 therein, this result was hidden in Struwe’s argument in [St p.255]). The difference between the two norms used in these references does not make much different for a finite time as it was explained in [HP] lines 23 to 25 of page 76. We initially applied our method here to [Gu7] (see our last section for example) and in that case Theorem 5.9 in [Kb p.212] was enough to apply.

We can also apply a different kind of linearization

$$\partial_t v = -\Delta^2 v + B(t, z, g)$$

with $B$ only related to the third derivatives of the given family of metrics $g$ such that we can get our equation back if we let $\partial \partial \partial \partial = g - g_0$. Above argument shows that $A = \int v^2 dV$ is bounded for a short time and if $A(0) < C_1$, then $A < C_1$ for sufficient short time interval $[0, T]$. In the same way, by multiplying $\Delta^2 v$ we can have $\Delta v$ is bounded in $L^2$, and so is $\nabla^2 v$. Similarly, by assuming good initial value conditions we can get also the higher order estimates. See [HP p.76]. This enable us to apply the linearization and contraction method for our original equation (22) (Cf. [St p.255]). We can do it as follows. Take the initial metric as the given family metrics $g_0(t) = g_0$ for the linearized equation we can get a solution of a family of metrics $g_1(t)$ in short time. We then use the new family $g_1$ as the given family of metrics in the linearized equation and get a solution of another family $g_2(t)$ of metrics. We apply the Newton method by iterating above argument and obtain $g_i(t)$. We can claim that the map from $g_{i+1} - g_i$ to $g_{i+2} - g_{i+1}$ is a contraction. More precisely, given any metric $g(t)$ we have a
family of metrics \( h(t) = F(g) \) such \( h(t) - g(t) \) is corresponding to the solution of the linearized equation. The map \( F_1(g, h) = (F(g), F(h)) \) is a contraction related to a seminorm which measures the difference \( h - g \). For example, as above we have \( B = \int_0^\delta (v_1 - v_2)^2 dt \leq \delta C \), where \( C \) is a constant related to the semi B-norm of the preimage of the pair \((g_0 + i\partial\bar{\partial}v_1, g_0 + i\partial\bar{\partial}v_2)\). We see that when \( \delta \) small enough it is a contraction. This should lead to another proof.

Therefore, we have the short time existence by the usual argument of linearization:

**Theorem 3.** The short time existence of the modified Calabi flow holds for any compact Kähler manifold with two given complex conjugates of Killing vector fields, i.e., holomorphic vector fields with real potential functions.

### 5 Setting up the Equation of the Modified Calabi Flow in Our Cases

In this and the following sections, we shall focus on the case in which \( \phi_2 = 0 \), i.e., the extremal metric case. Let us recall some results on the modified Calabi flow \((22)\). Let

\[
Cal(\omega) = \int_M (R - HR - \phi_E)^2 \omega^n
\]

be the modified Calabi functional, then:

**Lemma 12.** (Cf. [GC p.820], [Gu4 p.550]) The modified Calabi flow is the gradient flow of the modified Mabuchi functional respect to the Mabuchi metric on the space of Kähler metrics. Its derivative is just the negative of the modified Calabi functional. And the second derivative is just \( 2 \int_X |R_{\alpha\beta}|^2 \omega^n \).

In our case, the evolution equation of the Kähler potential function \( F \) along the modified Calabi flow is given by

\[
-\dot{G}(t, U) = \frac{\partial F}{\partial t}(t, s) = \dot{R} - a - bU. \tag{24}
\]

Recall that \( G = sF_s - F \) is the Legendre transformation of \( F \) (see the paragraphs before and after Lemma 10). Considering \( G \) as a function of \( t \) and \( U \), we use \( U \) as the free variable. We need to estimate the function \( G \) and its derivatives.
We use $'$ for the partial derivative to $U$ and $\cdot$ for the partial derivative to $t$.

Moreover, $\varphi$ also determines the metric as we have seen before, therefore we shall estimate $\varphi$ and its derivatives as functions of $t$ and $U$ instead.

Notice that if we take derivative to $U$ twice we obtain

$$-\hat{G}'' = \frac{\varphi'}{2\varphi^2} = \tilde{R}''.$$  \hfill (25)

In this way, we change the modified Calabi flow into a flow of function $\varphi$.

To make things simpler we assume that $D_0 = D_\infty = 1$  \hfill (26)

in sections from 5 to 10, i.e., the manifolds we considering are just $\mathbb{C}P^1$ bundles and shall deal with the other situations in another paper. Since the modified Calabi functional

$$\text{Cal} = \int_X (R - HR - \phi_E)^2 \omega^n = A \int_{-1}^1 (\tilde{R} - a - bU)^2 QdU$$ \hfill (27)

is decreasing with a constant $A = \frac{\int \omega^n}{\int Qd\omega} > 0$ and is always positive, by (5) we have:

$$\int (2\Delta - (Q\varphi)'' - 2(a + bU)Q)^2 \frac{dU}{Q} < C.$$

Now by $(Q\varphi^0)'' = 2\Delta - 2(a + bU)Q$ (see (14)) and $Q$ being a polynomial of $U$ we have:

$$\int ((Q(\varphi - \varphi^0))'')^2 dU < B \int (Q(\varphi - \varphi^0)')^2 \frac{dU}{Q} < C$$

with a constant $B > Q$ (since $Q$ is a polynomial of $U$).

By the Sobolev imbedding theorem we have $Q\varphi$ is $C^{1+\frac{1}{2}}$ since $\varphi(-1) = 0, \varphi'(-1) = 2$ by the last sentence of the Lemma 4. Therefore, a subsequence of $Q\varphi$ converges to $Q\varphi_1$ in $C^{1+\frac{1}{2}}$. We should prove that $\varphi$ converges to $\varphi_1$. First we want to see that $\varphi_1$ can not have zero in $(-1, 1)$ since the modified Mabuchi functional is bunted and $\varphi_1$ has continuous first derivative. If $\varphi_1(U_0) = 0$, by $U_0$ being the minimal point we have $\varphi'_1(U_0) = 0$ and $\varphi_1(U) < C(U - U_0)$ with a positive number $C$. $\varphi_1^{-1} > (C(U - U_0))^{-1}$ is not
integrable. This will be a contradiction to the boundness of the modified Mabuchi functional.

Then apply the Sobolev imbedding theorem to the modified Mabuchi functional which is just \( \int dt \int (G)^2 dV \), \( G \) is continuous on \( t \) almost everywhere since \( G \) is given and continuous near \( t = 0 \). It follows that if a subsequence of \( \varphi \) converges to another \( \varphi_2 \), then \( \varphi_2 = \varphi_1 \). We obtain that under the modified Calabi flow \( \varphi \) converges to \( \varphi_1 \) in \( C^{1+\frac{1}{2}} \).

Let \( S = \bar{R} - a - bU \), then the equation for \( \varphi \) is

\[
\dot{\varphi} = 2\varphi^2 S''
\]  

(28)

and \( \varphi(-1) = \varphi(1) = 0, \varphi'(-1) = -\varphi'(1) = 2 \).

\( \varphi \) has a disadvantage that \( \varphi(-1) = \varphi(1) = 0 \). Therefore, even if \( \varphi \) is smooth and bounded, it might be negative at some point. Hence we let \( \varphi = \varphi_0 e^\theta \), we have

\[
\dot{\varphi} = \varphi \dot{\theta}.
\]

Therefore, we have:

\[
\dot{\theta} = 2\varphi S''
\]

(29)

and \( \theta(-1) = \theta(1) = 0 \).

6 Estimate of the \( C^0 \) norm of \( \theta \) and the derivative of the Modified Calabi Functional on \([0,T)\)

Now the derivative of the modified Calabi functional is (Cf. [Gu5 p.281])

\[
-2 \int (\varphi S'')^2 QdU.
\]

We like to see that this is bounded for \( t \in [0,T) \). To see this we let \( A = \varphi S'' \).

Then

\[
\frac{d}{dt}(\int A^2 QdU) = 2 \int AAQdU
\]

\[
= 2 \int A(\dot{\varphi}S'' + \varphi(S''))QdU
\]

\[
= 2 \int A(2\varphi^2(S'')^2 - \varphi(\frac{1}{2Q}(\dot{\varphi}Q)'')Q)dU
\]  

(30)
\[ = 4 \int \varphi^3(S'')^3 QdU - 2 \int \varphi^2 S'' \left( \frac{(\varphi^2 S'' Q)''}{Q} \right)' QdU \]
\[ = 4 \int A^3 QdU - 2 \int [\varphi QA]'' \frac{dU}{Q}. \]

Similar to [CKN p.262 (A)] we have:

**Lemma 13.** Following inequality is true

\[ \|x^{-k} u\|_{L^r} \leq C \|x|^{-k+1} u'|_{L^r}, \]  

(31)

for \( u(x) = x^k v(x) \) and \( v(x) \in C_0(\mathbb{R}), \) \( 1 < r \leq +\infty, 1 \leq k. \) In particular, we have

\[ (\int_{-1}^{1} ((1 - x^2)^{-k} u)^r dx)^\frac{1}{r} \leq C (\int_{-1}^{1} ((1 - x^2)^{-k+1} u')^r dx)^\frac{1}{r}, \]  

(32)

for \( u(x) = (1 - x^2)^k w(x) \) and \( w(x) \in C[-1,1]. \)

Proof: Without lost the generality we may assume that \( u(x) = 0 \) if \( x < 0 \) and \( x > a > 0. \) Applying integration by parts we have:

\[ \int (x^{-k} u)^r dx = -\int_{0}^{a} x((x^{-k} u')^r dx \]
\[ = rk \int (x^{-k} u)^r dx - r \int x^{-r-k+1} u^{r-1} u' dx. \]

Therefore,

\[ (rk - 1) \int (x^{-k} u)^r dx = r \int x^{-r-k+1} u^{r-1} u' dx \]
\[ \leq r \left( \int (x^{-k} u)^r dx \right)^\frac{1}{r} \left( \int (x^{-k+1} u')^{\frac{r-1}{r}} dx \right)^\frac{r}{r-1}. \]

Therefore,

\[ \left( \frac{rk - 1}{r} \right)^{\frac{1}{r-1}} \|x^{-k} u\|_{L^r} \leq \|x|^{-k+1} u'|_{L^r}. \]

When \( r = +\infty \) we just take the limit and get

\[ \sup |x^{-k} u| \leq \sup |x^{-k+1} u'|. \]

We have the first part of our Lemma.
For the second part, we let \( v(x) = (1 - x)^k w(x) \). Then we apply above argument we have

\[
\left( \int_{-1}^{0} ((1 - x^2)^{-k} u')^r dx \right)^{\frac{1}{r}} \leq \left( \int_{-1}^{0} ((1 + x)^{-k} u')^r dx \right)^{\frac{1}{r}}
\]

\[
\leq \left( \int_{-1}^{1} ((1 + x)^{-k} u')^r dx \right)^{\frac{1}{r}}
\]

\[
\leq C(\int_{-1}^{1} ((1 + x)^{-k+1} u')^r dx)^{\frac{1}{r}}
\]

\[
\leq C(C_1(\int_{-1}^{0} ((1 - x^2)^{-k+1} u')^r dx)^{\frac{1}{r}} + (\int_{0}^{1} ((1 - x^2)^{-k+1} u')^r dx)^{\frac{1}{r}}).
\]

We apply the same argument to \([0, 1]\) and get the second part of our Lemma.

Q. E. D.

Here we apply condition (26), which implies:

\[
C_1(1 - x^2) \leq \varphi Q \leq C_2(1 - x^2)
\]  

(33)

for some constants \( C_1, C_2 \) depend on \( t \). We shall see that on \([0, T)\) (33) holds for two constants which are independent on \( t \).

Since \( \varphi = \varphi^0 e^\theta \) with \( \theta(-1) = \theta(1) = 0 \), the fact \((\varphi)^\prime\) having bounded \( L^2 \) norm implies that \((\varphi^0(e^\theta - 1))^{\prime}\) has a bounded \( L^2 \) norm. Therefore \((\varphi^0)^{-1}(\varphi^0(e^\theta - 1))'\) and \((\varphi^0)^{-1}(e^\theta - 1)\) have bounded \( L^2 \) norms, so is \((e^\theta - 1)' = e^\theta \varphi'\) and \( e^\theta \) is differentiable with uniform \( C^1 \) bounds. Therefore, \( e^\theta \) has no zero point, otherwise the modified Mabuchi functional \( \int (e^{-\theta} - 1 - \theta) dU \) is positive infinite. That is, \( e^\theta \) is also bounded from zero. We have \( \theta \) is bounded.

We have

\[
\int (\varphi^{-1} A)^2 dU \leq C_1 \int (\varphi^{-1} (\varphi QA))^{\prime 2} dU \leq C_2 \int ((\varphi QA)^\prime)^2 dU.
\]  

(34)

Now

\[
\varphi^{-1}(\varphi QA)' = QA' + \varphi^{-1} A(Q\varphi)',
\]

and

\[
(\varphi QA)^\prime = 2(\varphi Q)' A' + A(\varphi Q)' + \varphi QA',
\]

we have

\[
\int (A')^2 dU \leq C_1 \int ((\varphi QA)^\prime)^2 dU,
\]

\[
\int A^2 dU \leq C_2 \int (\varphi QA)^{\prime 2} dU,
\]

\[
\int (\varphi QA)^{\prime 2} dU \leq C_3 \int ((\varphi QA)^\prime)^2 dU.
\]  

(35)
The last inequality is true because
\[
\int (A(\varphi Q)'')^2 dU \leq \sup\{A^2\} \int ((\varphi Q)'')^2 dU \\
\leq C_1 \left( \int |A'| dU \right)^2 \\
\leq C_2 \int (A')^2 dU.
\]

Similarly we have
\[
\int A^3 dU \leq \sup\{|A|\} \int A^2 dU \leq \int |A'| dU \int A^2 dU \\
\leq (\int (A')^2 dU)^{\frac{1}{2}} \int A^2 dU \\
\leq C (\int ((\varphi QA)'')^2 dU)^{\frac{1}{2}} \int A^2 Q dU.
\]

Combining (30) and (36) as well as the Young’s inequality \(ab \leq \varepsilon a^2 + \epsilon^{-1}b^2\)
we have
\[
\frac{d}{dt}(\int A^2 Q dU) \leq C(\int A^2 Q dU)^2 - (1 - \epsilon) \int ((\varphi QA)'')^2 \frac{dU}{Q}.
\]

If we let \(L = \int A^2 Q dU\) Then \(\dot{L} \leq CL^2\)
\[
\frac{d}{dt}(\ln L) \leq CL.
\]

But \(-2L\) is the derivative of the modified Calabi functional, we have \(\int_0^T L dt\)
is bounded. By integrating (38) we have \(L\) is bounded on \([0,T)\).

This implies that \(\varphi_1\) is \(C^{3+\frac{1}{2}}\) for points other than \(-1, 1\). In particular, \(S'\) is continuous in \((-1, 1)\). So we have a good inner estimates.

7 Estimate of \(C^2\) norm of \(\varphi\) on \([0,T)\)

For the boundary estimate we shall apply again a Hardy inequality since \(\varphi\) has zeros at \(U = -1\) and 1. It is another place where the condition (26) applies, otherwise \(Q\) also has zeros at the boundary. We should have the Hardy inequality in our case that
\[
\int_{-1}^1 (S')^2 dU \leq C \int_{-1}^1 \varphi^2(S'')^2 Q dU
\]
(see e.g., [CKN p.262 (A)]). Since
\[
\int_{-1}^{1} S'(1 - U^2) dU = S(1 - U^2)|_{-1}^{1} + 2 \int_{-1}^{1} S U dU = 0,
\]
we observe that $S'(a) = 0$ for some $a \in (-1, 1)$. Then
\[
\int_{-1}^{a} (S')^2 dU = \int_{-1}^{a} (U + 1)'(S')^2 dU
\]
\[
= (U + 1)(S')^2|_{-1}^{a} - 2 \int_{-1}^{a} (U + 1)S'' dU
\]
\[
\leq 2(\int_{-1}^{a} (S')^2 dU)^{\frac{1}{2}} (\int_{-1}^{a} ((U + 1)S'')^2 dU)^{\frac{1}{2}},
\]
i.e., $\int_{-1}^{a} (S')^2 dU \leq 4 \int_{-1}^{a} ((1 + U)S'')^2 dU$. Similarly, $\int_{a}^{1} (S')^2 dU \leq 4 \int_{a}^{1} ((U - 1)S'')^2 dU$. Combining above two inequalities we obtain our Hardy inequality. That is, $S$ is in the Holder space $C^{1, 2}$ since $\int S dU = 0$ and is continuous. Therefore, $\varphi_1$ is $C^2$.

Even without using the fact that $S'(a) = 0$ for a point $a \in (-1, 1)$ we still can have our conclusion. In fact by [Fr p.19 (8.2)] we have:

**Lemma 14.** If $u \in L^2[-1, 1]$ and
\[
\int_{-1}^{1} (x^2 - 1)^2 (u^{(k)})^2 dx
\]
is bounded, then $u^{(k-1)}$ is $L^2$ bounded. In particular, $u^{(k-2)}$ is continuous.

Proof: We apply [Fr p.19 (8.2)] to the closed interval $[-0.5, 0.5]$ and see that both $\int u^{(k-1)} dx$ and $\int u^{(k)} dx$ are bounded. That is, the average of $u^{(k-1)}$ and the difference of $u^{(k-1)}$ at two points are bounded, so $u^{(k-1)}(0)$ is bounded. Applying above argument we have
\[
\int_{-1}^{0} (u^{(k-1)})^2 dx \leq (u^{(k-1)}(0))^2 + 2(\int_{-1}^{0} (u^{(k-1)})^2 dx)^{\frac{1}{2}} (\int_{-1}^{0} ((x + 1)u^{(k)})^2 dx)^{\frac{1}{2}}.
\]

Now with the Young’s inequality we have our first part of Lemma. The second part follows from the Sobolev embedding theorem.

Q. E. D.
Estimates of $H^4$ and $C^3$ norm of $\varphi$ on $[0, T)$

To obtain $H^4$ norm estimates of $\varphi$ we let $O$ be an operator such that $O(u) = \varphi u''$, then $A = O(S)$. We let $O^*$ be the dual of $O$. If $u \in C_0^\infty(\mathbb{R})$, we have

$$\int uO^*(v)QdU = \int O(u)vQdU = \int \varphi u''vQdU = \int u \frac{(\varphi v')''}{Q}QdU.$$

Therefore $O^*(v) = \frac{(\varphi v')''}{Q}$.

Let $B = Q^{-1}(\varphi QA)' = O^*(A)$ and $A_1 = O(B)$, $B_i = O^*(A_i)$, $A_i = O(B_i^{-1})$. We shall estimate $K = \int B^2QdU$. Let $L_i = \int A_i^2QdU$.

We have

$$\dot{K} = 8 \int A_1A_1^2QdU - 2 \int A_1^2QdU \leq C(\int A_1^2dU)^\frac{1}{2}(\int A_1^4dU)^\frac{1}{2} - 2L_1. \quad (40)$$

As before we have

$$\int A_1dU \leq \sup\{A^2\} \int A_1^2dU \leq \left(\int |A'|dU\right)^2 \int A_1^2dU \leq C \int (A')^2QdU \int A_1^2dU \leq C_1 \int B^2QdU. \quad (41)$$

Therefore,

$$\dot{K} \leq C_{L_1} L_1^\frac{1}{2}K^\frac{1}{2} - 2L_1. \quad (42)$$

By Young’s inequality again we have $\dot{K} \leq C_1 K$ and hence $\frac{\partial (\ln K)}{\partial t} \leq C_2$ where $C_2$ only depend the number $L$.

Therefore $K$ is bounded by a function of the bound of $L$. $(\varphi QA)'$ has uniform $C_1^\frac{1}{2}$ norm.

By (35) $\int (A')^2dU$ is bounded. Therefore $A$ is continuous. So is $\varphi QA'$.

By (34) $\int (S'')^2dU$ is bounded and the $H^4$ norm of $\varphi$ is bounded. Therefore $S'$ is continuous since $S'$ has a zero point. $\varphi^{(3)}$ is continuous.

By $\varphi''$ bounded we have that $(\varphi^0(e^\theta - 1))''$ is bounded. By Lemma 13 $(\varphi^0)^{-1}(\varphi^0(e^\theta - 1))'$ and hence $(\varphi^0)^{-1}(e^\theta - 1)$ are bounded. So $(e^\theta - 1)'$ is bounded, i.e., $(e^\theta)'$ and $\theta'$ is bounded.
By \( \varphi^{(3)} \) bounded we have that there are two numbers \( c, d \) such that 
\[
e^\theta - 1 - (c + dU)(1 - U^2) = (1 - U^2)^2 f_1 \text{ with } f_1 \text{ continuous. The number } c, d \text{ are determined by } \theta'(-1) \text{ and } \theta'(1) \text{ which are bounded by the } C^0 \text{ norm of } \theta'. \text{ Now } (\varphi^0(e^\theta - 1 - (c + dU)(1 - U^2)))^{(3)} \text{ is bounded. So are } 
\]
\[
(\varphi^0(e^\theta - 1 - (c + dU)(1 - U^2)))' \text{, } (\varphi^0(e^\theta - 1 - (c + dU)(1 - U^2)))'' \text{ and } (\varphi^0(e^\theta - 1 - (c + dU)(1 - U^2)))'' \text{. So are } (\varphi^0(e^\theta - 1 - (c + dU)(1 - U^2)))' \text{ and } (\varphi^0(e^\theta - 1 - (c + dU)(1 - U^2)))'' \text{. We have that } (e^\theta')'' \text{ is bounded and so is } \theta''.
\]

In the same way we have:

**Lemma 15.** If \( \varphi^{(k)} \) has \( L^r \) bound, so is \( \theta^{(k-1)} \).

Proof: We only need to prove the case in which \( r \) is a finite number. If \( \varphi^{(k)} \) is \( L^r \) bounded, \( \varphi^{(k-1)} \) is \( C^0 \) bounded. So is \( \theta^{(k-2)} \). By applying a similar argument as above with Lemma 13 and \( L^r \) norms we have our Lemma.

Q. E. D.

Therefore, the \( H^3 \) norm of \( \theta \) is bounded.

## 9 Estimate of Higher Order Derivatives on \([0, T]\)

To get higher order estimate, we try to estimate \( L_i \) and \( K_i = \int B_i^2 QdU \). We might regard these functionals as higher order Calabi functionals.

We need following:

**Lemma 16.** If \( L_i \), \( K_j \) are bounded for \( i \leq l, j < l \) (resp. \( i, j \leq l \)), then \( B_{l-1}' \) (resp. \( A_{l}^1 \)) is bounded in \( L^2 \) norm and \( B_{l-1} \) is bounded (resp. \( (\varphi QA)_{l-1}' \)) is bounded. Moreover, \( \varphi^{(2l+2)} \text{ (resp. } \varphi^{(2l+3)} \text{)} \text{ and } \varphi^{(2l+1)} \varphi^{(4l+3)} \text{ (resp. } \varphi^{(2l+2)} \varphi^{(4l+5)} \text{)} \text{ are bounded and continuous.}

Proof: This is an extension of Lemma 14. When \( L \) is bounded we already see that \( \varphi^{(2)} \) is bounded.

\[
(\varphi S')' = \varphi' S' + \varphi S''
\]
is \( L^2 \) bounded. So \( \varphi S' \) is bounded. So is \( \varphi \varphi^{(3)} \). When \( K \) is bounded, \( (\varphi QA)' \) is bounded. So are \( \varphi A' \) and \( \varphi^2 S''' \). Therefore, \( \varphi^2 \varphi^{(5)} \) is bounded. The Lemma is true for \( l = 0 \).

If \( L_1, L, K \) are bounded. Then \( B' \) is bounded in \( L^2 \) norm. \( B \) is bounded. \( \varphi^{-1}(\varphi QA)' \) is bounded. \( \varphi^{-1} A = S'' \) is bounded. So is \( \varphi^{(4)} \). \( A' = (\varphi S''')' \) is bounded, so is \( \varphi S''' \). But

\[
(\varphi QA)'' = (\varphi^2 Q)''' S'' + 2(\varphi^2 Q)' S''' + \varphi^2 QS^{(4)};
\]

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we have that $\varphi^2 S^{(4)}$ is bounded. So is $\varphi^2 \varphi^{(6)}$. But we also have that

$$(\varphi B')' = \varphi' B' + \varphi B''$$

is $L^2$ bounded. Therefore, $\varphi B'$ is bounded. While

$$\varphi B' = \varphi (\varphi QA)'' = \varphi ((\varphi^2 Q)'S'' + 2(\varphi^2 Q)'S''' + \varphi^2 QS^{(4)})'$$

we have that $\varphi^3 QS^{(5)}$ is bounded. So is $\varphi^3 \varphi^{(7)}$.

If $K_1, L, K, L_1$ are bounded, we have that $(\varphi QA_1)'$ is bounded. So is $QA_1$ and so is $\varphi^2 B''$ as well as $B'$. Now $B' = (Q^{-1}(\varphi^2 QS'')')'$, $(\varphi^2 QS'')'''$ is bounded. Since $S''$ are bounded, there are two numbers $c, d$ such that $\varphi^2 S'' - (c + dU)\varphi^2 = \varphi^2 f_1$ for a continuous function $f_1$. Then $\varphi^{-1}(S'' - (c + dU))$ is bounded as before, so is $(S'' - (c + dU))'$ = $S''' - d$. Therefore, $S'''$ and $\varphi^{(5)}$ is bounded. So is $\varphi S^{(4)}$ and so is $\varphi^2 S^{(5)}$. Therefore, by $A_1 = \varphi B''$ being bounded we have that $\varphi^3 S^{(6)}$ is bounded. Finally, from $(\varphi QA_1)'$ being bounded we have that $\varphi^4 S^{(7)}$ is bounded. So is $\varphi^4 \varphi^{(9)}$.

The same argument works for all $l$. We could also apply the proof of the next Lemma.

Q. E. D.

Furthermore if we let $O_1(u) = \varphi u'$ and $O_2(u) = u'$, we call $O_2$ the pure derivative and $O_1$ the coupled derivative, we have:

**Lemma 17.** If $L_i, K_i$ are bounded for $i, j \leq l$ (resp. $i \leq l, j < l$), then $O_{i_1}Q_{i_2} \cdots O_{i_k} \varphi$ is bounded for at most $2l + 3$ pure derivatives $O_2$ and at most $2l + 2$ coupled derivatives $O_1$ (resp. at most $2l + 2$ pure derivatives and $2l + 1$ coupled derivatives) in $(O_{i_1}, \cdots, O_{i_k})$. Moreover, it is $L^2$ bounded for at most $2l + 4$ pure derivatives and $2l + 2$ coupled derivatives (resp. $2l + 3$ pure derivatives and $2l + 1$ coupled derivatives).

Proof: If $L$ is bounded, then $\varphi^{(3)}$ is $L^2$ bounded and $\varphi''$ is bounded.

$$O_2^k O_1 O_2^{3-k} \varphi = \sum_{m \leq k} (O_2^m \varphi) (O_2^{3-m} \varphi).$$

When $0 < m \leq 2$ the first factor is bounded and the second factor is also bounded. When $m = 0$ we have $\varphi \varphi^{(3)}$ which is also bounded. We also have

$$O_2^k O_1 O_2^{3-k} \varphi = \sum_{m \leq k} (O_2^m \varphi) (O_2^{4-m} \varphi).$$

When $0 < m \leq 3$ the first factor is bounded and the second factor is $L^2$ bounded. When $m = 0$ the whole term is $L^2$ bounded since $L$ is bounded.
If $L, K$ are bounded we want to see that

$$O_{i_1} \cdots O_{i_k} O_1 O_2 O_{j_1} \cdots O_{j_{3-k}} \varphi$$

with three pure derivatives and two coupled derivatives is bounded if and only if

$$O_{i_1} \cdots O_{i_k} O_2 O_1 O_{j_1} \cdots O_{j_{3-k}} \varphi$$

is bounded. Actually the difference of them is

$$O_{i_1} \cdots O_{i_k} (\varphi' O_2 O_{j_1} \cdots O_{j_{3-k}} \varphi)$$

$$= \sum_{m \leq k \ (k_1, \ldots, k_m) \subset (i_1, \ldots, i_k)} (O_{k_1} \cdots O_{k_m} \varphi')(O_{i_1} \cdots O_{i_{k-m}} O_2 O_{j_1} \cdots O_{j_{3-k}} \varphi),$$

where $(k_1, \ldots, k_m)$ is an ordered subset of the ordered set $(i_1, \ldots, i_k)$ and $(l_1, \ldots, l_{k-m})$ is its ordered complement. When $0 \leq m < 3$ the first factor is bounded and in the similar way the second factor being a sum of product of two bounded factors is also bounded. When $m = k = 3$ the second factor is $\varphi'$ and is bounded, the first factor being a sum of products of two bounded factors is also bounded. A easier alternative proof is that we use 2 in the place of 3 in above argument first, then do the case of 3. Similarly for the $L^2$ bounded case.

The same argument can be carried out for all $l$ by induction.

Q. E. D.

Now we are ready to have:

**Lemma 18.** If $L_i, K_j$ are bounded for $i, j \leq l$ (resp. $i \leq l, j < l$), then $L_{l+1}$ (resp. $K_l$) is bounded.

Proof: If $L_i, K_j$ are bounded for $i, j \leq l$, then

$$L_{l+1} = -2K_{l+1} + 4 \int A_{l+1}^2 A \varphi dU$$

$$+ 8 \sum_{k=0}^{l} \int B_{l+1} ((O^* O)^{k} O^* (AO(\varphi O^{l-k} \varphi))) \varphi dU.$$  \hspace{1cm} (43)

Let $O = O_1 O_2$ with $O_1(u) = \varphi u'$, $O_2(u) = u'$. $O^* = O_3 O_4$ with $O_3(u) = Q^{-1} u'$ and $O_4(u) = (\varphi Q u)'$. We see that $O_i(\varphi u) = O_i(u)v + uO_i(v)$ for $i = 1, 2, 3$ and

$$O_4(\varphi u v) = (\varphi Q u v)' = (\varphi Q u)' v + \varphi Q u v' = O_4(u)v + Q u O_1(v).$$

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Let \( O_5 = QO_1 \), then
\[
(O^*O)^kO^*(AO((O^*O)^{l-k}S)) = \sum (O_{i_1}O_{i_2} \cdots O_{i_m}A)(O_{j_1}O_{j_2} \cdots O_{j_{4k-m+2}}O((O^*O)^{l-k}S)) \tag{44}
\]
with half, i.e., \( 2k + 1 \), of \( O_i \)'s and \( O_j \) being \( O_2 \) and \( O_3 \). If we forget the derivatives coupled with \( \varphi \), there are at most \( 2k + 1 + 3 \) pure derivatives of \( \varphi \) from the first factor. Also there are at most \( 2k + 1 + 1 \) coupled derivatives from the first factor. Therefore, if \( k < l \) then the first factor is bounded. The second factor has bounded \( L^2 \) norm as \( K_l \) being bounded.

So we only need to check the term in which \( k = l \) and the first factor has \( 2l + 4 \) pure derivatives. Then the second factor has only 3 pure derivatives and at most \( 2l + 2 \) coupled derivatives. Therefore in this case the second factor is bounded. However, again the first factor is \( L^2 \) bounded. Therefore,
\[
\dot{L}_{l+1} \leq C_1L_{l+1} + C_2.
\]
By integration we see that \( \ln(C_1L_{l+1} + C_2) \) is bounded. So is \( L_{l+1} \).

If \( L_i, K_j \) are bounded for \( i \leq l, j < l \), we have
\[
\dot{K}_l = -2L_{l+1} + 8 \sum_{k=0}^{l} A_{l+1}((OO^*)^k(A(OO^*)^{l-k}OS))QdU. \tag{45}
\]
And
\[
(OO^*)^k(A(OO^*)^{l-k}OS) = \sum (O_{i_1} \cdots O_{i_m}A)(O_{j_1} \cdots O_{j_{4k-m}}(OO^*)^{l-k}OS) \tag{46}
\]
with half, i.e., \( 2k \), of \( O_i \) and \( O_j \) being \( O_2 \) and \( O_3 \). If we forget the derivatives coupled with \( \varphi \), there are at most \( 2k + 3 \) pure derivatives of \( \varphi \) from the first factor. Also there are at most \( 2k + 1 + 1 \) coupled derivatives from the first factor. Therefore, if \( k < l \) then the first factor is bounded. The second factor is \( L^2 \) bounded since \( L_l \) is bounded.

Therefore, we only need to check the term in which \( k = l \) and the first factor has \( 2l + 3 \) pure derivatives. Then the second factor has only 3 pure derivatives and at most \( 2l + 1 \) coupled derivatives. In the case there are \( 2l + 1 \) coupled derivatives we can treat the first one of them as pure derivative if \( l > 0 \), then we treat this term as with \( 2l \) coupled derivatives with 4 pure derivatives. Therefore, the second factor is bounded. But the first factor is again \( L^2 \) bounded. Therefore
\[
\dot{K}_l \leq C
\]
and hence $K_t$ is bounded. Q. E. D.

Therefore the $C^k$ norm of the solutions are uniformly bounded on $[0,T)$. The solution can be extended to $[0,T]$.

10 Long Time Existence and the Convergency

By the general short time existence we obtain the long time existence.

The derivative of the modified Calabi functional is $-2L$ and has a sub-sequence of $t$ such that $L$ turns to zero. Therefore the modified Calabi flow converges to $\varphi_1$ in $C^1$ and in $C^2$ with a subsequence. The modified Calabi functional is decreasing and tend to zero, but then $\varphi_1 = \varphi^0$.

Moreover, by (39) and

$$ Cal = -2L \leq C_1 \int (S')^2 dU \leq -C_2 \int S^2 dU = -C_2 Cal $$

Therefore $Cal \leq Cal(t_0)e^{-kt}$ with $k = C_2$ and converges to zero in an exponential rate.

By integrating (38) we have:

$$ L(t) \leq L(t_0) \exp C \int_{t_0}^t Ldt $$

(47)

For any $\epsilon$ we can pick up a $\delta$ such that $\delta \exp \delta < \epsilon$ and a $t_0$ such that both $2 \int_{t_0}^{+\infty} Ldt < \frac{2}{C}\delta$ as the modified Calabi functional at $t_0$ and $L_{t_0} < \delta$, then $L(t) < \epsilon$ for any $t > t_0$. Therefore, $\lim_{t \to +\infty} L = 0$. Now, $\int_t^{t+1} Ldt$ converges to zero in an exponential rate. With (47) we have that $L$ also converges to zero in an exponential rate.

In this case (41) becomes

$$ \int A^4 dU < \epsilon K $$

(48)

when $t$ is big enough. Therefore, (42) becomes

$$ \dot{K} \leq \epsilon L_1^{\frac{1}{2}} K^{\frac{3}{2}} - L_1 $$

(49)

when $t$ is big enough. Therefore

$$ \dot{K} \leq \epsilon K. $$

(50)
But from (37) we have:

$$2(1 - \epsilon_1) \int_{t_0}^{+\infty} K dt + L(t_0) \leq C e \int_{t_0}^{+\infty} L dt \leq C e^{-kt_0}. \quad (51)$$

In particular,

$$\int_{t_0}^{+\infty} K dt \leq C e^{-kt_0} \quad (52)$$

when \( t_0 \) is big enough. Therefore,

$$K(t) \leq K(t_0) + \epsilon \int_{t_0}^{t} K dt \leq K(t_0) + C e^{-kt_0}$$

for \( t > t_0 \). Picking up a \( t_0 \) such that \( K(t_0) \) small enough we see that

$$K \leq \epsilon.$$

So we also have

$$\lim_{t \to +\infty} K = 0.$$ 

Moreover, by \( \int_{t}^{t+1} K dt \leq C e^{-kt} \) we have \( K \leq C e^{-kt} \) for some positive \( C \) and \( k \).

In particular, \( \lim_{t \to +\infty} A = 0 \) and \( A \) converges in an exponential rate.

Furthermore, the argument in the proof of Lemma 18 shows that:

**Lemma 19.** If \( L_i, K_i \leq C e^{-kt} \) for \( i \leq l \) (resp. \( L_{i+1}, K_i \leq C e^{-kt} \) for \( i < l \)), then

$$\dot{L}_{l+1} \leq -2K_{l+1} + C e^{-kt}(K_{l+1}^2 + L_{l+1}) \quad (53)$$

(resp.

$$\dot{K}_{i} \leq -2L_{l+1} + C e^{-kt}L_{l+1}^{\frac{3}{2}}). \quad (54)$$

Therefore, if \( L_i, K_i \leq C r^{-kt} \) for \( i \leq l \). then

$$\dot{L}_{l+1} \leq \epsilon(L_{l+1} + C e^{-kt}) \quad (55)$$

So

$$L_{l+1} \leq (L_{l+1}(t_0) + C e^{-kt_0})e^\epsilon \quad (56).$$

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But we also have
\[ \dot{K}_l \leq -2(1 - \epsilon_1)L_{l+1} + Ce^{-kt}, \tag{57} \]
so
\[ 2(1 - \epsilon_1) \int_t^{+\infty} L_{l+1} dt + K_l \leq \frac{C}{k}e^{-kt}. \tag{58} \]
We have
\[ \int_t^{t+1} L_{l+1} dt \leq Ce^{-kt} \tag{59} \]
and \( L_{l+1} \leq Ce^{-kt} \) for some \( C, k > 0 \).

Similarly if \( L_{i+1}, K_i \leq Ce^{-kt} \) for \( i < l \), then \( \dot{K}_l \leq Ce^{-kt} \) and \( K_l \leq K_l(t_0) + \frac{C}{k}e^{-kt_0} \). But we also have \( \dot{K}_l \leq -2(1 - \epsilon_1)K_l + Ce^{-kt} \), that implies
\[ 2(1 - \epsilon_1) \int_t^{+\infty} K_l dt + L_l \leq \frac{C}{k}e^{-kt}. \]
Therefore \( \int_t^{t+1} K_l dt \leq \frac{C}{k}e^{-kt} \) and \( K_l \leq C_1e^{-kt} \) for some \( C_1, k > 0 \).

From these estimates we see that \( \varphi \) converges to \( \varphi^0 \) exponentially in any \( C^m \) norm.

Therefore, we have:

**Theorem 4.** In the cases of \( D_0 = D_\infty = 1 \), i.e., with the hypersurface ends, the modified Calabi flow converges exponentially in \( C^m \) norm for any \( m \) to the extremal metric on our manifolds whenever the Kähler class is stable.

### 11 Evolution of Metrics in a Kähler Class along the Modified Ricci Flow

Now we consider the problem of finding (generalized) quasi-Einstein metrics (Cf. [Gu3]) in a given Kähler class by following modified Ricci flow equation:
\[ \frac{\partial}{\partial t} g = -\text{Ric}(g) + H\text{Ric}(g) + L_{\text{Vir}}g. \tag{60} \]

By contraction we get:
\[ \frac{\partial}{\partial t} \log(\det(g)) = \Delta \log(\det(g)) + H + tr_g L_{\text{Vir}}g. \tag{61} \]
We can easily see that the above two equations are equivalent. Once we have a solution \( g_t \) for the (61), we have

\[
-Ric(g) + HRic(g) + L_{\nabla_R} g = \partial \bar{\partial} f
\]

for some \( f \) with \( \int_M f dV_g = 0 \) and \( \Delta f = \frac{\partial}{\partial t} \log(det(g)) \). Let \( g_{t;i} = g_{0,i} + \partial_t \partial_j u \), then \( f = \frac{\partial}{\partial t} u + C(t) \) where \( C(t) \) is a function which only depends on \( t \). We have \( \frac{\partial}{\partial t} g_{ij} = \partial_i \partial_j f \). This means that \( g_t \) is a solution of (60).

Now we consider the short time existence of (61), we let \( h = (\exp(-Vt))_* g \), then (61) is equivalent to

\[
\frac{\partial}{\partial t} \log(det(h)) = \Delta \log(det(h)) + HR.
\]

Let \( h_t = h_0 + \partial \bar{\partial} F_t \), then the linearized equation of (63) is

\[
\frac{\partial}{\partial t} \Delta v = \Delta^2 v + R^{ij} v_{ij}
\]

where \( v_{ij} = \partial_i \partial_j v \).

Now we want to prove that the equation (64) has unique solution. We multiply (64) by \( 2\Delta v \) and integrate, we get

\[
\frac{d}{dt} \int (\Delta v)^2 = -2 \int |\nabla \Delta v|^2 + 2 \int \Delta v R^{ij} v_{ij} + 2 \int \Delta F(\Delta v)^2 \\
\leq C_1 \left( \int (\Delta v)^2 \right)^{\frac{1}{2}} \left( \int \left( \sum_{i,j} |v_{ij}|^2 \right)^{\frac{3}{2}} \right) + C_2 \int (\Delta v)^2 = C \int (\Delta v)^2
\]

where \( C_i, i = 1, 2 \) are constants which are not dependent on \( t \) and \( C = C_1 + C_2 \). Let \( v(t) = \int (\Delta \phi)^2 \), we get \( \frac{d}{dt} v - Cv \leq 0 \), that is, \( \frac{d}{dt} e^{-Ct} v \leq 0 \). We see that \( e^{-Ct} v \) is decreasing, so \( v = 0 \) if \( v(0) = 0 \). We have the short time existence for the evolution.

In general case of the extremal-soliton metrics we consider the equation

\[
\frac{\partial}{\partial t} \log(det(g)) = -R + HR + \phi_1 + \Delta \phi_2.
\]

The linear equation is

\[
\frac{\partial}{\partial t} \Delta v = \Delta^2 v + (R^{ij} - \phi_2^{ij}) v_{ij} + \Delta(E_2(v)) + \frac{1}{2}(E_1^1 v_1 + E_1^2 v_2).
\]
The proof for short existence still holds since for the extra terms we have

\[
\int \Delta v \Delta E_2(v) dV = - \int (\bar{\partial} \Delta v, \bar{\partial} E_2(v)) dV
\]

\[
\leq C(\int |\nabla \Delta v|^2 dV)^\frac{1}{2} (\int |v_{ij}|^2 dV)^\frac{1}{2} = C(\int |\nabla \Delta v|^2 dV)^\frac{1}{2} (\int (\Delta v)^2 dV)^\frac{1}{2}
\]

and

\[
\int \Delta v E_1(v) dV \leq C(\int (\Delta v)^2 dV)^\frac{1}{2} (\int |v_i|^2 dV)^\frac{1}{2}
\]

\[
\leq C(\int (\Delta v)^2 dV)^\frac{1}{2} (\int |v|^2 dV)^\frac{1}{2}
\]

\[
\leq \lambda_1 \frac{1}{2} C \int (\Delta v)^2 dV,
\]

where \( \lambda_1 > 0 \) is the first eigenvalue of the Laplacian \( \Delta \). We also apply Young’s inequality in the proof.

In this paper we only consider the special situation of the completions of \( C^* \) bundles which we studied in previous sections. To avoid confusion we use \( \partial_t \) to denote the partial derivative of \( t \) to a function of \( s \) and \( t \), i.e., respect to the coordinates of the manifold. And use \( \frac{d}{dt} \) to denote the partial derivative of \( t \) to a function of \( U \) and \( t \), i.e., respect to the moving coordinates. The equation (60) can be written as:

\[
\frac{1}{\varphi} \partial_t \varphi(s,t) + \frac{1}{Q} \partial_t Q(s,t) = -\frac{\Delta}{Q} + \frac{1}{2Q} (\varphi Q)'' + a + bU + \frac{c}{2Q} (\varphi Q)'
\]

(68)

And by \( \partial_t H(U_0(s,t)) = 0 \) we get \( \partial_t(\varphi(s,t) \frac{dU_0}{dU}(s,t)) = 0 \), this implies that

\[
\frac{d}{dU}(\partial_t U) = \varphi^{-1} \partial_t \varphi(s,t)
\]

(69)

Combining above two equations we get:

\[
\frac{d}{dU}(\partial_t U(s,t)) + \frac{Q'}{Q} \partial_t U(s,t) = \frac{\Delta}{Q} + \frac{1}{2Q} (\varphi Q)'' + a + bU + \frac{c}{2Q} (\varphi Q)'
\]

(70)

We regard above equation as a first order equation of \( \partial_t U \), we have solution:

\[
\partial_t U = -\frac{1}{Q} \int_{-1}^{U} \Delta(x) dx + \frac{1}{2Q} (\varphi Q)'
\]

\[
+ \frac{1}{Q} \int_{-1}^{U} (a + bU) Q(x) dx + \frac{c}{2\varphi} - \frac{d}{Q}
\]

(71)
Now we let $U = -1$, we get $d = Q(-1)$. Therefore,
\[
\partial_t U = -\frac{p}{2Q} + \frac{(\varphi Q)'}{2Q} + \frac{c}{2} \varphi
\]
with $p$ in (16).

Now we let $\varphi = (1 + \theta)\varphi^0$ with $\varphi^0$ the extremal-soliton solution, then combine above equations (68) and (71) we get:
\[
2\varphi^0 \frac{d}{dt} \theta = 
\varphi \varphi^0 \theta'' + (1 + \theta) \theta (\varphi^0 \frac{p}{Q})' - (\varphi^0)' (\frac{p}{Q})' 
+ \frac{\varphi^0}{Q} p \theta' - (\varphi^0 \theta')^2
\]
(72)
this is basically a nonlinear heat equation, the short time solution exists. Now we want to prove that the long time solution of (72) exists and uniformly converges to 0 with an exponential rate. We let $p_1 = \frac{c}{Q}$ and it can be proven by maximum principle as in [Ki2] and under following condition C:

The function $\varphi^0 (p_1)' - (\varphi^0)' p_1$ is negative on $[-1, 1]$.

In general, this is very difficult to check since $p$ might not be a product of linear factors as in [Ki2].

Therefore, we see another evident that the modified Calabi flow is more natural than the modified Ricci flow.

Moreover, it was known for a long time that the generalized Mabuchi functional is decreasing under the modified Ricci flow but the Calabi functional is not. For example, we let our manifold to be a $\mathbb{C}P^1$, then $Q = 1$, $\Delta = 0$, $\varphi^0 = 1 - U^2$, $p(U) = -2U$,
\[
2\dot{\varphi} = (1 - U^2)^2 (1 + \theta) \theta'' - 2U (1 - U^2) \theta' - (1 - U^2)^2 (\theta')^2 - 2(1 + \theta) \theta (1 + U^2).
\]
And $R = -2^{-1} \varphi''$, $Cal = 2^{-2} \int_{-1}^{1} (\varphi'')^2 dU$,
\[
C' al = 2^{-1} \int_{-1}^{1} \varphi'' \dot{\varphi}'' dU 
= 2^{-1} \left[ \int_{-1}^{1} \varphi^{(4)} \varphi dU + \varphi'' \varphi' \right]_{-1}^{1}.
\]
We also let $\theta = A(1 - U^2)$, then $\varphi^{(4)} = 24A$, $\theta' = -2AU$, $\theta'' = -2A$. Therefore,

$$2\dot{\varphi} = -2(1 - U^2)^2[A(1 + A(1 - U^2)) + 2A^2U^2 + A^2(1 + U^2)]$$
$$- 2(1 - U^2)[-2AU^2 + A(1 + U^2)]$$
$$= -2A(1 - U^2)^2[1 + A(1 - U^2 + 2U^2 + 1 + U^2) + 1]$$
$$= -4A(1 - U^2)^2(1 + A(1 + U^2)).$$

We have $\dot{\varphi}'|_{-1} = 0$. Since $1 + \theta > 0$ we have

$$1 > -A(1 - U^2),$$

and this holds if and only if $-A < 1$, i.e., $A > -1$. We have

$$\lim_{A \to -1} \text{Cal} = \lim_{A \to -1} (\text{Cal}) = \int_{-1}^{1} (1 + A(1 + U^2))(1 - U^2)^2dU > 0.$$ 

Therefore, there is a negative $A$ near $-1$ such that the Calabi functional Cal (same for the modified Calabi functional since they are only different by adding a constant) is not decreasing under the Modified Ricci flow.

**Reference**


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