# Type II Almost-Homogeneous Manifolds of Cohomogeneity One

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**Abstract:** This paper is the continuation of [20] on the existence of extremal metrics of the general affine and type II almost-homogeneous manifolds of cohomogeneity one. In this paper, we deal with the general type II cases with hypersurface ends. More precisely, we deal with manifolds with certain $\mathbf{C}P^n \times (\mathbf{C}P^n)^*$ or $\mathbf{C}P^2$ bundle structures. These manifolds are the direct generalization of the manifolds we dealt with in [12], [13]. In particular, we study the existence of Kähler-Einstein metrics on these manifolds and obtain new Kähler-Einstein manifolds as well as Fano manifolds without Kähler-Einstein metrics.

## 1 Introduction

The theory of simply connected compact Kähler homogeneous manifolds has applications in many branches of mathematics and physics. These complex manifolds possess significant properties: they are projective, Fano, Kähler-Einstein, rational, etc..

One class of more general Kähler manifolds which would be useful is the class of almost-homogeneous compact Kähler manifolds with two orbits, especially those manifolds of cohomogeneity one.

If we assume that they are simply connected, then they are automatically projective. Some of many interesting questions of them are when they are

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Fano, Kähler-Einstein, etc., see [20].

This paper is one of a series of papers in which we answer above questions and we finished the project of the existence of Calabi extremal metrics in any Kähler class on any compact almost-homogeneous manifolds of cohomogeneity one. That is, we dealt with all the compact Kähler manifolds on which we could use ordinary differential equations instead of partial differential equations for these geometric analysis problems.

There are three types of these kind of manifolds. We refer the readers to [12] for the details. The type III compact complex almost homogeneous manifolds of real cohomogeneity one were dealt with in [10] about twenty years ago. There is no much stability involved there. However, see [13] for the stability of the related constructions.

We shall deal with the type I case in [17] and the type II case in [20] and this paper. This is the first class of manifolds for which the existence is completely understood and it is equivalent to the geodesic stability. Originally, we had [20] and this paper as one paper. But it was too long to publish. Therefore we separated it into two papers. We take this opportunity to thank all the people and referees who helped.

In this paper, we finish the task of the proof that there is a Kähler metrics of constant scalar curvature on the type II almost-homogeneous manifold of cohomogeneity one if the generalized Futaki invariant is positive, see Theorem 9, Theorem 9' and Theorems 12, 13. We shall prove the converse in [14]. In [8] and [12], [13], [16] we dealt with some examples, and in [20] we dealt with two most conceptually difficult series of manifolds.

We should mention that our concept of generalized Futaki invariant might not be the same as the one in [7] although it looks similar in our case. The generalized Futaki invariant in this paper comes from some kind of combination of the generalized Futaki invariants along the maximal geodesic rays in the moduli space of Kähler metrics but does not necessarily come directly from any one of them as we have described and observed in [13], [16].

In this paper, we shall first treat the manifolds which are fiber bundles with typical fibers of the first and fifth cases in [1] p.73 as one situation. Let $G$ be a complex Lie subgroup of the automorphism group of our manifold $M$ and $G$ has an open orbit $O$ on $M$. $M$ is a fiber bundle over a compact homogeneous space $Q$. We have that $Q = G/P$ with $P$ a parabolic subgroup of $G$ and $P = SS_1R$ with $S$, $S_1$ semisimple factors of $G$ and $R$ the radical of $P$. $S_1R$ acts on the fiber $F$ trivially. In our case $S = A_n$ acts on the central fiber. The fiber is just $\mathbf{C}P^n \times (\mathbf{C}P^n)^*$ that is isotropic and is the first manifold in the list of [1] p.67. It is also in the case of affine type and therefore is of type II. Therefore, to finish the affine case and the type II case we have to deal with this case. But this seems, as individual, easier than those in [20]. However, we have more of them and it turns out that as a group and analytically, they are technically more involved.

We should also notice the difference of the open orbits of the manifolds with the $S = A_n$ actions from those of the manifolds we treated in [12], [13]. For example, the isotropic group $U$ of the $A_n$ action case is $GL(n, \mathbf{C})$ as that of the first manifold in the table 2 of [1] p.67, while the isotropic groups of the manifolds in [12], [13] are not reductive at all. Another point is that the manifolds in [1] p.67 are actually all homogeneous, that is not true for the examples in [12], [13]. We shall come to some generalizations of those examples from [12], [13] in the ninth section (see Theorems 10, 11). See also some similar calculations in the third and fourth sections. However, the manifolds we considered in [12], [13] are manifolds with $S = A_1$ actions on the fiber and are special cases of what are treating in this paper. It is amazing that the first examples we treated in [8], [12], [13] are both type II and isotropic (that has a similar complex structure as the type I case) that they served as sample cases of both type I and type II manifolds which led us to the breakthroughs for both cases.

What we have done in this paper also take care of the type II case (see section 8). We only need to take care of manifolds with certain $\mathbf{C}P^2$ bundle structure, which have same structure as that of the cases with $S = A_1$ at the Lie algebra level and are a direct generalization of what we dealt with in [13].

As in [16], [20], we take our original method in [12], [13]. From Lie group point of view our method can be regarded as a nilpotent path method, i.e., we consider a path, starting from the singular real orbit, generated by the action of a 1-parameter subgroup generated by a nilpotent element. One could also consider the path as a path generated by a semisimple element $H_\alpha$, where $\alpha$ is the root which generates the $sl(2)$ Lie algebra $\mathcal{A}$ (see section 2).

In this paper, we first look back to what we did in [8], [12], [13] from a Lie group point of view in the second section. Then we apply the same argument in the third section of [16], [20] to the affine $A_n$ action case. We found that the same method works for the complex structure of both the affine and the type II cases. We deal with the $A_n$ action case we mentioned above. At the end of the second section, we use a similar method as in [12] to give a comparation of two different methods for the homogeneous case. Similar comparations for the homogeneous case will be carried out also in the third and fourth sections to give more confirmations to the readers that our arguments are trustable.

In the third section, we found that the same argument works for the Kähler structure. This is a section in which we deal with many different possibilities of the pairs of groups $(A_n, G)$. This also shows that the affine and type II classes are very big and are not extraordinary at all (see also the proof of the Lemma 6 for a huge amount of this kind of manifolds). A new ingradient is that being different from [20] our $B$ here can be either positive or negative.

The fourth section is one of the major part of this paper. To calculate the Ricci curvature we apply a modified Koszul's trick which was motivated by [25] p.567–570 as we did in [16], [20]. The formula we used from [5] 4.11 is due to Professor Dorfmeister.

We calculate the scalar curvature in the fifth section and setting up the equations in the sixth section. The pattern of these equations make it possible to reduce a fourth order ODE to a second order ODE as in [16], [20].

We finally prove our Theorem 9 in the seventh section.

We then treat the type II case in the eighth section and the Kähler Einstein case in the ninth section. We also generalize our results in [12], [13]. At the end of the ninth section we give a very uniform description for the generalized Futaki invariant, see Theorems 12 and 13. The result there also confirmed our calculation in [16].

In all our calculations we also need to take care carefully of the change of the invariant inner products when we restrict our calculation to a typical subgroup $S$ in $G$.

## 2 The complex structures of the isotropic affine almost homogeneous manifolds

In this section we will deal with the complex structure of the isotropic affine almost-homogeneous manifolds. Let us recall some basic notations of the Lie groups and Lie algebras.

In general, as in [1] we let $G$ be a semisimple complex Lie group, $U_G$ be the 1-subgroup. There is a parabolic subgroup

$$P = SS_1R \tag{1}$$

with $S, S_1$ semisimple and $R$ solvable such that

$$U_G = US_1R \tag{2}$$

where $U$ is a 1-subgroup of $S$. The manifold is a fibration over $G/P$ with the completion of

$$P/U_G = S/U \tag{3}$$

as the affine almost homogeneous fiber $F$. In this case, the root system of $S$ is a subsystem of the root system of $G$.

Let $\mathcal{H}$ be the corresponding Cartan subalgebra of $G$. The Lie algebra $\mathcal{G}$ of $G$ has a decomposition $\mathcal{H} + \sum_{\alpha\in\Delta} \mathbf{C}E_\alpha$ with a Chevvalley lattice generated by $h_\alpha,\ E_\alpha$ (cf. [23] p.147). Assume that a maximal compact Lie subalgebra is generated by

$$F_\alpha = E_\alpha - E_{-\alpha}, \quad G_\alpha = i(E_\alpha + E_\alpha), \quad H_\alpha = i[E_\alpha,\ E_{-\alpha}] = ih_\alpha. \tag{4}$$

We have that

$$[H_\alpha,\ E_\alpha] = 2iE_\alpha. \tag{5}$$

Let $\mathcal{A} = su(2)$ be the commutator of a generic compact isotropic subgroup and $p_t$ be a curve generated by a nilpotent element in the complexification of $\mathcal{A}$. In the Lie algebra of $G$, we have $F_\alpha, G_\alpha$ for those roots of $G$ which are not in $S$. The tangent space of $G/U_G$ along $p_t$ is decomposed into irreducible $\mathcal{A}$ representations. $F_\alpha, G_\alpha$ are in the complement representation of $\mathcal{S}$. But $JF_\alpha = -G_\alpha(\mathrm{mod}\mathcal{S})$ as it is in the tangent space of $G/P$. Therefore, we have $JF_\alpha = -G_\alpha$ for any $\alpha$ which is not in the root system of $S$. This discussion is corresponding to the discussion in the last paragraph of the second section of [16].

As it is stated in [24] p.38, we can always identify the Lie algebra as the left invariant vector fields on the Lie group. For example, if $G$ is $GL_n(\mathbf{C})$, $B(t)$ a curve on $G$ with tangent vector $X_0$ at $B(0) = I$. Then $AB(t)$ is a curve started at $A$ and $AX_0$ with $A \in G$ is a left invariant vector field on $G$. That is, the left invariant vector fields can be described as $AX_0$ for some $X_0$.

Let $X_0 = (b_{ij})$ and $Y_0 = (c_{ij})$. Then the Lie bracket of two left invariant vector fields $AX_0$ and $AY_0$ is

$$\begin{aligned}
[AX_0,\ AY_0] &= [a_{ij}b_{jl}\frac{\partial}{\partial a_{il}},\ a_{ks}c_{st}\frac{\partial}{\partial a_{kt}}] \\
&= a_{ij}b_{jl}c_{lt}\frac{\partial}{\partial a_{it}} - a_{ks}c_{st}b_{tl}\frac{\partial}{\partial a_{kl}} \\
&= a_{ij}(b_{jl}c_{lt}\frac{\partial}{\partial a_{it}} - c_{jt}b_{tl}\frac{\partial}{\partial a_{il}}) \\
&= a_{ij}(b_{jl}c_{lt} - c_{jl}b_{lt})\frac{\partial}{\partial a_{it}} \\
&= A[X_0,\ Y_0],
\end{aligned}$$

which is comparable with the Lie bracket of the Lie algebra $gl_n(\mathbf{C})$.

In our case we have $S = A_n = SL(n+1, \mathbf{C})$ action of [1] p.73, which includes both the first case and the fifth case there.

Let us look at the case for $n = 1$ first. The action is

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} \times [1, 0]A^{-1} \tag{6}$$

where $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, [1.0] represent the points in $\mathbf{C}P^1$. We have

$$E_{\alpha_1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{-\alpha_1} = E_{\alpha_1}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = H_{\alpha_1} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{7}$$

And

$$\exp(tE_{\alpha_1})\begin{bmatrix} 1 \\ 0 \end{bmatrix} \times [1.0]\exp(-tE_{\alpha_1}) = \begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix} = p_t, \tag{8}$$

$$p_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times [0, 1]. \tag{9}$$

We let

$$F = F_{\alpha_1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad G = G_{\alpha_1} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{10}$$

Using the coordinates $[1, z]^T \times [1, w]$ we can check that along $p_t$ $H$ acts as vector $(z, w) = (0, -2it)$. The tangent vector $T$ of $p_t$ is $(0, -1)$. $F$ acts as $(-1, -1-t^2)$ and $G$ acts as $(i, -i(1-t^2))$ along $p_t$. $F + (1+t^2)T$ is $(-1, 0)$. Therefore, we have that

$$JF = i(-1, -1-t^2) = -(i, -i(1-t^2) + 2i) = -G + \frac{H}{t} \tag{11}$$

and

$$JH = i(0, -2it) = -2tT. \tag{12}$$

In general, if $S = SL(n+1, \mathbf{C}) = A_n$, $S$ has simple roots $\alpha_i = e_i - e_{i+1}$. the affine fiber $\mathbf{C}^n$ is generated by the root vectors with the roots $e_1 - e_j, 1 < j \leq n+1$. The action is

$$A[1, 0, \cdots, 0]^T \times [1, 0, \cdots, 0]A^{-1}. \tag{13}$$

We can choose

$$E_{e_i - e_j} = E_{ij} \tag{14}$$

as a square metrix $(a_{kl})_{(n+1)\times(n+1)}$, that is, all the elements $a_{kl}$ are zero except $a_{ij} = 1$. We also let $H_{e_i - e_j} = iE_{ii} - iE_{jj}$. $[E_{ij},\ E_{kl}] = 0$ if $j \neq k$, $i \neq l$ and

$$[E_{ij},\ E_{jk}] = E_{ij}E_{jk} - E_{jk}E_{ij} = E_{ik} - 0 = E_{ik}$$

if $i \neq k$. As above $F = F_{\alpha_1}$, $G = G_{\alpha_1}$ and $H = H_{\alpha_1}$.

$$p_t = \exp(tE_{\alpha_1})[1, 0, \cdots, 0]^T \times [1, 0, \cdots, 0]exp(-tE_{\alpha_1}) \tag{15}$$
$$= \quad [1, 0, 0, \cdots, 0]^T \times [1, -t, 0. \cdots, 0]. \tag{16}$$

$$JF = -G + \frac{H}{t},\ JH = -2tT. \tag{17}$$

$$JF_{e_2 - e_j} = G_{e_2 - e_j}, \quad JF_{e_1 - e_j} = -G_{e_1 - e_j} - \frac{2G_{e_2 - e_j}}{t} \quad 2 < j, \tag{18}$$

$$F_{e_k - e_j} = G_{e_k - e_j} = 0 \quad 2 < k < j. \tag{19}$$

One also have that

$$J(F_{e_1-e_j} + \frac{F_{e_2-e_j}}{t}) = -(G_{e_1-e_j} + \frac{G_{e_2-e_j}}{t}). \tag{20}$$

Actually, if we let $[1, z_1, \cdots, z_n] \times [1, w_1, \cdots, w_n]$ be the coordinate, then $F_{1j}$ is the same as $z_k = w_k = 0 \quad k \neq j$ and $z_j = w_j = -1$. $F_{2j}$ has $z_k = w_l = 0 \quad l \neq j$ and $w_j = t$. Therefore, $F_{1j} + t^{-1}F_{2j}$ has $z_k = w_j = 0 \quad k \neq j$ and $z_j = -1$. At $p_\infty$,

$$JF_{e_1-e_k} = -G_{e_1-e_k}, \quad JF_{e_2-e_k} = G_{e_2-e_k}, \tag{21}$$

$$F_{e_i-e_k} = G_{e_i-e_k} = 0 \quad 2 < i < k. \tag{22}$$

Let

$$F_{ij} = E_{ij} - E_{ji}, \quad G_{ij} = i(E_{ij} + E_{ji}), \tag{23}$$

we have

$$[F_{ij},\ G_{jk}] = G_{ik} \tag{24}$$

if $i \neq k$.

In our case of $S = A_n$, the bigger complex Lie group $G$ can be any complex semisimple Lie group. That is quite different from that in [20]. This make our argument more involved in this paper starting from the next section.

We can also use a similar method in [8], [12], [13] to understand the complex structure. Let

$$[z, w] = ([z_0, z_1, \cdots, z_n]; [w_0, w_1, \cdots, w_n]) \in \mathbf{C}P^n \times (\mathbf{C}P^n)^*.$$

We let

$$(z, w) = z_0w_0 + z_1w_1 + \cdots + z_nw_n \tag{25}$$

be the complex bilinear form. This is different from the one in [8], [12], [13] where $(z, w)$ respresents the inner product. Then the hypersurface end is just $(z, w) = 0$ and the singular $SU(n+1)$ orbit is $w = \bar{z}$ or if we let

$$\gamma = \frac{|(z, w)|^2}{|z|^2|w|^2}, \tag{26}$$

the singular orbit is just $\gamma = 1$. Notice that our $\gamma$ here is different from $\theta$ in [8], [12], [13]. Actually the $\theta$ there is similar to our $1-\gamma$, which we shall call $\theta$ (compare our case with [16] section 3). $\theta$ is like the square of the cosine and $\gamma$ is like the square of sine. We might call $\theta$ the phase angle (or the square phase angle), $\gamma$ the dual phase angle (or the dual square phase angle).

## 3 The Kähler structures

Now we should calculate the Kähler form by different methods. First, if $G = S = A_n$, we let

$$\omega = a\omega_1 + b\omega_2 + i\partial\bar{\partial}F$$

with $\omega_1 = \partial\bar{\partial}\log|z|^2$ and $\omega_2 = \partial\bar{\partial}\log|w|^2$, $F$ is a $SU(n+1)$ invariant smooth function. We see that $F = F(\gamma)$.

Let $f = \gamma F'$ with the derivative respect to $\gamma$, then at $p_t$ we have $\gamma = \frac{1}{1+t^2}$ and

$$\partial\bar{\partial}\log\gamma = -\partial\bar{\partial}(\log|z|^2 + \log|w|^2),$$

$$\partial\log\gamma = \partial(\log(z,w) - \log|z|^2 - \log|w|^2) = -t(dz_1 - \frac{dw_1}{|w|^2}).$$

$$\begin{aligned}
\omega &= a\omega_1 + b\omega_2 + \gamma f'\partial\log\gamma\wedge\bar{\partial}\log\gamma + f\partial\bar{\partial}\log\gamma \\
&= (a-f)dz\wedge d\bar{z} + (b-f)(\frac{dw_1\wedge d\bar{w}_1}{|w|^4} + |w|^{-2}\sum_{j>1} dw_j\wedge d\bar{w}_j) \\
&+ \gamma f'|w_1|^2(dz_1 - |w|^{-2}dw_1)\wedge(d\bar{z}_1 - |w|^{-2}d\bar{w}_1).
\end{aligned}$$

The difference of our formula from that in [13] is that we do not have the second term to the right since $(z,w)$ here is holomorphic. We notice that the subspaces $W = \{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial w_1}\}$ and $\mathbf{C}\frac{\partial}{\partial z_j}$, $\mathbf{C}\frac{\partial}{\partial w_j}$ $j > 1$ are othogonal to each other. Let us calculate the determinant $\tau$ of $W$.

We have

$$\begin{aligned}
\tau &= \begin{vmatrix} a-f+(1-\gamma)f' & -(1-\gamma)f'|w|^{-2} \\ -(1-\gamma)f'|w|^{-2} & (b-f+(1-\gamma)f')|w|^{-4} \end{vmatrix} \\
&= \frac{1}{|w|^4}((a-f)(b-f) + (1-\gamma)(a+b-2f)f'),
\end{aligned}$$

In the same way, we observe that for the standard metric $a = b = n+1$, $f = 0$ and $\tau_0 = \frac{(n+1)^2}{|z|^4|w|^4}$. Therefore,

$$\tau = -\frac{1}{|z|^4|w|^4} D' \tag{27}$$

with

$$D = (a-f)(b-f)(1-\gamma). \tag{28}$$

The determinant of $\mathbf{C}\frac{\partial}{\partial z^i} \quad i > 1$ is $|z|^{-2}(a-f)$. The determinant of $\mathbf{C}\frac{\partial}{\partial w^i} \quad i > 1$ is $|w|^{-2}(b-f)$. Therefore, the volume form is

$$\begin{aligned} V = & \frac{-D^{n-1}D'}{(|z||w|)^{2n+2}(1-\gamma)^{n-1}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ \wedge & \quad dw^1 \wedge d\bar{w}^1 \wedge \cdots \wedge dw^n \wedge d\bar{w}^n. \end{aligned} \tag{29}$$

Second, by regarding the open $A_n$ orbit as a homogeneous space, the vector fields which corresponding to the Lie algebra are the pushdown of the right invariant vector fields on the Lie group $A_n$. As we did in [16], we study the corresponding left invariant vector fields on the Lie group. To make the things simpler, we still use our original notation for the left invariant vector fields. Since the Kähler form is (left)invariant under the action of the maximal compact Lie subalgebra $\mathcal{K}$ of the complex Lie algebra $\mathcal{A}_n$, the pullback of this Kähler form is left $\mathcal{K}$ invariant form on $A_n$. We also extend $t$ to be $\mathcal{K}$ invariant, and so is $T$ as the derivative of $t$. Therefore, we have (cf. [24] p.36 and [21] p.283, here we use the convention in [21])

$$\begin{aligned} 0 = & \, d\omega(T, X, Y) \\ = & \quad T(\omega(X,Y)) - X(\omega(T,Y)) + Y(\omega(T,X)) \\ - & \quad \omega([T,X],Y) + \omega([T,Y],X) - \omega([X,Y],T) \\ = & \quad T(\omega(X,Y)) - \omega([X,Y],T). \end{aligned}$$

$T(\omega(X,Y)) = -\omega(T,[X,Y])$ for any two left invariant $X, Y \in \mathcal{K}$.

Now,

$$\begin{aligned} T(\omega(G,\ H)) = & -2\omega(T,\ F) \\ = & \quad -2\omega(JT,\ JF) \\ = & \quad -\omega(\frac{H}{t},\ -G + \frac{H}{t}) \\ = & \quad -t^{-1}\omega(G,\ H), \end{aligned}$$

that is, $\omega(G,\ H) = Ct^{-1}$ for a constant $C$. Then $C = 0$, otherwise $\omega(G,\ H)$ is infinite at $p_0$. Therefore, $\omega(G,\ H) = \omega(T,\ F) = 0$.

Similarly,

$$
\begin{aligned}
tT(\omega(H,\ F)) - T(\omega(F,\ G)) &= 2\omega(tT,\ -G + \frac{H}{t}) \\
&= 2\omega(tJT,\ J^2F) \\
&= -\omega(H,\ F),
\end{aligned}
$$

i. e., $T(t\omega(H,\ F) - \omega(F,\ G)) = 0$. We have

$$\omega(F,\ G) = t\omega(H,\ F) + A.$$

Let $(\ ,\ )_A$ be an invariant metric on $\mathcal{K}$ such that $(H, H)_A = 1$. If there is no confusion we write $(\ ,\ ) = (\ ,\ )_A$. Then $H,\ G,\ F$ is an unitary basis of the Lie algebra $\mathcal{A}$. Therefore

$$
\begin{aligned}
[X, Y] = ([X, Y], H)H + [X, Y], F)F + ([X, Y], G)G \\
+ \quad [X, Y]_l + [X, Y]_{(\mathcal{A}+l)^\perp}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\omega(T, [X, Y]) = ([X, Y], H)\omega(T, H) + ([X, Y], G)\omega(T, G) \\
+ \quad \omega(T, [X, Y]_{(\mathcal{A}+l)^\perp}).
\end{aligned}
$$

But

$$\omega(T, [X, Y]_{(\mathcal{A}+l)^\perp}) = \omega((2t)^{-1}H, J([X, Y]_{(\mathcal{A}+l)^\perp})) = 0,$$

since $JX \in (\mathcal{A} + l)^\perp$ if $X \in (\mathcal{A} + l)^\perp$. We also have that

$$\omega(X, Y) = (g_1H + g_2F + g_3G + I, [X, Y])$$

with $I$ in the center of $l$.

$$\omega(G, H) = (g_1H + g_2F + g_3G + I, [G, H]) = 2(g_2F, F) = g_2 = 0,$$

i.e., $g_2 = 0$. Therefore, using $\cdot$ for the derivative with respect to $t$, for left invariant $X, Y$ we have

$$
\begin{aligned}
T(\omega(X, Y)) &= (\dot{g}_1H + \dot{g}_3G + \dot{I}, [X, Y]) \\
&= -\omega(T, [X, Y]) \\
&= -([X, Y], \omega(T, H)H + \omega(T, G)G),
\end{aligned}
$$

i.e., $\dot{I} = 0$ and $\dot{g}_1 = -\omega(T, H),\ \dot{g}_3 = -\omega(T, G)$. The last two equalities are actually already known to us. We actually obtained

$$\begin{aligned} \omega(T, -G + \frac{H}{t}) = \dot{c} - \frac{\dot{a}}{t} \\ &= \omega(JT, J^2F) \\ &= -\omega(\frac{H}{2t}, F) \\ &= -t^{-1}(g_1H + g_3G, G) \\ &= -g_3t^{-1}, \end{aligned}$$

that is, $t\dot{g}_3 + g_3 = \dot{g}_1$. Therefore, $g_1 = tg_3 + C$. That is,

$$\omega(F, G) = 2g_1 = 2tg_3 + 2C = t\omega(H, F) + 2C.$$

Therefore, we already have this equality with $A = 2C$. We also see that $g_3(0) = 0$ since $H(0) = 0$. The first equality $I' = 0$ means that $I$ does not depend on $t$, i. e., if we let $I_0 = \frac{n-1}{n+1}(e_1 + e_2) - \frac{2}{n+1}\sum_{i=3}^{n+1} e_i$, then

$$I = BiI_0$$

for some constant $B$. Denote $g = g_3$. Then, $g_1 = tg + C$ and the Kähler form is

$$\begin{aligned} \omega(X, Y) &= ((tg(t) + C)H + g(t)G + BiI_0, [X, Y]) \\ &= (H(t), [X, Y]) \end{aligned}$$

for left invariant $X, Y$, where $H(t) = g_1H + gG + I = (tg + C) + gH + H$.

As an observation, we see that if

$$V_1 = \mathbf{span}(T, F_\alpha),$$

$$V_2 = \mathbf{span}(H, G_\alpha),$$

then

$$JV_1 = V_2$$

and

$$V_1^\perp = V_2$$

with respect to $\omega$. Moreover,

$$[V_1, V_1], [V_2, V_2] \subset V_1,$$

$$[V_1, V_2] \subset V_2.$$

The Kähler metric is a direct sum of its restriction on the subspaces

$$W = \mathbf{span}(T, H, F, G), \tag{30}$$

$$W_1 = \mathbf{span}(E_\alpha | \alpha = e_i - e_j, i \neq j, \{i, j\} \cap \{1, 2\} \neq 0). \tag{31}$$

On $W$ the metric is

$$\begin{aligned} \begin{bmatrix} \omega(T, JT) & \omega(T, JF) \\ \omega(F, JT) & \omega(F, JF) \end{bmatrix} &= \begin{bmatrix} \omega(T, \frac{H}{2t}) & \omega(JT, -F) \\ \omega(F, \frac{H}{2t}) & \omega(F, -G + \frac{H}{t}) \end{bmatrix} \\ = & \begin{bmatrix} -\frac{t\dot{g}+g}{2t} & -\frac{g}{t} \\ -\frac{g}{t} & -\frac{2(1+t^2)g}{t} - 2C \end{bmatrix}. \end{aligned}$$

The determinant is equal to

$$\begin{aligned} & (2t)^{-1} \det \begin{bmatrix} \omega(T, H) & \omega(T, -G) \\ \omega(F, H) & \omega(F, -G) \end{bmatrix} \\ = & (2t)^{-1} \det \begin{bmatrix} -\dot{g}_1 & \dot{g} \\ -2g & -2g_1 \end{bmatrix} \\ = & t^{-1}(g_1 \dot{g}_1 + g\dot{g}) \\ = & \frac{\dot{U}}{2t}, \end{aligned}$$

where

$$U = g_1^2 + g^2. \tag{32}$$

We notice that $U$ is the square norm $(H(t), H(t))$ up to a constant, i.e., the energy of $H(t)$ up to a constant.

We also see that $U$ is increasing. We also see that $g(0) = 0, -\dot{(tg)} > 0$ when $t > 0$, therefore, $-g > 0$ when $t > 0$ and $-tg$ is increasing. We also notice that $\frac{g(-t)}{-t} = \frac{g(t)}{t}$, that is, $g(t)$ is an odd function.

Now we consider $n = 2$, then

$$W_1 = \mathbf{span}(E_\alpha|_{\alpha = \pm\alpha_2, \pm(\alpha_1+\alpha_2)}).$$

On $W_1$ we have that:

$$\begin{bmatrix} \omega(F_{\alpha_2}, JF_{\alpha_2}) & \omega(F_{\alpha_2}, JF_{\alpha_1+\alpha_2}) \\ \omega(F_{\alpha_1+\alpha_2}, JF_{\alpha_2}) & \omega(F_{\alpha_1+\alpha_2}, JF_{\alpha_1+\alpha_2}) \end{bmatrix}$$
$$= \begin{bmatrix} -g_1 + B & g \\ g & -g_1 - B - \frac{2g}{t} \end{bmatrix}.$$

The determinant is equal to

$$U - B^2.$$

Since $F_{\alpha_2}(0) = 0$, we have that $g_1(0) = C = B$ and $U(0) = B^2$. By $U$ increasing, we have that $U - B^2 > 0$.

When $n > 2$ we have 2-stings $e_2 - e_j, e_1 - e_j$ of $\alpha_1$. The calculation is exactly the same and the determinant is $U - B^2$. Therefore, the volume form is

$$\dot{U}(2t)^{-1}(U - B^2)^{n-1}. \tag{33}$$

This fits well with our earlier volume formula (29).

Now we also have that along $p_t$

$$\omega(F_{23},\ JF_{23}) = 2t^2 \frac{b-f}{|w|^2} = \frac{2t^2(b-f)}{1+t^2}, \tag{34}$$

$$\omega(F_{13} + \frac{F_{23}}{t},\ J(F_{13} + \frac{F_{23}}{t})) = 2(a-f). \tag{35}$$

Then we have

$$-g = \frac{2t(b-f)}{1+t^2}, \tag{36}$$

$$-t^{-1}(1+t^2)g - 2B = 2(a-f). \tag{37}$$

Therefore, $2(b-f) + 2B = (a-f)$, i. e.,

$$B = b - a. \tag{38}$$

We also have

$$-t^{-1}g = 2\gamma(b-f)$$

and

$$-tg = \frac{2t^2}{1+t^2}(b-f). \tag{39}$$

Therefore, when $t \to 0$ we get $-\dot{g}(0) = 2(b - f(1))$ and $\lim_{t\to+\infty} tg = -2b$. That is, $-tg$ is nonnegative and increasing with a limit $2b$. In particular, both $B$ and $l = \lim_{t\to+\infty} tg = -2b$ are topological invariants of the given Kähler class.

Moreover, we have

$$\begin{aligned} D &= (1-\gamma)(a-f)(b-f) \\ &= 4^{-1}(1-\gamma)(t\gamma)^{-2}g(g-2Bt\gamma) \\ &= 4^{-1}g((1+t^2)g+2tB) = 4^{-1}(U-B^2). \end{aligned} \tag{40}$$

When $n=1$ we have

$$\begin{aligned} \omega(T, JT) &= (2t)^{-1}\omega(T, H) \\ &= -(2t)^{-1}\dot{g}_1 \\ &= 2\frac{b-f+\theta f'}{(1+t^2)^2}, \end{aligned}$$

$$\begin{aligned} \omega(T, J(F-(1+t^2)T)) &= \omega(T, -G + \frac{H}{t} - \frac{1+t^2}{2t}H) \\ &= \dot{g} + \frac{t^2-1}{2t}\dot{g}_1 \\ &= -2\theta f'(1+t^2)^{-2}. \end{aligned}$$

Therefore,

$$-(2t)^{-1}\dot{g}_1 = 2\frac{b-f}{(1+t^2)^2} - \frac{2t\dot{g}+(t^2-1)\dot{g}_1}{2t(1+t^2)}.$$

We have

$$2\frac{b-f}{1+t^2} = \dot{g} - t^{-1}\dot{g}_1 = -t^{-1}g$$

as above.

To get the formula for $B$, we similarly have

$$2(a-f+\theta f') = \omega(F-(1+t^2)T,\ J(F-(1+t^2)T))$$

$$
\begin{aligned}
&= -2g_1 + \frac{t^2-1}{t} g - (1+t^2)\dot{g} - \frac{t^4-1}{2t}\dot{g}_1 \\
&= -2g_1 + \frac{t^2}{t} g + 2\theta f' \\
&= -\frac{t^2 - 2B + 1}{t} g + 2\theta f'.
\end{aligned}
$$

That is,

$$
2(a-f) = -\frac{t^2+1}{t} g - 2B = 2(b-f) - 2B
$$

as before. Hence, again we get $B = b - a$.

As we notice in [20] that all the $I$ and therefore the coefficients $B$ depend on the inner product $( \, . \, )$ we choose. In general, $G$ might be bigger than $S = A_n$. And, we can write the volume formula as

$$
M\dot{U}t^{-1}(U - B^2)^{k-1} \prod (a_i^2 - U).
$$

For each string, by change the sign of the eigenvalues we can exchange the eigenvectors. This induces a *mirror symmetry* of the eigenvectors. Formally, we can let $c = 0$ (and assume $a \neq 0$), then we have for each eigenvector $\beta_i$ $(aH + I, \beta_i) = k_{\beta_i}(a_i \pm a)$. Therefore, we can choose $a_i = -\left|\frac{(I,\beta_i)}{(H,\beta_i)}\right|$ if $(H, \beta_i) \neq 0$. And if $\beta_{i_1}$, $\beta_{i_2}$ are mirror symmetry to each other, then we have the same $a_i$. We say that a *mirror symmetry class* is the set $[i]$ of two different roots which are mirror symmetry to each other and denote $a_{[i]} = a_i$ for $i \in [i]$. We also let $\mathcal{I}$ be the all mirror symmetry classes.

Similar to what we have in [16], [20] we have that:

**Theorem 1.** *For the affine isotropic case, i. e., when $S = A_n$, the volume is*

$$
V = \frac{M\dot{U}}{t}(U - B^2)^{n-1} \prod_{[i] \in \mathcal{I}} (a_i^2 - U) \tag{41}
$$

*for some positive numbers $M$ and $a_i^2$ with*

$$
a_i = -\left| \frac{(I_G, \beta_i)}{(H, \beta_i)} \right|.
$$

*Moreover, $U(0) = B^2$ and $B^2 \leq U < a_i^2$. In particular, if $G = S$, we have that $V = Mt^{-1}\dot{U}(U - B^2)^{n-1}$.*

Proof: We need to take care of the case in which $S = A_n, \quad G \neq S$.

If $G = A_{m+n+k}$ and $S = A_n$ is generated by simple roots

$$e_{m+1} - e_{m+2}, \cdots, e_{m+n} - e_{m+n+1},$$

then $\alpha_{m+1}$ has other 2-strings with determinants $a_j^2 - U$ for some constants $a_j$.

As we see in the last section that in the general case of $S = A_n$, $G$ can be any semisimple Lie group. To see that the Theorem 1 still holds we have to deal with pairs of roots. There is a classification in [23] p.44–45. We have following three Lemmas:

**Lemma 1.** *If $\alpha$ has a 1-string, then the 1-string and $\alpha$ generate an $A_1 \times A_1$ type of complex Lie subalgebra. In this case, the determinant is a positive constant.*

Proof: The Lie algebra is a rank 2 algebra. Since the action of $\alpha_1$ is trivial on the 1-string $\beta$, the minimal Lie algebra including both triples must be $A_1 \times A_1$. The restricted $\omega$ is $(aH + cG + M\beta, [X, Y])$ for a constant $M$. The positivity comes from the positivity of the metric.

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**Lemma 2.** *If $\alpha$ has a 3-string generated by $\beta$, then $\beta$ has the twice length as $\alpha$ and $\alpha, \beta$ generate an $B_2$ type of complex Lie subalgebra, which has an induced cohomogeneity one action. The determinant is $-8M(M^2 - U)$ for a real negative number $M$.*

Proof: The Lie algebra has a rank 2. Since the representation of $\mathcal{A}$ has a length 3, it can not be $A_1 \times A_1$, $A_2$ nor $G_2$. It must be a $B_2$. The calculation of the volume follows from a similar argument for 3-strings in [20].

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Before we go further, we check that the other possible strings are 4-strings and 2-strings. While the 4-strings can only occur in $G_2$, the 2-stings are more complicated comparing to what we considered above which only involves Lie subalgebras of type $A_2$.

We have basically dealt with the $G_2$ case in [16]. The only possible case for a 4-string is $G = G_2$ and $S = A_1$ is generated by the short root $\alpha = \alpha_1$. In this case, the 4-string is

$$\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2.$$

The restricted metric $\omega$ is $(aH + cG + B_1 i(3\alpha_1 + 2\alpha_2), [X, Y])$. The determinant is equal to

$$\det(\omega(F_{\alpha_i}, -G_{\alpha_j})) = (B_1^2 - U)(9B_1^2 - U)$$

(cf. [16]). We let $a_1 = B_1$ and $a_2 = 3B_1$.

If a simple root $\alpha$ has a 2-string generated by $\beta$ and the length of $\beta$ is the same of $\alpha$, then they generate an $A_2$. This case includes all the case for $G = A_n, D_n, E_k$.

If a simple root $\alpha$ has a 2-string generated by $\beta$ and the length of $\beta$ is half of that of $\alpha$, then they generate a $B_2$ type of complex Lie subalgebra. Assuming that $\alpha = e_1 - e_2$, $\beta = e_2$ the 2-string is $e_2, e_1$. Then the restricted metric $\omega$ is

$$(aH + cG + B_1 i(e_1 + e_2), [X, Y]).$$

The determinant is $B_1^2 - U$. This includes the long simple roots in $B_n, C_n, F_4$. Together with above paragraph we dealt with all the possibilities except the case in which $G = G_2$.

If a simple root $\alpha$ has a 2-string generated by $\beta$ and the length of $\beta$ is a third of that of $\alpha$, then $\alpha$ and $\beta$ generate a $G_2$ type of complex Lie algebra. This only occurs in $G_2$. $\alpha = \alpha_2$ is the long simple root. $\beta$ could be either $\alpha_1$ or $3\alpha_1 + \alpha_2$. The last case can not occur, since $3\alpha_1 + \alpha_2$ has the same length as $\alpha_2$ and they generate an $A_2$ type of complex Lie subalgebra. Therefore, $\beta = \alpha_1$. We have that $H = \frac{1}{3} H_{\alpha_2}$ and by $(H, H)_A = 1$ we have that $(H_{\alpha_2}, H_{\alpha_2})_A = 9$ and.

$$\omega(X, Y) = (g_1 H + gG + B_1 i(2\alpha_1 + \alpha_2), [X, Y]).$$

The restricted metric is

$$\begin{aligned}
&\begin{bmatrix} \omega(F_{\alpha_1}, JF_{\alpha_1}) & \omega(F_{\alpha_1}, JF_{\alpha_1+\alpha_2}) \\ \omega(F_{\alpha_1+\alpha_2}, JF_{\alpha_1}) & \omega(F_{\alpha_1+\alpha_2}, JF_{\alpha_1+\alpha_2}) \end{bmatrix} \\
&= \begin{bmatrix} \omega(F_{\alpha_1}, -G_{\alpha_1}) & \omega(F_{\alpha_1}, -G_{\alpha_1+\alpha_2}) \\ \omega(F_{\alpha_1+\alpha_2}, -G_{\alpha_1}) & \omega(F_{\alpha_1+\alpha_2}, -G_{\alpha_1+\alpha_2}) \end{bmatrix} \\
&= \begin{bmatrix} 3g_1 - 3B_1 & -3g \\ -3g & -3g_1 - 3B_1 \end{bmatrix}.
\end{aligned}$$

Therefore, the determinant is $9(B_1^2 - U)$.

We have that:

**Lemma 3.** *If $\alpha$ has a 2-string, the determinant is $M(d-U)$ for some numbers $M$ and $d>0$. If $\alpha$ has a 4-string, the determinant is $(d-U)(9d-U)$ for a positive number $d$.*

By these three Lemmas, we obtain our Theorem 1.

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# 4 Calculating the Ricci curvature

Now, we calculate the Ricci curvature. Let $\alpha_1$ be the root which generates $\mathcal{A}$ and $h=\log V$. Following Koszul [25] p.567, we have that

$$\rho(X,JY)=\frac{L_{J[X_r,JY_r]}(\omega^n)(T,JT,F,JF,F_\alpha,JF_\alpha)}{2\omega^n(T,JT,F,JF,F_\alpha,JF_\alpha)}, \tag{42}$$

where $X_r, Y_r$ are the corresponding right invariant vector fields and here we use $F_\alpha, JF_\alpha$ to represent

$$F_{\alpha_2}, JF_{\alpha_2}, \cdots, F_{\alpha_l}, JF_{\alpha_l}$$

the array of $F_\alpha$ with its conjugate for positive roots $\alpha$ other than $\alpha_1$ which have nonzero $F_\alpha$ and $G_\alpha$.

We can also use a similar method in [8], [12], [13] to calculate the Ricci curvature for the case $S=G=A_n$. Let us do this first. Then we shall compare the conclusion to Koszul's method. By the volume formula (29), or (33) (see also (41)) we have

$$a_\rho = n+1 = b_\rho \tag{43}$$

and $F_\rho = -(n-1)(\log D - \log(1-\gamma)) + \log(-D')$. Therefore,

$$\begin{aligned} f_\rho = \gamma F'_\rho &= -\gamma[(n-1)(D'D^{-1}+(1-\gamma)^{-1}) + D''(D')^{-1}] \\ &= -\frac{n-1}{t^2} + 2 + \frac{1+t^2}{2t}\dot{h} \qquad (44) \\ & \qquad\qquad (45) \end{aligned}$$

and by (36), (38) we have:

$$g_\rho = -\frac{2t}{1+t^2}(b_\rho - f_\rho) = \dot{h} - \frac{2(n-1)}{t},\ B_\rho = 0. \tag{46}$$

To use Koszul's method we need to consider $X, Y$ for first $H, G - \frac{H}{t}$, and then $F, F$. We have that

$$[H, J(G - \frac{H}{t})] = [H, F] = 2G,$$

$$J[H_r, J(G - \frac{H}{t})_r] = -2JG = -2J(G - \frac{H}{t} + \frac{H}{t}) = 2(2T - F).$$

$$[F, JF] = [F, -G + \frac{H}{t}] = -2H - \frac{2G}{t}.$$

$$\begin{aligned} J[F_r, JF_r] &= J(2H + \frac{2G}{t}) \\ &= 2J\left(-2tT + \frac{F - 2T}{t}\right) \\ &= 2\frac{F - 2(1 + t^2)T}{t}. \end{aligned}$$

Again as what happened in [25] p.567–570, usually it is not clear how to find $JX$ for a right invariant vector field $X$ along $p_t$ and to deal with the left invariant form with right invariant vector fields. Therefore, the argument in [Si] does not work as we can see for our situation. We need something similar to the Koszul's trick in [25] p.567–570. It turns out that all the arguments there still go through for our situation once both $X, JY$ are in the maximal compact Lie algebra $\mathcal{K}$. Therefore, we have that:

$$\begin{aligned} \rho(H, J(G - \frac{H}{t})) &= 2\dot{h} + \frac{1}{2\omega^n(T, JT, F, JF, F_\alpha, JF_\alpha)}\cdot \\ [ \quad & \omega^n([2(F - 2T), T] - J[2G, T], JT, F, JF, F_\alpha, JF_\alpha) \\ + \quad & \omega^n(T, [2(F - 2T), JT] - J[2G, JT], F, JF, F_\alpha, JF_\alpha) \\ + \quad & \omega^n(T, JT, [2(F - 2T), F] - J[2G, F], JF, F_\alpha, JF_\alpha) \\ + \quad & \omega^n(T, JT, F, [2(F - 2T), JF] - J[G, JF], F_\alpha, JF_\alpha) \\ + \quad & \omega^n(T, JT, F, JF, [2(F - 2T), F_\alpha] - J[2G, F_\alpha], JF_\alpha) \\ + \quad & \omega^5(T, JT, F, JF, F_\alpha, [2(F - 2T), JF_\alpha] - J[2G, JF_\alpha])] \\ = \quad & 2\dot{h} - \frac{4(n-1)}{t}, \end{aligned}$$

here we use the notation

$$\omega^n(\cdots, [A, F_\alpha] - J[B, F_\alpha], JF_\alpha),$$

to represent

$$\begin{aligned}
&\omega^n(\cdots,[A,F_{\alpha_2}]-J[B,F_{\alpha_2}],JF_{\alpha_2},\cdots,F_{\alpha_l},JF_{\alpha_l})+\cdots\\
+&\quad \omega^n(\cdots,F_{\alpha_2},JF_{\alpha_2},\cdots,[A,F_{\alpha_l}]-J[B,F_{\alpha_l}],JF_{\alpha_l})
\end{aligned}$$

which is the sum of

$$\omega^n(\cdots,F_{\alpha_2},JF_{\alpha_2},\cdots,[A,F_\alpha]-J[B,F_\alpha],JF_\alpha,\cdots,F_{\alpha_l},JF_{\alpha_l})$$

for all the positive roots $\alpha$ other than $\alpha_1$, and we use the notation

$$\omega^n(\cdots,F_\alpha,[A,JF_\alpha]-J[B,JF_\alpha])$$

to represent a similar sum.

Another way to understand the calculation is regarding the volume tensor formally as a product of the two determinant tensors. When $n=2$, these determinants are $\tau$, $\tau_1$ of the subspaces $W$, $W_i$. We have the formula

$$\rho(X,JY)=\frac{1}{2}J[X_r,JY_r](h)+\frac{A_{X,Y}(\tau)}{2\tau}+\frac{A_{X,Y}(\tau_1)}{2\tau_1}, \tag{47}$$

where

$$A_{X,Y}(\tau)=\sum_i \tau(\cdots,[J[X,JY],X_i]-J[[X,JY],X_i],\cdots). \tag{48}$$

Applying this formula, we have the components which come from the determinants $\tau$ and $\tau_1$:

$$\frac{A_{H,G-\frac{H}{t}}(\tau)}{2\tau}=0$$

since

$$[F-2T,T]=-J[G,T]=0,$$

$$[F-2T,JT]=[F-2T,\frac{H}{2t}]=-\frac{G}{t}+\frac{H}{t^2}=t^{-1}JF,$$

$$-J[G,JT]=-J[G,\frac{H}{2t}]=-t^{-1}JF,$$

$$[F-2T,F]=0,\quad -J[G,F]=-2JH=4tT,$$

$$[F-2T,JF]=[F-2T,-G+t^{-1}H]=-2H-2t^{-1}G+2t^{-2}H=2t^{-1}JF-2H,$$

$$-J[G,JF]=-t^{-1}J[G,H]=-2t^{-1}JF;$$

and

$$\frac{A_{H,G-\frac{H}{t}}(\tau_1)}{2\tau_1} = -\frac{4}{t}$$

since

$$[F-2T, F_{23}] = F_{13},$$

$$-J[G, F_{23}] = -JG_{13} = -J(G_{13} + 2t^{-1}G_{23} - 2t^{-1}G_{23}) = -2t^{-1}F_{23} - F_{13},$$

$$[F-2T, JF_{23}] = [F-2T, G_{23}] = G_{13} = -JF_{13} - 2t^{-1}JF_{23},$$

$$-J[G, JF_{23}] = -J[G, G_{23}] = JF_{13},$$

$$[F-2T, F_{13}] = -F_{23}, \quad -J[G, F_{13}] = -JG_{23} = F_{23},$$

$$\begin{aligned} &[F-2T, JF_{13}] = [F-2T, -G_{13} - 2t^{-1}G_{23}] \\ &= \quad G_{23} - 2t^{-1}G_{13} - 4t^{-2}G_{23} \\ &= \quad JF_{23} + 2t^{-1}JF_{13}, \end{aligned}$$

$$-J[G, JF_{13}] = -J[G, -G_{13} - 2t^{-1}G_{23}] = -JF_{23} - 2t^{-1}JF_{13}.$$

Similarly, we have

**Theorem 2.** *If the fiber with $S = A_n$ action is affine and isotropic, then $g_\rho = \dot{h} - \frac{2(n-1)}{t}$. Moreover, $B_\rho = 0$. Other coefficients, i.e., other part of $I_\rho$, come from the Ricci curvature of $G/P$ which is $-(q_{G/P}, [X,Y])_0$ with $q_{G/P} = \sum_{\alpha \in \Delta^+ - \Delta_P} H_\alpha$ with the standard inner product.*

Proof: As above, we consider $X, Y$ for $H, G - \frac{H}{t}$ and $F, F$.
First,

$$[H, J(G - \frac{H}{t})] = 2G, \quad J[H_r, J(G - \frac{H}{t})_r] = 2(2T - F).$$

As above, the contribution of $T, JT, F, JF$ is zero. The contribution of $e_2 - e_j, e_1 - e_j$ is $-\frac{4}{t}$. When $G \neq S$, the contribution from the roots outside $S$ is zero. Therefore,

$$\begin{aligned} 2g_\rho &= \rho(H, F) \\ &= \quad \rho(H, -J^2 F) \\ &= \quad \rho(H, J(G - \frac{H}{t})) \\ &= \quad 2(\dot{h} - \frac{2(n-1)}{t}). \end{aligned}$$

That is, $g_\rho = \dot{h} - \frac{2(n-1)}{t}$.

Second,

$$[F, JF] = -2H - \frac{2G}{t}, J[F_r, JF_r] = \frac{2}{t}(F - 2(1+t^2)T).$$

The contribution of $T, JT, F, JF$ is zero. The contribution of $e_2 - e_j, e_1 - e_j$ is $4(t + \frac{1}{t})$. When $G \neq S$, the contribution from the roots outside $S$ is zero. Therefore, $\rho(F, JF) = -2(t + \frac{1}{t})(\dot{h} - \frac{2(n-1)}{t})$, and $B_\rho = 0$.

Other coefficients come from the $q_{G/P}$ as above.

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## 5 Calculating the scalar curvature

To calculate the scalar curvature we separate our subspaces into five kind of spaces. The first $W$ is generated by $T, JT, F, JF$. The second , third, fourth and fifth are the subspaces of 1, 2, 3 and 4-strings. The Ricci curvature is a sum of its restriction to each subspaces

$$\rho = \sum_i \rho_i. \tag{49}$$

Similarly

$$\omega = \sum_i \omega_i. \tag{50}$$

Then, by Theorem 1 we have that

$$V = \frac{M\dot{U}Q(U)}{t} = \frac{M\dot{U}}{t}(U - B^2)^{k-1}Q_1(U), \tag{51}$$

$$\rho \wedge \omega^{n-1} = \sum_i \Omega_i \tag{52}$$

where

$$\Omega_i = \rho_i \wedge \omega^{n-1}. \tag{53}$$

Let

$$U_\rho = (g_1 H + gG, tg_\rho H + g_\rho G), \tag{54}$$

then

$$U_\rho(0) = 0. \tag{55}$$

$$\Omega_W = (n-1)! K \dot{U}_\rho Q(U)/t$$

if the determinant of $W$ is $K\dot{U}/t$. For 1-strings,

$$\Omega_i = K_i \dot{U} Q(U)/t.$$

For 2-strings,

$$\Omega_i = -2(n-1)!(U_\rho - a_i a_{\rho,i}) \frac{V}{q_i}$$

where $q_i = a_i^2 - U$ is the linear factor of $Q$ introduced from the given 2-string. Similarly, we can see, by a direct calculation, that for a 3-string

$$\Omega_i = -(2U_\rho - 2a_i a_{\rho,i} + \frac{a_{\rho,i}}{a_i}(U - a_i^2)) \frac{(n-1)!V}{q_i}.$$

For the case of 4-strings, it only occurs when $G = G_2$ and $H$ correspond to the short root. In this case, we have

$$\begin{aligned}
\Omega_1 &= \rho_1 \wedge \omega^{n-1} \\
&= -4(U_\rho(5B_1^2 - U) + B_1 B_{\rho,1}(5U - 9B_1^2)) \frac{(n-1)!V}{(B_1^2 - U)(9B_1^2 - U)} \\
&= -2[U_\rho[B_1^2 - U) + (9B_1^2 - U)] \\
&- B_1 B_{\rho,1}[9(B_1^2 - U) + (9B_1^2 - U)]] \frac{(n-1)!V}{(B_1^2 - U)(9B_1^2 - U)} \\
&= -2(U_\rho - 9B_1 B_{\rho,1}) \frac{(n-1)!V}{9B_1^2 - U} - 2(U_\rho - B_1 B_{\rho,1}) \frac{(n-1)!V}{B_1^2 - U}.
\end{aligned}$$

Therefore,

$$\rho \wedge \omega^{n-1} = (n-1)! M \frac{(U_\rho \dot{Q}(U)) + p_0(U)\dot{U}}{t}. \tag{56}$$

**Theorem 3.** *The scalar curvature is* $\frac{2(\dot{U_\rho Q}) + p\dot{U}}{\dot{U}Q}$ *with a polynomial* $p$ *of* $U$*. Moreover,* $p(U) = (U - B^2)^{n-1} P_1(U)$*, where* $P_1(U)$ *is a polynomial of* $U$ *and is a positive linear sum of (1)* $Q_1$ *and (2) the products of* $\deg Q_1 - 1$ *linear factors of* $Q_1$*. Only 1-strings and 3-strings have contributions to (1);*

*the contribution of each 1-string and 3-string is* $\frac{c_{\rho,l}}{c_l}$ *for the* $Q_1$ *term, where* $c_i = \omega(F_{\alpha_i}, JF_{\alpha_i})$ *for 1-strings and* $c_i = a_i$ *for 3-strings. Only 2-strings, 3-strings and 4-strings have contributions to (2); the contribution of each 2-string and 4-string related to the products of* $\deg Q_1 - 1$ *linear factors of* $Q_1$ *is* $2\frac{a_{\rho,i}a_iQ_1}{q_i}$. *In particular, if* $G = S$, *we have that* $p(U) = 0$.

# 6 Setting up the equations

Now, we shall set up the equations for the metrics with constant scalar curvature. Before we do that, we shall understand more about the metrics. We have that:

**Theorem 4.** If $S = A_n$, $\omega$ is a metric on the open orbit if and only if $B < -\frac{\dot{g}(0)}{2}$ and $g$ is an odd function with $\dot{g}(0) < 0$, $t^{-1}\dot{U} > 0$ and $U < a_i^2$.

Proof: from the metric formula for the metrics we need that

$$\lim_{t\to 0}\frac{t\dot{g}+g}{t} = 2\dot{g}(0) < 0,$$

$$\lim_{t\to 0}\left(\frac{(1+t^2)g}{t} + B\right) = \dot{g}(0) + B < 0,$$

$$\lim_{t\to 0}(tg + 2B + 2t^{-1}g) = 2B + 2\dot{g}(0) < 0,$$

$$\lim_{t\to 0} t^{-1}g = \dot{g}(0) < 0,$$

$$\lim_{t\to 0} t^{-1}\dot{U} = 2\dot{g}(0)B + (\dot{g}(0))^2 > 0$$

and

$$t^{-1}U > 0.$$

Q. E. D.

This result is somehow quite different from those in [Gu8] and [Gu12]. Therefore, with also the property that $B_\rho = 0$ in Theorem 2 we prefer to call the manifolds in the case $S = A_n$ the *type IV manifolds*.

To understand the metrics near the hypersurface orbit, we can let $\theta = \frac{t^2}{1+t^2}$, and we see that $\dot{\theta} = \frac{2t}{1+t^2} - \frac{2t^3}{(1+t^2)^2} = \frac{2t}{(1+t^2)^2}$. We can also see that $U_\theta(1) = \lim_{t\to+\infty}\frac{(1+t^2)^2\dot{U}}{2t} > 0$ exists. In particular, $U$ is bounded, so is $tg$. This was done in the third section. Let $l = \lim_{t\to+\infty} tg$.

We also notice that the closure $D$ of the orbit $\Omega$ of the complex Lie group $SL(2, \mathbf{C})$ generated by $\alpha_1$ is a cohomogeneity one fiber bundle with a

$\mathbf{C}P^1$ as the base and another $\mathbf{C}P^1$ as the fiber. Since $\Omega$ is a $\mathbf{C}$ bundle over $\mathbf{C}P^1$, $D$ is affine compact almost homogeneous manifold with the $SL(2,\mathbf{C})$ action. That is, $D$ is exactly the $S = A_1$ action manifold and is $\mathbf{C}P^1 \times \mathbf{C}P^1$. A calculation in section 3 for the $S = A_1$ action also gives the bounded property of $U$ and $l$. The restriction of the metric to $D$ also gives us the same topological invariants $B$ and $l$.

**Theorem 5.** $\omega$ in Theorem 1 extends to a Kähler metric over the exceptional divisor if and only if $\lim_{t\to+\infty} tg = l > a_i - B$ and $U_\theta(1) > 0$.

Now, for any given pair $B, l$ with $0 > l > a_i - B$ we can check that $g(t) = \frac{lt}{1+t^2}$ satisfies Theorems 4 and 5. We shall see later on that this actually gives us the solutions of our equations for the homogeneous cases, i.e., when $G = S$. So we have that:

**Theorem 6.** The Kähler classes are in one to one correspondence with the elements in the set $\Gamma = \{(B,l)|_{0>l>a_i-B \quad \text{and} \quad B<-\frac{l}{2}}\}$.

To calculate the total volume, we notice that

$$T \wedge JT \wedge F \wedge JF \bigwedge_{\alpha=\alpha_2}^{\alpha_l} (F_\alpha \wedge JF_\alpha) = M \frac{T \wedge H \wedge F \wedge G \bigwedge_{\alpha=\alpha_2}^{\alpha_l} (F_\alpha \wedge G_\alpha)}{t} \tag{57}$$

with a possitive number $M$.

$$U(0) = B^2, \; U(+\infty) = (l+B)^2. \tag{58}$$

Therefore, the total volume is

$$V_T = \int_{B^2}^{(l+B)^2} Q(U) dU. \tag{59}$$

We also see that

$$g_\rho = \dot{h} - \frac{2(n-1)}{t} = \frac{\ddot{U}}{\dot{U}} + \frac{Q'(U)\dot{U}}{Q(U)} - \frac{2n-1}{t}. \tag{60}$$

One can easily check that

$$\left(\frac{\ddot{U}}{\dot{U}} - \frac{1}{t}\right)(0) = 0,$$

$$\left(\frac{\dot{U}}{U - B^2} - \frac{2}{t}\right)(0) = \dot{U}(0) = 0$$

by $g$ being an odd function and therefore $g_\rho(0) = 0$.

Now, from

$$\begin{aligned} U &= (tg+B)^2 + g^2 \\ &= (t^2+1)g^2 + 2Btg + B^2 \\ &= (t^2+1)\left(g + \frac{Bt}{t^2+1}\right)^2 + \frac{B^2}{1+t^2} \end{aligned}$$

we have that

$$\left(g + \frac{Bt}{t^2+1}\right)^2 = \frac{1}{(1+t^2)^2}((1+t^2)U - B^2).$$

We have that

$$-g - \frac{Bt}{1+t^2} = \frac{\sqrt{(1+t^2)U - B^2}}{1+t^2}.$$

That is

$$g = -\frac{\sqrt{(1+t^2)U - B^2} + Bt}{1+t^2}.$$

To make the things clearer, we replace $t$ by $\theta = \frac{t^2}{1+t^2}$. We have that

$$\begin{aligned} tg_\rho &= [[\log[U_\theta Q(U)(1-\theta)^2]]_\theta 2\theta(1-\theta) - 2(n-1)] \\ &= [2\theta(1-\theta)\left[\frac{U_{\theta\theta}}{U_\theta} + \frac{Q'(U)U_\theta}{Q(U)}\right] - 4\theta - 2(n-1)], \end{aligned}$$

which has a limit $-2(n+1)$ at $\theta = 1$ so

$$l_\rho = -2(n+1). \tag{61}$$

Therefore, the Ricci class is $(0, -2(n+1))$.

We also have that

$$U_\rho(1) = l_\rho(B+l) = -2(n+1)(B+l). \tag{62}$$

Now, we have the Kähler Einstein equation

$$\begin{aligned} &[2\theta(1-\theta)\left[\frac{U_{\theta\theta}}{U_\theta} + \frac{Q'(U)U_\theta}{Q(U)}\right] - 4\theta - 2(n-1)] = tg \\ &= -\frac{t\sqrt{(1+t^2)U}}{1+t^2} \\ &= -\sqrt{\theta U}. \end{aligned} \tag{63}$$

The total scalar curvature is

$$R_T = \int_0^{+\infty} [p(U)\dot{U} + 2(U_\rho \dot{Q}(U))]dt. \tag{64}$$

And from this, we have the average scalar curvature

$$\begin{aligned} R_0 &= \frac{R_T}{V_T} \\ &= \frac{\int_{B^2}^{(B+l)^2} p(U)dU + 2(U_\rho Q(U))|_{B^2}^{(B+l)^2}}{\int_{B^2}^{(B+l)^2} Q(U)dU} \\ &= \frac{\int_{B^2}^{(B+l)^2} p(U)dU + 2l_\rho(B+l)Q((B+l)^2)}{\int_{B^2}^{(B+l)^2} Q(U)dU}. \end{aligned}$$

If $G = S = A_n$ (we see in [20] that this is the same as the assumption that the manifold being homogeneous), then $Q = (U - B^2)^{n-1}$ and $p = 0$. Therefore,

$$R_0 = \frac{l_\rho(B+l)}{n^{-1}((B+l)^2 - B^2)} = 2n\frac{Bl_\rho + ll_\rho}{2Bl + l^2}.$$

The equation of constant scalar curvature is $\frac{R}{V} = R_0$. Therefore, we have that

$$\begin{aligned} &2U_\rho Q(U) + \int_{B^2}^{U} p(U)dU \\ &= R_0 \int_{B^2}^{U} Q(U)dU + A_0 \end{aligned} \tag{65}$$

with $A_0$ a constant.

Let $\theta = 0$, we have that

$$0 = 2BB_\rho Q(B^2) = A_0.$$

If we put $\theta = 1$ in we get the same $A_0$.

We have that

$$U_\rho = \frac{R_0 \int_{B^2}^{U} QdU - \int_{B^2}^{U} pdU}{2Q(U)} \tag{66}$$

where $Q(U) = (U - B^2)^{n-1} Q_1(U)$.

Applying Theorem 3 and integration by parts, we have that

$$\begin{aligned}U_\rho &= \frac{R_0\int_{B^2}^{U} QdU - \int_{B^2}^{U}(U-B^2)^{n-1}P_1 dU}{2Q} \\ &= \frac{\int_{B^2}^{U}(R_0 Q - (U-B^2)^{n-1}P_1)dU}{2Q} \\ &= \frac{R(U)}{2Q_1(U)},\end{aligned}$$

where $R(U)$ is a polynomial of $U$. Therefore,

$$g_\rho((t^2+1)g+Bt) = \frac{um(u)}{Q_1(u)}$$

where we let $R(U) = 2um(U)$.

If $G = S = A_n$, we have that

$$U_\rho = \frac{R_0}{2n}(U-B^2).$$

And $R(U) = \frac{R_0}{n}(U-B^2)$, $m(u) = \frac{R_0}{2n}$.

Now, by

$$tg = -B\theta - \sqrt{\theta(u+B^2\theta)}$$

we have that

$$(1+t^2)tg + Bt^2 = -\frac{\sqrt{\theta(u+B^2\theta)}}{1-\theta},$$

and therefore if we use $'$ for the derivative with respect to $\theta$ we have that

$$\begin{aligned}&\theta(1-\theta)\left[\frac{u''}{u'} + \frac{Q'(u)u'}{Q(u)}\right] - 2\theta - n + 1 \\ &= -2^{-1}\sqrt{\frac{\theta}{u+B^2\theta}}\,u\frac{\dot{m}(u)}{Q_1(u)}.\end{aligned} \tag{67}$$

Comparing with (63), we see that

$$m(u) = Q_1(u)$$

if the Kähler metric is in the Ricci class.

If $G = S = A_n$, then we have that $\frac{m(u)}{Q_1}$ is a constant. There is a solution with $u = c\theta$. Actually, if we use the $g = \frac{lt}{1+t^2}$ in the proof of the Theorem 6 we obtain that $u = (2B+l)l\theta$ which solves our equation.

From (67), we have that

$$[\log[u'Q(u)]]' = \frac{P}{\theta(1-\theta)}.$$

We also have

$$2\theta + n - 1 - A_{B,l}\theta^{\frac{1}{2}} \leq P \leq 2\theta + n - 1 + C_{B,l}\theta^{\frac{1}{2}}.$$

for some positive constant $A_{B,l}, C_{B,l}$ which only depend on $B$ and $l$. Since $P(1) = n + 1 + 2^{-1}l_\rho = 0$, we have that $A_{B,l} \geq n + 1$.

By integration, we have that

$$\begin{aligned}\frac{a^{n-1}(1-a^{\frac{1}{2}})^{A_{B,l}-n-1}(1+\theta^{\frac{1}{2}})^{A_{B,l}+n+1}}{\theta^{n-1}(1-\theta^{\frac{1}{2}})^{A_{B,l}-n-1}(1+a^{\frac{1}{2}})^{A_{B,l}+n+1}} &\leq \frac{u'(a)u^{n-1}(a)Q_1(u(a))}{u'(\theta)u^{n-1}(\theta)Q_1(u(\theta))} \\ &\leq \frac{a^{n-1}(1-\theta^{\frac{1}{2}})^{n+1+C_{B,l}}(1+\theta^{\frac{1}{2}})^{n+1-C_{B,l}}}{\theta^{n-1}(1-a^{\frac{1}{2}})^{n+1+C_{B,l}}(1+a^{\frac{1}{2}})^{n+1-C_{B,l}}}\end{aligned} \tag{68}$$

for $0 < \theta \leq a < 1$. We let $V = u^n$ and $x = \theta^n$, and obtain the following Harnack inequality:

$$\begin{aligned}\frac{(1-a^{\frac{1}{2}})^{A_{B,l}-n-1}(1+\theta^{\frac{1}{2}})^{A_{B,l}+n+1}}{(1-\theta^{\frac{1}{2}})^{A_{B,l}-n-1}(1+a^{\frac{1}{2}})^{A_{B,l}+n+1}} &\leq \frac{V_x(a)Q_1(u(a))}{V_x(\theta)Q_1(u(\theta))} \\ &\leq \frac{(1-\theta^{\frac{1}{2}})^{n+1+C_{B,l}}(1+\theta^{\frac{1}{2}})^{n+1-C_{B,l}}}{(1-a^{\frac{1}{2}})^{n+1+C_{B,l}}(1+a^{\frac{1}{2}})^{n+1-C_{B,l}}}.\end{aligned} \tag{69}$$

Arguing as in [12], we have that

**Theorem 7.** *If there is a solution $0 \leq u \leq l(l+2B)$ of above equation with $u(0) = 0$ and $u(1) = l(l+2B)$. Then there is a Kähler metric with constant scalar curvature in the considered Kähler class.*

**Theorem 8.** *For any small positive number $f$, we have a solution $u(0) = 0, u(1-f) = l(l+2B)$. This corresponds to a Kähler metric with constant scalar curvature on the manifold with boundary $\theta \leq 1-f$.*

# 7 Global solutions

In this section, we shall extend our solutions to the hypersurface orbit. We shall let $f \to 0$. As we did in [12], we let $\tau = -\log(1-\theta)$ and have that

$$[\log[u_\tau Q(u)]]_\tau = \frac{P-\theta}{\theta}.$$

Therefore, we have that

$$
\begin{aligned}
\left[\log\left[\frac{nu^{n-1}u_\tau}{\theta^{n-1}}Q_1(u)\right]\right]_\tau &= \frac{P-\theta}{\theta} - \frac{(n-1)\theta_\tau}{\theta} \\
&= \frac{P-\theta}{\theta} - (n-1)\left(\frac{1}{\theta}-1\right) \\
&= \frac{P-n+1+(n-2)\theta}{\theta} \qquad (70)\\
&= n - \frac{2^{-1}u}{\sqrt{\theta(u+B^2\theta)}}\frac{\dot{m}(u)}{Q_1(u)} \\
&= T(u,\theta) \\
&\to n - \frac{um(u)}{2Q_1(u)\sqrt{u+B^2}} \\
&= n-\alpha,
\end{aligned}
$$

when $\theta$ turns to 1 and it converges unformly for $u \geq u_0$ with any $u_0 > 0$.

If $\omega$ is in the Ricci class, then $m(u) = Q_1(u)$ and

$$\alpha = 2^{-1}\sqrt{u}.$$

Let $u_i$ be a series of solutions corresponding to $f_i \to 0$. By $P(1) = 0$, for any $e_0 \in (n, n+1)$ there are two numbers $A(e_0) < l(l+2B)$ and $B(e_0) > 0$ such that if $u > A(e_0)$ and $\tau > B(e_0)$ then $\alpha > e_0 > n$ and $T(u, \theta(\tau)) < n - e_0$. Let $\tau_i$ be a point of $\tau$ such that $u_i(\tau_i) = A(e_0)$, and if we also have $\tau_i > B(e_0)$ then

$$\left[\log\left[\frac{nu_i^{n-1}u_{i,\tau}}{\theta^{n-1}}Q_1(u_i)\right]\right]_\tau = \frac{P-n+1+(n-2)\theta}{\theta} = T(u,\theta) < n - e_0$$

for $\tau \geq \tau_i$.

Let $w = \frac{nu^{n-1}u'}{\theta^{n-1}}Q_1(u)$, then

$$w_i \leq e^{(n-e_0)(\tau-\tau_i)}w_i(\tau_i).$$

If there is no subsequence of $\tau_i$ which tends to $+\infty$, then there is a subsequence of $\tau_i$ which tends to a finite number $\tau_0$. By the left side of the Harnack inequality (69), we see that $V_{i,x}(\theta(\tau_0))$ must be bounded from above, otherwise $V_{i,x}$ will be bounded from below by a very large number such that $V_i$ will be bigger than $l(l+2B)$ before $x$ reaching the point 1. That is, there is a subsequence of $u_i$ converging to a solution $u$ of our equation with $u(1) > A(e_0)$.

We shall observe that there is no subsequence of $\tau_i$ which tends to $+\infty$ under *certain* condition below.

If there is a subsequence of $\tau_i$ which tends to $+\infty$, we might assume that

$$\lim_{i\to+\infty} \tau_i = +\infty,$$

and $\tau_i > B(e_0)$. To make the things simpler, we should avoid the cases in which $G = S = A_n$. In those cases, the second Betti numbers are 2 and the manifolds are homogeneous. By Calabi's result, all the extremal metrics are homogeneous and therefore they are unique since there is only one invariant metric in the the given Kähler class. As we see before in the last section in the paragraph after (67), $u = c\theta$ will solve the equations.

Thus, we can assume that $G \neq S$, and therefore there is at least one $a_i$. From the equation (67), we observe that if

$$u_{i,\tau}(\tau_i)u_i^{n-1}(\tau_i) > 2(l(2B+l))^{n-1}(a_1^2 - B^2)A_{B,l} > 2u^{n-1}(a_1^2 - U)A_{B,l},$$

then

$$\frac{u_{i,\tau}(\tau_i)}{a_1^2 - U(\tau_i)} > 2A_{B,l}$$

and we have that $v_\tau = u_i^{n-1}u_{i,\tau}$ is increasing for $\tau \geq \tau_i$. This can not happen. Therefore, $u_{i,\tau}(\tau_i)$ is bounded from above.

We shall see that in this circumstance there is a subsequence of

$$\tilde{u}_i(\tau) = u_i(\tau + \tau_i)$$

which converges in $C^1$ norm to a nonconstant function $\tilde{u}$. We see that for each $\tau \geq 0$, $w_i$ is decreasing and $\tilde{u}_{i,\tau}$ are uniformly bounded. For each $\tau < 0$, $-A_{B,l} < [\log w_i]_\tau < n + C_{B,l}$ when $i$ big enough, that is, $\tilde{V}_{i,\tau}$ are also bounded uniformly on $i$ over any closed intervals. Therefore, a subsequence of $\tilde{V}_i$ converges in the $C^1$ norm to a function $\tilde{u}$. Thus, the same thing happens for a subsequence of $\tilde{u}_i$.

To observe that $\tilde{u}$ is not a constant, we notice that

$$\frac{nu_i^{n-1}u_{i,\tau}}{\theta^{n-1}} \leq C_i \frac{nu_i^{n-1}(\tau_i)u_{i,\tau}(\tau_i)}{\theta^{n-1}(\tau_i)} e^{(n-e_0)(\tau-\tau_i)}$$

for $\tau \geq \tau_i$, where $C_i$ does not depend on $u_i$. That is,

$$nu_i^{n-1}u_{i,\tau} \leq Cu_{i,\tau}(\tau_i)e^{(n-e_0)(\tau-\tau_i)}.$$

By integrating both side we have that

$$(l(l+2B))^n - A(e_0)^n \leq -\frac{C}{n-e_0} u_{i,\tau}(\tau_i),$$

i.e., $u_{i,\tau}(\tau_i)$ is bounded from below. Therefore, $\tilde{u}_{i,\tau}(0)$ are bounded from below. We have that $\tilde{u}_\tau(0) > 0$. This implies that $\tilde{u}$ is not a constant.

Then, $\tilde{u}$ satisfies the equation

$$[\log[x^{n-1} x_\tau Q_1(x)]]_\tau = -\alpha + n$$

on $(-\infty, +\infty)$. Therefore,

$$[x^{n-1} x_\tau Q_1(x)]_\tau = (-\alpha + n) x^{n-1} Q_1(x) x_\tau.$$

Integrating as in [12], we have that

$$\int_{x(-\infty)}^{x(+\infty)} f_l dx = 0,$$

where

$$f_l = (-\alpha + n) x^{n-1} Q_1(x).$$

As in [12], we see that $x(+\infty) = l(l+2B)$.

As in [12], we shall prove:

**Lemma 5.** $n - \alpha$ has only one zero.

Proof: As in [12], we may expect that $x$ is related to a Kähler metric of constant scalar curvature on the normal line bundle over the hypersurface orbit. Hence, we may apply the method of counting zeros in [10], [12] to this circumstance. $x^{n-1} x' Q_1(x)$ is proportional to "$\varphi Q$" in [10]. Therefore, the counting of zeros of $n - \alpha$ should be the same as counting the zeros of the derivative of "$\varphi Q$" to "$U$" there.

Let $v = \sqrt{u + B^2}$, then $u = v^2 - B^2$ and $a_i^2 - u = (-a_i + v)(-a_i - v)$. We observe that $g_l = 2v f_l$ is actually a polynomial of $v$ and should be proportional to the derivative of "$\varphi Q$" in [10]. Therefore, we may expect that

$$y = \frac{2}{l}(-B - v) - 1$$

corresponds to the "$U$" in [10]. We let

$$q = 2vQ(v),$$

and observe that $q$ is proportional to the "$Q$" in [10].

We see that

$$\begin{aligned} g_l &= nq - m(u)u^n \\ &= nq - \frac{R(U)}{2}u^{n-1} \\ &= nq - \frac{R_0}{2}\int QdU + \frac{1}{2}\int pdU. \end{aligned} \tag{71}$$

Let $g_l'$ be the derivative of $g_l$ to $v$, we have that

$$\begin{aligned} g_l' &= nq' - vR_0Q + vp \\ &= nq' + vP_2 - vR_0Q + vP_3 \\ &= \Delta - mq, \end{aligned} \tag{72}$$

where $P_3 = 2m_1Q$ is the $Q$ term in $p$ and $P_2 = p - P_3$ is the positive linear combination of $\frac{Q}{q_i}$,

$$\Delta = nq' + vP_2,$$

$m = \frac{R_0}{2} - m_1$. Therefore,

$$g_l = \int_0^v (\Delta - mq)dv.$$

**Lemma 6.** *The coefficients of $\Delta$ are always positive.*

Proof of Lemma 6: From Theorem 3, we see that the 1-strings do not have any contribution to $\Delta$.

The contibution to $P_2$ of each 2-string and 3-string, 4-string of the $U-B^2$ factor is in the first term of the $p(U)$ in the Theorem 3.

The contribution to $P_2$ of each 2-string and 3-string, 4-string related to the $Q_1$ factors is $\frac{a_{\rho,i}a_i}{q_i}q$.

For the first term of $\Delta$, we have $2nQ$ (one might call it the term of $v$ factor since $Q = \frac{q}{2v}$) with $2n > 0$.

Then, we have the $U - B^2$ term (or the term of $\frac{q}{U-B^2}$)

$$\begin{aligned} &2(n-1)v(2nv)(U-B^2)^{n-2}Q_1 \\ &= (n-1)v[2n(v-B) + 2n(v+B)](U-B^2)^{n-2}Q_1 \end{aligned}$$

with both $2n$ positive.

Similarly, we have $q_s$ factor of $Q_1$ term (or the term of $\frac{q}{q_s}$)

$$\begin{aligned} & 2v[-2nv + a_s a_{\rho,s}]\frac{Q}{q_s} \\ = \quad & v[(2n - a_{\rho,s})(a_s - v) - (2n + a_{\rho,s})(a_s + v)]\frac{Q}{q_s} \end{aligned}$$

with coefficients $2n - a_{\rho,s} > 0$ and $-2n - a_{\rho,s}$.

So we need to check that the last coefficient is also positive. There are two ways to prove this. First we notice that this actually is the same to check that the coefficients

$$2n, 2n, 2n$$

and

$$2n - a_{\rho,s}, -2n - a_{\rho,s}$$

are all positive. We claim that these are the components of the Ricci curvature of the exceptional divisor, then the positivity comes from the positivity of the Ricci curvature of the compact rational homogeneous spaces. The point is that $v$ is corresponding to an $H$ in the calculation of the metric and the volume form, and we should prove that the contribution of $H$ to the Ricci curvature is exactly $2n$, i.e.,

$$(q_{G/P_\infty}, H)_0 = (q_{S/(S\cap P_\infty)}, H)_0 = 2n,$$

where $P_\infty$ is the isotropic group of the exceptional divisor at $p_\infty$. Notice that $P_\infty$ is parabolic.

For $S = A_n$, the semisimple part of $P_{\infty,1}$ is generated by $\alpha_3, \cdots, \alpha_n$ with an orientation $e'_1 = e_1, e'_i = e_{i+1}$ $n+1 > i > 1, e'_{n+1} = e_2$. Therefore,

$$(q_{S/P_{\infty,1}}, H)_0 = n + n = 2n.$$

This gives a proof of our Lemma 5.

Secondly, we could *also* check the positivity of the last coefficient with a case by case checking. That will also give all the $a_{\rho,s}$ in concrete calculations. This is extremally useful when we check the Fano property of the manifolds and classify the manifolds with higher codimensional end (see [17]). For example, we can check that

**Proposition 1.** *In the affine isotropic case the manifold is Fano if and only if*

$$-2(n+1) - a_{\rho,s} > 0$$

*holds.*

We could give another proof that the last coefficient $-2n - a_{\rho,s} > 0$.

This is a little bit long, since there are so many cases. We shall check the last inequality

$$2n + a_{\rho,s} < 0$$

for the cases of

$$G = A_{m+n+k}, B_{m+n+k+1}, C_{m+n+k+1}, D_{m+n+k+1}$$

first, then $G_2$. We will leave the cases of $G = F_4$ and of $E_8$ to another paper, since the proof is too tedious. The cases of $G = E_6$ and $E_7$ will follow from those of $E_8$. If $G = A_{m+n+k}$, we have that

$$\begin{aligned}\rho_{G/P}(F_{e_l-e_{m+1}}, JF_{e_l-e_{m+1}}) &= -(q_{G/P}, -2H_{e_l-e_{m+1}}) \\ &= 2(-l_1 - l_2 + 2m + n + 2).\end{aligned}$$

We also have that $-2H_{e_l-e_{m+1}} = -2H_{e_l} - H - H_{e_{m+1}+e_{m+2}}$, and therefore

$$a_{\rho,l} = -2(-l_1 - l_2 + 2m + n + 2) \leq -2(n+2).$$

The corresponding affine manifolds are Fano.

If $G = B_{m+n+k+1}$, we have that (1) $(q_{G/P}, e_l)_0 = -l_1 - l_2 + 2(m+n+k) + 3$ in the standard inner product, but we took an inner product such that $(e_l, e_l) = \frac{1}{2}$, therefore, $B_{\rho,l} = 2(l_1 + l_2 - 2(m+n+k) - 3)$ if $l_1 \leq l \leq l_2$ and there is a $S_1$ factor $A_{l_2-l_1}$ or $l$ is not in any $S_1$ factor in which case we let $l_1 = l = l_2$; or (2) $B_{\rho,l} = 0$ if $l$ is in a $S_1$ factor of type $B$. We have 2-strings generated by $e_l - e_{m+1}, e_l + e_{m+2}, e_{m+2}$ with $l \leq m$, $e_{m+2} + e_i$ with $m + 2 < i \leq m + n + 1$ and $e_{m+2} \pm e_j$ with $m + n + 1 < j \leq m + n + k + 1$. The corresponding $a_{\rho,s}$ are

$$\begin{aligned}&-2(-l_1 - l_2 + 2(m+n+k) + 3 - 1 - n - 2k) \\ &= -2(2m + 2 + n - l_1 - l_2) \leq -2(n+2),\end{aligned}$$

$$\begin{aligned}&-2(-l_1 - l_2 + 2(m+n+k) + 3 + 1 + n + 2k) \\ &= -2(2(m + 2k + 2) + 3n - l_1 - l_2) \leq -2(3n+4),\end{aligned}$$

$$-2(1 + n + 2k) \leq -2(n+1),$$

$$-2(2 + 2n + 4k) \leq -4(n+1),$$

and

$$\begin{aligned}
&-2(1+n+2k-(l_1+l_2-2(m+n+k)-3))\\
&\quad\leq\ -2(1+n+2k+1)\\
&\quad\leq\ -2(n+4),
\end{aligned}$$

$$\begin{aligned}
&-2(1+n+2k+(l_1+l_2-2(m+n+k)-3))\\
&\quad\leq\ -2(1+n+2k-2k+1)\\
&\quad=\ -2(n+2)
\end{aligned}$$

in the (1) case or

$$-2(1+n+2k)\leq -2(n+1)$$

in the (2) case. The corresponding manifolds are nef and Fano if and only if $k>0$.

If $G=B_{m+1}$ and $S=A_1$ generated by $e_{m+1}$, $H=2H_{e_{m+1}}$. By $(H,H)_A=1$, we get $(e_{m+1},e_{m+1})_A=\frac{1}{4}$. We have 3-strings generated by $e_l-e_{m+1}$, we have that

$$a_{\rho,l}=\frac{B_{\rho,l}}{2}=-2(-l_1-l_2+2m+3)\leq -6=-2(n+2).$$

The corresponding affine manifold is Fano.

If $G=C_{m+n+k+1}$, then (1) $B_{\rho,l}=-2(-l_1-l_2+2(m+n+k+2))$ if $l_1\leq l_2$ and there is a $S_1$ factor $A_{l_2-l_1}$ or $l$ is not in any $S_1$ factor (in this case $l_1=l=l_2$); or (2) $B_{\rho,l}=0$ if $l$ is an $S_1$ factor of type C. We have 2-strings generated by $e_l-e_{m+1}, e_l-e_{m+2}$ with $l\leq m$, $e_{m+2}+e_i$ with $m+2<i\leq m+n+1$, $e_{m+2}\pm e_l$ with $m+n+1<l\leq m+n+k+1$, and 3-string generated by $2e_{m+2}$. The corresponding $a_{\rho,s}$ are

$$\begin{aligned}
&-2(-l_1-l_2+2(m+n+k+2)-2-n-2k)\\
&\quad=\ -2(-l_1-l_2+2m+n+2)\\
&\quad\leq\ -2(n+2),
\end{aligned}$$

$$\begin{aligned}
&-2(-l_1-l_2+2(m+n+k+2)+2+n+2k)\\
&\quad=\ -2(-l_1-l_2+2(m+2k+3)+3n)\\
&\quad\leq\ -6(n+2),
\end{aligned}$$

$$-2(2n+4+4k)\leq -4(n+2),$$

$$-2(n+2+2k-l_1-l_2+2(m+n+k+2)) \leq -2(n+4+2k) \leq -2(n+6)$$
$$(\text{or } -2(n+2+2k) \leq -2(n+2)),$$
$$-2(n+2+2k+l_1+l_2-2(m+n+k+2)) \leq -2(n+2+2k-2k) = -2(n+2)$$
$$(\text{or } -2(n+2+2k) \leq -2(n+2)),$$
and
$$-2(2n+4+4k) \leq -4(n+2).$$
The corresponding affine manifolds are Fano.

If $S = A_1$ and $G = C_{m+1}$, then $\alpha = 2e_{m+1}$. But by $[H_{2e_{m+1}}, F_{2e_{m+1}}] = 4G_{2e_{m+1}}$ we have that $H = \frac{1}{2}H_{2e_{m+1}}$, since $[H, F] = 2G$. By $(H, H)_A = 1$, we have that $(e_{m+1}, e_{m+1})_A = 1$. The only strings we need to consider are the 2-strings generated by $e_l - e_{m+1}$. We have that
$$\omega(F_{e_l-e_{m+1}}, JF_{e_l-e_{m+1}}) = (\frac{a}{2}H_{2e_{m+1}} + iB_l e_l, -2H_{e_l-e_{m+1}})_A = 2a - 2B_l$$
and
$$a_{\rho,l} = B_{\rho,l} = -(-l_1 - l_2 + 2(m+2)) \leq -4 = -2(n+1).$$
The corresponding affine manifold is nef but not Fano.,

If $S = A_n$ and $G = D_{m+n+k+1}$, then (1)
$$B_{\rho,l} = -2(-l_1 - l_2 + 2(n+m+k+1))$$
if $l_1 \leq l \leq l_2$ and there is an $S_1$ factor $A_{l_2-l_1}$ or $l$ is not related to the Dynkin graph of any $S_1$ factor ($l_1 = l = l_2$ in this case); or (2) $B_{\rho,l} = 0$ if $l$ is in an $S_1$ factor of type D. There are 2-strings generated by $e_l - e_{m+1}, e_l + e_{m+2}$ with $l \leq m$, $e_{m+2} + e_i$ with $m+2 < i \leq m+n+1$ (if $n > 1$) and $e_{m+2} \pm e_j$ with $m+n+1 < j$. The corresponding $a_{\rho,s}$ are
$$\begin{aligned} & -2(-l_1 - l_2 + 2(n+m+k+1) - n - 2k) \\ = & \ -2(-l_1 - l_2 + 2(m+1) + n) \\ \leq & \ -2(n+2), \end{aligned}$$
$$\begin{aligned} & -2(-l_1 - l_2 + 2(m+n+k+1) + n + 2k) \\ = & \ -2(-l_1 - l_2 + 2(m+2k+1) + 3n) \\ \leq & \ -2(3n+2), \end{aligned}$$
$$-2(2n+4k) \leq -4n \leq -2(n+2),$$

and

$$-2(n+2k-l_1-l_2+2(m+n+k+1)) \leq -2(n+2k) \leq -2(n+2)$$

$$(\text{or } -2(n+2k) \leq -2(n+2)),$$

$$\begin{aligned} &-2(n+2k+l_1+l_2-2(m+n+k+1)) \\ \leq \quad & -2(n+2k+2-2k) \\ = \quad & -2(n+2) \end{aligned}$$

$$(\text{or } -2(n+2k) \leq -2(n+2)),$$

The corresponding affine manifolds are Fano.

If $S = A_3$ is generated by $e_{m+1} - e_{m+2}, e_{m+2} - e_{m+3}, e_{m+2} + e_{m+3}$ in $D_{m+3}$, we let $\alpha = e_{m+2} - e_{m+3}$. We have 2-strings generated by $e_l - e_{m+2}, e_l + e_{m+3}$ $l \leq m$ and

$$a_{\rho,l} = B_{\rho,l} = -2(-l_1 - l_2 + 2(m+3)) \leq -12 = -2(n+3).$$

The corresponding affine manifold is Fano.

If $G = G_2$ and $\alpha = \alpha_1$, then $a_1 = B_1, a_2 = 3B_1$.

$$(aH + cG + B_1 i(3\alpha_1 + 2\alpha_2), -2H_{3\alpha_1+2\alpha_2}) = -6B_1.$$

And,

$$\Big(\sum_{\alpha \in \Delta^+ - \{\alpha_1\}} \alpha, 2(3\alpha_1 + 2\alpha_2)\Big)_0 = (3(\alpha_1 + 2\alpha_2), 2(3\alpha_1 + 2\alpha_1)) = 36,$$

we have that $B_{\rho,1} = -6 = -2(n+2)$. The corresponding affine manifold is Fano.

If $G = G_2$ and $\alpha = \alpha_2$, $H = \frac{1}{3} H_{\alpha_2}$. By $(H,H)_A = 1$, we get that $(H_{\alpha_2}, H_{\alpha_2})_A = 9$, then

$$\omega(X,Y) = (aH + cG + B_1 i(2\alpha_1 + \alpha_2), [X,Y]).$$

$$\omega(F_{2\alpha_1+\alpha_2}, JF_{2\alpha_1+\alpha_2}) = -6B_1.$$

There are two 2-strings generated by $\alpha_1$ and $3\alpha_1 + \alpha_2$. We have that $a_1 = B_1$ and $a_2 = 3B_1$. But we also have that

$$\sum_{\alpha \in \Delta^+ - \{\alpha_2\}} \alpha(2(2\alpha_1 + \alpha_2)) = 5(2\alpha_1 + \alpha_2)(2(2\alpha_1 + \alpha_2)) = 20 = -6B_{\rho,1}.$$

Therefore, we have that $B_{\rho,1} = -\frac{10}{3}$ and

$$a_{\rho,1} = -\frac{10}{3}, a_{\rho,2} = -10 < -3 = -2n - 1.$$

But the corresponding manifold is not even nef.

Before we go further, we shall make an observation. If $G' \subset G$ is a subgroup of $G$ such that the Dynkin graph of $G'$ is a subgraph of that of $G$ and $S \subset G'$ fits with the Dynkin graph of $G'$, then, if the last inequality holds for $G, S$, so does it for $G', S$. Actually, let $\beta$ be a positive root in $G'$ which generates a nontrivial string (i.e., either 2, 3 or 4-string), then

$$(q_{G/P}, \beta) = (q_{G/P_1}, \beta) + (q_{G'/P_2}, \beta) = (q_{G'/P_2}, \beta)$$

where $P_1$ is the minimal parabolic subgroup of $G$ containning $G'$ and $P_2 = P \cap G'$, since $(q_{G/P_1}, )$ is trivial on $G'$. Therefore, once the last inequality is true for $E_8$, it is also true for both $E_6$ and $E_7$. Similarly, the inequality $a_{\rho,s} \leq -2(n+2)$ holds for $G = E_k \quad 6 \leq k \leq 8$. Therefore, the last inequality holds for the left cases of $G = F_4, E_6, E_7, E_8$ by a further calculation with $F_4$ and $E_8$.

Q. E. D.

Therefore, as we argued in [12, p. 73], if $n - \alpha$ has two zeros, then $\Delta - mq$ has $\deg q - 3 + 4 = \deg q + 1$ zeros. That will be a contradiction to the degree of this polynomial which is $2 \deg Q + 1$. Thus, we obtain our Lemma 5.

Q. E. D.

Now, we have that $f_l$ has a unique zero. Therefore, if

$$\int_0^{l(l+2B)} f_l dx < 0, \tag{73}$$

we can not have that

$$0 = \int_{x(-\infty)}^{l(1+2B)} f_l dx \leq \int_0^{l(l+2B)} f_l dx.$$

Otherwise, we have a contradiction.

By choosing $A(e_0)$ close to $l(l + 2B)$ we have that $u(1) = l(l + 2B)$. Arguing as in [12], we have that $u'(1)$ exists and is finite. Similarly, $u''(0)$ and $u''(1)$ exist and are finite.

Also, we already see that if $G = S = A_n$, the manifold is homogeneous and admits unique extremal metric in any given Kähler class. Therefore, we have that:

**Theorem 9.** *There is a Kähler metric of constant scalar curvature in a given Kähler class if the condition (73) is satisfied.*

We shall prove the converse in [14].

**Corollary 1.** *If $G = A_k$ or $D_k$, then $a_{\rho,s} \leq -2(n+2)$ and therefore the manifolds are Fano.*

We could easily argue as in [13] p.273–274 and [12] that the right side of (73) is the Ding-Tian generalized Futaki invariant for a (possibly singular) completion of the normal line bundle of the exceptional divisor, although we do not really know that there is an actually analytic degeneration with this completion as the central fiber. Our condition here is stronger than the Ross-Thomas version of Donaldson's version of K-stability (cf. [17]).

# 8 Type II cases

Now, we consider the case of type II, i.e., the case in which the centralizer of the isotropic group containing a three dimensional simple Lie algebra $\mathcal{A}$. Since most cases are affine and other cases are actually homogeneous, we actually only need to consider the case in which $S = A_1$. We denote the manifold by $N$.

In that case, the involution induces an involution in $\mathcal{A}$ and $d = 1$. The argument after the Theorem 4 and [13] Theorem 3.1 tell us that $U_\theta(1) = \lim_{t\to+\infty} \frac{(1+t^2)^2 U'}{2t} = 0$. We actually have that $U_\theta = (1-\theta)h(\theta)$ with $h(1) > 0$. We also have that $B_\rho = 0 = B,\ k = 1$ and $l_\rho = -4 - 2 = -6$.

The Kähler-Einstein equation is

$$(1-\theta)(\frac{u''}{u'} + \frac{Q'(u)u'}{Q(u)}) = 2 - 2^{-1}(\frac{u}{\theta})^{\frac{1}{2}}.$$

The constant scalar curvature equation is

$$(1-\theta)(\frac{u''}{u'} + \frac{Q'(u)u'}{Q(u)}) = 2 - 2^{-1}(\frac{u}{\theta})^{\frac{1}{2}}\frac{m(u)}{Q(u)} = P\theta^{-1},$$

where $m(u) = \frac{R_0 \int Q du - \int p du}{u}$. We also notice that our $P\theta^{-1}$ here is the $P$ in [13].

If $G = S = A_1$, we have $R_0 = \frac{2l_\rho}{l} = -12l^{-1}$, $Q = Q_1 = 1$, $m(u) = \frac{l_\rho}{l} = -6l^{-1}$. The equation is

$$(1-\theta)u'' = (2 + \frac{3}{l}(\frac{u}{\theta})^{\frac{1}{2}})u.$$

We have that

$$2(1 - A_l \theta^{-\frac{1}{2}}) \leq P\theta^{-1} \leq 2$$

with a constant $A_l \geq \frac{3}{2}$ since $P(1) = -1$ as in [13].

The difference of this case from those in section 6 before can be summarized in following two theorems:

**Theorem 5'.** *$\omega$ in Theorem 1 extends to a Kähler metric over the exceptional divisor of $N$ if and only if $\lim_{t \to +\infty} tf = l > a_i$ and $U_\theta(1) = 0$.*

Let $f(t) = \frac{2lt}{1+2t^2}$, then $U = 4l^2\theta(1+\theta)^2$ satisfies the assumption of Theorem 4 and 5'. Actually, one can check that this $U$ is the solution of the equation when $G = S = A_1$.

Therefore, we have that:

**Theorem 6'.** *The Kähler classes on $N$ are in one to one correspondence with the elements in the set $\Gamma = \{l|_{0>l>a_i}\}$.*

We also have that $f_l = (1-\alpha)Q$ with $\alpha = 2^{-1}u^{\frac{1}{2}}\frac{m(u)}{Q(u)}$.

If $G = S = A_1$, $f_l = 1 + 3l^{-1}u^{\frac{1}{2}}$. The integral is

$$\int_0^{l^2} (1 + 3l^{-1}u^{\frac{1}{2}})du = l^2 - 2l^2 = -l^2 < 0$$

always.

In general, we have:

**Theorem 9'.** *For a nonaffine type II cohomogeneity one manifold there is a Kähler-Einstein metrics if $M$ is Fano and*

$$\int_0^{36} (1 - 2^{-1}u^{\frac{1}{2}})Qdu < 0.$$

*There is a Kähler metric of constant scalar curvature if*

$$\int_0^{l^2} f_l du < 0$$

*holds.*

We shall prove the converse in [14].

# 9 Kähler-Einstein metrics, Fano properties and further comments

If the Kähler class is the Ricci class, we have that

$$B = B_\rho = 0,\ l = l_\rho = -2(n+1), \tag{74}$$

$$m(u) = Q_1(u),$$

$$\alpha = 2^{-1}\sqrt{u}. \tag{75}$$

Therefore,

$$f_l = [n - 2^{-1}\sqrt{u}]u^{n-1}Q_1(u). \tag{76}$$

In this section, we show how we can check the Kähler-Einstein property case by case on the pairs of groups $(S, G)$.

We also notice in [20] that if $S = B_n$ or $C_n$ the manifolds are always Fano.

Now, we consider the case in which $S = A_n$ and $G = A_{m+n+k}$ such that $S$ is generated by $e_{i+1} - e_i$ with $m+1 \le i \le m+n$. We shall see that the manifolds are Fano for the compact affine almost-homogeneous manifolds of cohomogeneity one. For the case of type II manifolds other than the affine case, we shall see that they have numerical effective anticanonical line bundle and are Fano if every $e_i$ which is not in $A_n$ is in some $A_l$ factor in $S_1$, here we say that $e_i$ is in an $A_l$ if $e_i - e_j \in A_l$ for some $e_j$.

By our formula, we have

$$\rho_{G/P}(F_{e_l - e_{m+1}}, JF_{e_l - e_{m+1}}) = 2(-l_1 - l_2 + 2m + n + 2)$$

if $l_1 \le l \le l_2 \le m$ induces an $A_{l_2 - l_1}$ in $S_1$. We also have that

$$\begin{aligned}[][F_{e_l - e_{m+1}}, JF_{e_l - e_{m+1}}] &= [F_{e_l - e_{m+1}}, -G_{e_l - e_{m+1}}] \\ &= -2H_{e_l - e_{m+1}} = -2H_{e_l} - H - H_{e_{m+1} + e_{m+2}}\end{aligned}$$

and the coefficient of $H$ is $-1$. Therefore, we have that

$$a_{\rho,l} = -2(-l_1 - l_2 + 2m + n + 2) < l_\rho = -2(n+1)$$

if the manifold is affine. If the manifold is of type II but not affine, then $n = 1$ and

$$-2(-l_1 - l_2 + n + 2) = -2(-l_1 - l_2 + 2m + 3) \le l_\rho = -6$$

with a equality only when $l_2 = l_1 = m$. Similarly for $l > m + n$.

We have our claim.

When $k = m = 0$, we have the product of two projective spaces. Therefore, there are Kähler-Einstein metrics. Indeed, one can easily check that

$$\begin{aligned} K_{0,0}^n = \int_0^{2(n+1)} (2n - v)v^{2n-1} dv &= (v^{2n} - \frac{v^{2n+1}}{2n+1})|_0^{2(n+1)} \\ &= (1 - \frac{2(n+1)}{2n+1})(2(n+1))^{2n} \\ &\leq 0. \end{aligned}$$

When $k = 0$ and $n = 1$ with a maximal parabolic subgroup $P$, we have the examples $M_{m+1}$ and $N_{m+1}$ in [12], [13].

Similarly, we can consider the general case with the maximal parabolic subgroup, in which $S_1 = A_m A_k$, then we have the integral

$$K_{m,k}^n = \int_0^{2(n+1)} v(2n - v)v^{2(n-1)}(4(m+n+1)^2 - v^2)^m(4(k+n+1)^2 - v^2)^k dv$$

for the affine case and

$$K'_{m,k} = \int_0^6 v(2 - v)(4(m+2)^2 - v^2)^m(4(k+2)^2 - v^2)^k dv$$

for the nonaffine case in which $n = 1$ and $m, k \neq 1$.

For the case of $k = 0$ and $n = 1$, if we let $v = 4x$, we have that

$$K_{m,0}^1 = \int_0^1 4^2 x \cdot 2(1 - 2x) \cdot 2^m((m+2)^2 - 4x^2)^m dx,$$

similarly for $K'_{m,0}$, which is exactly the integrals in [8] and [13] up to a multiplication of a constant $2^{m+5}$.

We first have that:

**Lemma 7.** *$K_{i,j}^1 < 0$ for $i, j = 0, 1, 2$.*

Proof: By the method in [8] or by using Mathematica. We actually only need to check the case with $i = j = 2$, $i = j = 1$ and $i = 1, j = 2$.

For example, with Mathematica we use

```
Integrate[v(2-v)(64-v^2 )^4 , {v, 0, 4}]

Integrate[v(2-v)(36-v^2 )^2 , {v, 0, 4}]
```

and

```
Integrate[v(2-v)(64-v^2 )^2 (36-v^2 ), {v, 0, 4}].
```

Q. E. D.

We could call the related manifolds $M^n_{m,k}$ and $N_{m,k}$ (not to be confused with the similar notations in last section). We have that:

**Theorem 10.** *$M^1_{m,k}$ and $N_{m,k}$ are nef. $M^1_{m,k}$ are Kähler-Einstein for all $m, k$. $N_{m,k}$ are Fano if and only if $m, k \neq 1$, in which case $N_{m,k}$ are Kähler-Einstein.*

Proof: We have that

$$K^1_{m,k} \leq CK^1_{2,k}$$

if $m \geq 2$ by applying the comparasion method we used in [17], [20], and as follows:

We can compare the change rate of the factor $h(v) = (4(n+m+1)^2 - v^2)^m$. We let

$$\begin{aligned} t(m) &= (\log h)' \\ &= m(\frac{1}{2n+2m+1+v} - \frac{1}{2n+2m+1-v}). \end{aligned}$$

Then,

$$t(m+1) - t(m) = \frac{-2v[4(n+1)^2 - 4m(m+1) - v^2]}{(4(n+m+1)^- v^2)(4(n+m+2)^2 - v^2)} > 0$$

if $m > n$. Therefore, if $K^n_{m,k} \leq 0$ with $m > n$, then $K^n_{m+1,k} < 0$.

And, we have that

$$K'_{m,k} < K^1_{m,k} < 0.$$

Q. E. D.

We also notice that

$$\lim_{m \to +\infty} (2m)^{-2m} K^n_{m,k} = e^{4(n+1)} K^n_{0,k}.$$

We shall prove that $K^n_{0,k} < 0$. This would imply that (1) $M^n_{0,k}$ admits Kähler-Einstein metrics, which also generalizes our results in [8]; and (2) for any given $n, k$ there is a integer $N(n,k)$ such that if $m > N(n,k)$ then $M^n_{m,k}$ admit Kähler-Einstein metrics.

**Lemma 8.** *Let $m = l(n+1), k = sm$, then*

$$K^n_{m,k} = -CI_{n,l,s},$$

*With*

$$\begin{aligned} I_{n,l,s} = \int_0^1 & x^{2n}(1-x)((1+l)^2 - x^2)^{m-1}((1+sl)^2 - x^2)^{k-1} \\ [ \quad & (1-x^2)[(1+l+sl)(1-x^2) \\ + \quad & l^2(s(2+l+sl) + (1+s)^2) + l(1+s)(1-x)] \\ + \quad & sl^3(1+s)(1-x) + sl^2(sl^2 - 4x)]dx \end{aligned}$$

*and a constant $C > 0$. In particular, (1) $K^n_{0,k} < 0$ and (2) $K^n_{m,k} < 0$ if $mk \geq 4(n+1)^2$. Therefore, (3) $K^n_{m,k} < 0$ if $m \geq 4(n+1)^2$.*

Proof: We let $n = l^{-1}m - 1$, $k = sm$ and $v = 2l^{-1}m$ we have that

$$\begin{aligned} K^n_{m,k} &= C_1 \int_0^1 x^{2l^{-1}m-3}(l^{-1}m(1-x) - 1)((1+l)^2 - x^2)^m((1+sl)^2 - x^2)^{sm}dx \\ &= C_1[l^{-1}m \int_0^1 x^{2n-1}(1-x)(((1+l)^2 - x^2)((1+sl)^2 - x^2)^s)^m dx \\ &- \int_0^1 x^{-3} m \int_0^x (y^{2l^{-1}}((1+l)^2 - y^2)((1+sl)^2 - y^2)^s)^{m-1} \\ &[ \quad 2l^{-1}y^{2l^{-1}-1}((1+l)^2 - y^2)((1+sl)^2 - y^2)^s - 2y^{2l^{-1}+1}((1+sl)^2 - y^2)^s \\ &- 2sy^{2l^{-1}+1}((1+l)^2 - y^2)((1+sl)^2 - y^2)^{s-1}]dydx] \\ &= C_2[\int_0^1 x^{2n-1}(1-x)((1+l)^2 - x^2)^m((1+sl)^2 - x^2)^k dx \\ &- 2\int_0^1 y^{2n+1}((1+l)^2 - y^2)^{m-1}((1+sl)^2 - y^2)^{k-1}[(1+l)^2(1+sl)^2 \\ &- (1+l)(1+sl)(2+l+sl)y^2 + (1+l+sl)y^4] \int_y^1 x^{-3}dxdy]. \end{aligned}$$

This is exactly what we need.

Q. E. D.

This lemma also shows that if $l, s$ are constants and $0 < sl^2 < 4$, then $I_{n,l,s}$ are increasing with $\lim_{n\to+\infty} I_{n,l,s} > 0$. In particular, $K^n_{n+1,n+1} > 0$ when $n$ big enough.

Actually, using Mathematica with:

```
Integrate[v^(2m-3)(m(1-v)-1)(4-v^2 )^(2m) , {v, 0, 1}]
```

we obtain that:

**Lemma 9.** *$K^n_{n+1,n+1} > 0$ if $n \geq 10$. Otherwise, $K^n_{n+1,n+1} < 0$.*

Similarly, we can use Mathematica to calculate $K^{11}_{12,13}$, $K^{11}_{12+k,12-k}$ $1 \leq k \leq 7$, $K^{11}_{13+k,12-k}$ $1 \leq k \leq 5$, $K^{11}_{19,k}$ $k \leq 4$ and obtain that:

**Lemma 10.** *$K^{11}_{12+k,13-k} < 0$, $K^{11}_{19,k} < 0$ always and $K^{11}_{12+k,12-k} > 0$ if $0 \leq k \leq 6$.*

Therefore, we can check that $K^{11}_{m,k} < 0$ $m \leq 18$ if $m = 1$ and if $k \leq n_m$ or $k \geq N_m$ $m > 1$ with $n_2 = 2, n_k = 1$ $3 \leq k \leq 11, n_l = 2$ $12 \leq l \leq 15, n_{16} = n_{17} = 3, n_{18} = 4$; $N_2 = 12, N_3 = 16, N_4 = 18, N_5 = 19, N_{6+k} = 19 - k$ $0 \leq k \leq 12$. One might conjecture that the open set $K^n_{m,k} > 0$ is a convex set with an asymptotic cone $mk \leq 4(n+1)^2$.

Similarly, we check that $K^n_{m,k} < 0$ if $n = 5, 7$ and $K^8_{m,k} < 0$ if $m \leq 2$ or $m \geq 7$. $K^8_{m,k} < 0$ if $k \leq n_m$ or $k \geq N_m$ for $3 \leq m \leq 6$ with $n_3 = 3 = n_6, n_4 = 2 = n_5$ and $N_3 = 6, N_4 = 7 = N_5 = N_6$.

Furthermore, we have $K^9_{m,k} < 0$ if $m \leq 2$ or $m \geq 11$. $K^9_{m,k} < 0$ when $3 \leq m \leq 10$ for $k \leq n_m$ or $k \geq N_m$ with $n_k = 2$ $3 \leq k \leq 9, n_{10} = 3$; $N_3 = 10 = N_7, N_4 = N_5 = N_6 = 11, N_8 = 9, N_9 = 8, N_{10} = 7$.

We can also check that $K^n_{2,2} < 0$ if $n \leq 13$ and $K^{14}_{2,2} > 0$. $K^n_{1,n+1} < 0$ if $n \leq 33$ and $K^{34}_{1,35} > 0$, $K^{34}_{1,36} < 0$. Therefore, $K^n_{1,k} < 0$ if $n \leq 33$ $k \geq n+1$, $K^{34}_{1,k} < 0$ if $k \geq 36$.

$K^n_{1,1}, K^n_{1,2} < 0$ always and $K^n_{1,3} > 0$ $25 \leq n \leq 34$. $K^{15}_{1,k} < 0$ always.

$K^{10}_{m,k} < 0$ for $k \leq n_m$ or $k \geq N_m$ with $n_2 = 2 = n_i$ $9 \leq i \leq 12, n_i = 1$ $3 \leq i \leq 8$, $n_{13} = n_{14} = 3$ and $N_2 = 9$, $N_3 = 13 = N_8$, $N_i = 15$ for $i = 4, 5, 6$, $N_{6+i} = 15 - i$ for $1 \leq i \leq 8$.

We finally check that $K^n_{k,m} < 0$ for $n = 6, 4, 3, 2$:

**Theorem 11.** *$M^n_{k,m}$ are nonhomogeneous with Kähler-Einstein manifolds for $n \leq 7$. $M^n_{k,m}$ admit Kähler-Einstein metric for $8 \leq n \leq 11$ if $k \leq k_n$ or $k \geq K_n$ with $k_8 = 2 = k_9, k_{10} = 1 = k_{11}$; $K_n = 7 + 4(n-8)$. For $k_n < k < K_n$, there are two numbers $m^n_k > k_n$ and $M^n_k < K_n$ such that $M^n_{k,m}$ are Kähler-Einstein for $m \leq m^n_k$ or $m \geq M^n_k$; and $M^n_{k,m}$ are non-Kähler Einstein Fano manifolds for $m^n_k < m < M^n_k$.*

So far, I could not find any manifold such that the integral is zero. Otherwise, it might provid a counter example for being weakly K-Stable and Mumford stable but not Kähler-Einstein.

Our manifolds might not always be Fano in general. For example, if $S = A_n$ and $G = B_{m+n+k+1}$ such that $S$ is determined by $e_i, m+1 \leq i \leq$

$m+n+1$, with the minimal parabolic subgroup $P$, the manifolds are not Fano when $k=0$. For example, $a_{\rho,s}$ for the 2-string generated by $e_{m+2}$ is $-2(1+n+2k) = -2(n+1)$ and $l_\rho = -2(n+1)$, therefore, $a_{\rho,s}^2 - v^2 = 0$ at $v = -2(n+1)$. The manifold is not Fano. That is, affine type does not imply Fano in general in the case of $S = A_n$. However, from the proof of the Lemma 6 we have that

$$l_\rho - 2 + a_{\rho,s} < 0,$$

that is, the manifolds are not far from being Fano.

When the manifold is Fano, we notice that in the affine case, the manifold is a $\mathbf{C}P^n$ bundle over a rational projective homogeneous manifold. Let $D$ be the hypersurface line bundle of $\mathbf{C}P^n$, then $K_F = -(n+1)D$ is just the canonical line bundle of $\mathbf{C}P^n$. We denote $K_F = -(n+1)$ and $D = 1$, let $x = \frac{1}{2}v$ and we still denote $Q(v)$ by $Q(x)$, our integral is proportional to

$$\int_0^{-K_F} (-K_F - D - x)Q(x)dx.$$

For the nonaffine Type II case, $F = \mathbf{C}P^2$ as a double branched quotient of $\mathbf{C}P^1 \times (\mathbf{C}P^1)^*$, the exceptional divisor $D$ is a quardraic. Let $H$ be the hypersurface divisor, then $K_F = -3H$, $D = 2H$. As above we denote $K_F = -3$ and $D = 2$, then the integral is proportional to

$$\int_0^{-K_F} (-K_F - D - x)Q(x)dx$$

again. Moreover, by adjunct formula we have $K_D = K_F + D$ on $D$ and we write $K_D = K_F + D$ also as numbers.

Combinning with [16], [20] we have:

**Theorem 12.** *If a type II manifold $M$ is Fano, then it admits a Kähler-Einstein metric if and only if*

$$\int_0^{-K_F} (K_F + D + x)Q(x)dx = \int_0^{-K_F} (K_D + x)Q(x)dx > 0$$

*holds, where $Q(x)dx$ is the volume element.*

Proof: Let us deal with the integral in [16] page 166 first. If we let $v = \sqrt{u+1} - 1$, the integral is proportional to

$$\int_0^{\frac{3}{2}} (1-v)Q(v)dv.$$

The open orbit is a $\mathbf{C}^2$ bundle. The manifold is a $\mathbf{C}P^2$ bundle and $K_F = -3, D = 1$. Let $v = \frac{x}{2}$, then the integral is

$$\int_0^3 (2-x)Q(x)dx = \int_0^{-K_F} (-K_F - D - x)Q(x)dx$$

as desired. This also give us another confirmation that our calculation in [16] is correct.

The cases in [20] can be found in the Theorem 10.2 there.

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Combinning with [17] we have:

**Theorem 13.** *A cohonogeneity one two orbits Fano manifold with an codimension $m$ close orbit and a semisimple group action is Kähler-Einstein if and only if*

$$\int_0^{-K_F+m-1} (K_F + D + x)Q(x)dx = \int_0^{-K_F+m-1} (K_D + x)Q(x)dx > 0$$

*holds, where $Q(x)dx$ is the volume element.*

Here, we can understand the $F$ to be as the fiber in [22] but not the one in [1]. Then $K_F$ is exactly the correspondence of the canonical divisor and $D$ the exceptional divisor.

Combinning with Corollary 1 with some further calculations with exceptional Lie algebras, we have:

**Corollary 2.** *If the roots of $G$ has the same length, then $a_{\rho,s} \leq -2(n+2)$. Therefore, the affine manifolds are Fano and the nonaffine type II manifolds are nef.*

This also provide more Kähler Einstein metrics.

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