Type I Almost-Homogeneous Manifolds of Cohomogeneity One—II

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Abstract

This is the second part of [Gu1] on the existence of Kähler Einstein metrics of the general type I almost homogeneous manifolds of cohomogeneity one. We actually carry out all the results in [Gu3] to the type I cases. We also prove the existence of smooth geodesic connecting any two given metrics on the Mabuchi moduli space of Kähler metrics, which leads to the uniqueness of our Kähler metrics with constant scalar curvatures if they exist. We obtain a lot of new Kähler-Einstein manifolds as well as Fano manifolds without Kähler-Einstein metrics. Furthermore, in this paper we also deal with the cases with a higher codimensional end, then obtain more Kähler-Einstein manifolds as well as Fano manifolds without Kähler-Einstein metric. As an offshot, we are able to classify compact Kähler manifolds which are almost homogeneous of cohomogeneity one with a higher codimensional end. With applying our results to the canonical circle bundles we also obtain Sasakian manifolds with or without Sasakian-Einstein metrics. That also give some open Calabi-Yau manifolds.

In memory of Professor Hong-You Wu

Key Words: Kähler manifolds, Einstein metrics, Ricci curvature, constant scalar curvature, fibration, almost-homogeneous, cohogeneity one, semisimple Lie group, geodesic stability, sasakian Einstein, Calabi-Yau metrics, blow down criterion, contraction, exceptional divisor, classification.

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1. Introduction

This paper is the second part of [Gu1]. In [Gu1], we prove that:

Main Theorem 1. *For any simply connected type I compact Kähler complex almost homogeneous manifolds of cohomogeneity one with a hypersurface end, there*

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is an extremal metrics in a given Kähler class if and only if the condition (7) in [Gu1] holds.

We shall finish the similar results for the higher condimensional end case and the general case, in our sections 3 and 4. See also Theorem 15 in the last section for the Kähler-Einstein case.

We can also say that the existence is equivalent to the positivity of the generalized Futaki invariant, and is equivalent to the geodesic stability as we stated in [Gu3]. We shall prove the equivalence of the existence and the geodesic stability as well as the strictly slope stability in the third part of theis paper by the request of some earlier referees.

As an application, considering the canonical circle bundle, we also obtained Sasakian manifolds with and without Sasaki-Einstein metrics (with the same Reeb vector field and CR structure, see [BG Theorem 2.4 (iv)], also [Kb], [WZ]).

We shall prove the converse for the type II cases in [Gu4].

Therefore, we finished all the possible case in which the existence of the extremal metrics could be reduced to an ordinary differential equation problem. We also give many examples for both the stable and the unstable cases, as we promised in [Gu3]. It is difficult for us to find an example which is semistable but not stable. We note that since the automorphism group is semisimple, the original Futaki invariants are zero for the most manifolds we considered in this paper. Therefore, we give many and more classical examples than the example in [Ti].

In [Gu5,6,8], we dealt with the affine cases and the type II cases. To finish the program we dealt with the type I cases in [Gu1].

[PS] studied a few of the first case in [Gu1], i.e., the cases with $F = F(OP_n)$ in which $S = SO(4, \mathbf{C}), SO(6, \mathbf{C}), SO(8, \mathbf{C}), SO(10, \mathbf{C})$, where S is the induced group action on the fibers.

A classification which we refer to in this paper can be found in [Gu3 section 12]. We put these manifolds into three types there, type I, II and III.

In [Gu1] we use very explicit and elementary calculation to avoid the Cauchy-Riemann structure and other very abstract tools in [Si] and [PS] which also cumulating on results from other papers of Spiro. As we could not check the proofs of all those papers here we feel that our approach is safer[1].

We deal with the uniqueness in the second section where we prove the existence of smooth geodesics in the Mabuchi moduli spaces of the Kähler metrics.

Main Theorem 2. *For any two smooth Kähler metrics which are equivariant under the maximal compact subgroup in a given Kähler class on a type I com-*

[1] As they did make some mistakes or miss some major proofs. For example, as we shall see in section 5 that the two manifolds in the number 4 item of their table in [PS] are either not Fano or not Kähler Einstein. The integrals they had are also different from ours. When $F = Q^7$ it is just our M_7 (that is, $M_{7,1}$ in Theorem 3) in the last section and is one of the first compact almost homogeneous manifolds of cohomogeneity one with vanished Futaki invariants but without Kähler Einstein metrics. It is a Q^7 bundle over Q^8. When $F = \mathbf{C}P^7$, it is just $\mathbf{P}(T^*_{Q^8})$. It is a complex contact manifold and by [LB Thereom 2.3] (1) it can not be Fano, also by [LB Theorem A] (2) if it admits a Kähler Einstein metric it will be a twistor space of a quaternion-Kähler manifold with positive Ricci curvature. Both (1) and (2) contradict to their results. Compare also [Ye Corollary 3] for the case of the $\mathbf{C}P^1$ bundle over Q^2.

pact complex almost homogeneous manifold of cohomogeneity one, there is a smooth geodesic connecting them in the Mabuchi moduli space of Kähler metrics.

The same result was proven for the toric manifolds and type III manifolds in [Gu7] and for some type II manifolds in [Gu3], then for all type II manifolds in [Gu4].

The equivalence between the existence and the geodesic stability follows as exactly as what we did in [Gu3]. The proof is actually simpler than that in [Gu3] since we apply the orthogonal basis. Although there are efforts of Donaldson, Mabuchi, Chen and Tian on the uniqueness, we are not able to go through their proofs here. Our proof is much simpler and earlier, also more explicit. Actually, I have obtained this proof in 1999 (Cf. [Gu7] and [Gu3]) and showed both works to both Donaldson and Mabuchi in 2000 before I left Princeton including the discussions of the geodesic stability on the toric varieties.

We then treat the higher codimensional end cases and the general cases in the third and fourth sections. In the third section, we obtain a necessary and sufficient condition for the possible blow down of the hypersurface orbit. This enable us to actually classify the cases of manifolds with a higher codimensional end. This is a surprising bonus for us. We also obtain many new Kähler Einstein manifolds as well as Fano manifolds without Kähler Einstein metric of this type in the fifth section.

We finally treat the Kähler Einstein cases in the fifth section. We obtain many Kähler Einstein manifolds and Fano manifolds without Kähler Einstein metrics. This also gives what we promised in [Gu3] of the examples of positive integrals. We notice that all the Futaki invariants are zero in our case since the automorphism groups are semisimple. It turns out that our method is easier than that in [KS] for finding a Kähler-Einstein metrics since they depended on the zero of a Futaki invariant which might be very rare. Therefore, their method is more suitable for finding manifolds without Kähler Einstein metric. In our case, both the Kähler Einstein manifolds and manifolds without Kähler Einstein metrics are dense in the Zariski sense. It seems to me that I hardly see an example with vanishing generalized Futaki invariant. Also, most of our manifolds are Fano, it is not that simple in [KS]. Moreover, it is very easy to check that [KS] can be a Corollary of [Na] and does not involve much stability.

To understand better our examples M_n in our section 5, or $M_{n,1}$ in our Theorem 3, it is not difficult to see that M_n is a Q^n bundle over

$$Q^{n+1} = \{[z_0, z_1, \cdots, z_n, z_{n+1}, z_{n+2}] \in \mathbf{C}P^{n+2}|_{z_0^2+\cdots+z_{n+2}^2=0}\}.$$

Let $N_n = \mathbf{P}(T^*_{Q^{n+1}})$, then it is just our manifold with $F = \mathbf{C}P^n$. Therefore, N_n is never Fano (see [Ye], [LB]). To construct M_n we notice that by [KO p.590–593] there is a holomorphic conformal structure on Q^{n+1} with respect to the line bundle $N = \mathcal{O}(2)$. Therefore, $N_n = \mathbf{P}(T^*_{Q^{n+1}}) = \mathbf{P}(N \otimes T^*_{Q^{n+1}}) = \mathbf{P}(T_{Q^{n+1}})$. The exceptional divisor comes from the zero set of the corresponding holomorphic symmetric 2-tensor in N_n. Therefore, the vector bundle $\mathcal{O}(1) \oplus T_{Q^{n+1}}$ also has a conformal structure with respect to N. M_n is just the zero set of the corresponding holomorphic symmetric 2-tensor in $\mathbf{P}(\mathcal{O}(1) \oplus T_{Q^{n+1}})$. The branch double covering map $M_n \to N_n$ is just introduced by the projection $\mathcal{O}(1) \oplus T_{Q^{n+1}} \to T_{Q^{n+1}}$.

We only check some examples in the final section which also address and cover the examples in [PS] such that there will be a comparison and it will be easier for the readers. We also cover some of the cases in which $S = C_k$ and most of the major $Spin(7, \mathbf{C})$ cases. For example, we find that for any Fano manifolds with this $Spin(7, \mathbf{C})$ action and $F = \mathbf{C}P^7$, we call these manifolds the $SpinP$ Fano manifolds, there are always Kähler Einstein metrics. The picture in the case of C_k structure is quite different from that in the case of $SO(n+1, \mathbf{C})$ or $Spin(7, \mathbf{C})$ structures.

2. On Uniqueness

Now, we apply our arguments in [Gu3,7] to prove the uniqueness. We first consider the case in which $G = S = SO(n+1, \mathbf{C})$. The calcuatation in this section eventually led us to the solution of the Kähler-Einstein problem for the cohomogeneity two Fano manifolds in [Gu9]. We have a local coordinate

$$z = (1, z_1, \cdots, z_n).$$

Let $\theta_1 = \frac{|(z,z)|}{(z,\bar{z})}$ with the standard *real* inner product (,). As in [GC] and [Gu2,3], the metric is

$$\omega = m\omega_0 + \partial\bar{\partial}F = m\partial\bar{\partial}\log(z, \bar{z}) + \theta_1 f_{\theta_1}\partial\log\theta_1 \wedge \bar{\partial}\log\theta_1 + f\partial\bar{\partial}\log\theta_1$$

with $f = \theta_1 F_{\theta_1}$.

$$\partial\log\theta_1 = \frac{z_i dz_i}{(z_i, z_i)} - \frac{\bar{z}_i dz_i}{(z, \bar{z})}.$$

At p_t, we have that

$$\theta_1 = \frac{1-\tanh^2 s}{1+\tanh^2 s} = \frac{1-\theta}{1+\theta}, \quad \partial\log\theta_1 = \frac{2i\theta^{\frac{1}{2}}dz_1}{1-\theta^2},$$

$$\partial\bar{\partial}\log\theta_1 = -\frac{dz_1 \wedge d\bar{z}_1}{(1+\theta)^2}, \quad \theta_1\partial\log\theta_1 \wedge \bar{\partial}\log\theta_1 = \frac{4\theta dz_1 \wedge d\bar{z}_1}{(1-\theta)(1+\theta)^3}.$$

$$\theta_1' = \left(\frac{1-\tanh^2 s}{1+\tanh^2 s}\right)' = -\frac{4\theta^{\frac{1}{2}}(1-\theta)}{(1+\theta)^2},$$

$$\partial F = \theta_1 F_{\theta_1}\partial\log\theta_1 = -\frac{2\theta^{\frac{1}{2}}(1-\theta)}{(1+\theta)^2}F_{\theta_1}ds = \frac{1}{2}F'ds$$

$$f\partial\bar{\partial}\log\theta_1 = -F_{\theta_1}\frac{(1-\theta)dz_1 \wedge d\bar{z}_1}{(1+\theta)^3} = \frac{F'}{4\theta^{\frac{1}{2}}(1+\theta)}dz_1 \wedge d\bar{z}_1,$$

$$f_{\theta_1}\theta_1\partial\log\theta_1 \wedge \bar{\partial}\log\theta_1 = -\frac{\theta^{\frac{1}{2}}(\theta_1 F_{\theta_1})'dz_1 \wedge d\bar{z}_1}{(1-\theta^2)(1-\theta)} = \frac{\tanh s\left(\frac{1+\tanh^2 s}{4\tanh s}F'\right)'}{(1-\theta^2)(1-\theta)}dz_1 \wedge d\bar{z}_1$$

$$= \frac{1}{4(1-\theta^2)(1-\theta)}\left(2\tanh s(1-\theta)F' - \frac{(1+\theta)(1-\theta)}{\tanh s}F' + (1+\theta)F''\right)dz_1 \wedge d\bar{z}_1$$

$$= \left[\frac{F''}{4(1-\theta)^2} - \frac{F'}{4(1+\theta)\theta^{\frac{1}{2}}}\right]dz_1 \wedge d\bar{z}_1.$$

Therefore, the $dz_1 d\bar{z}_1$ term of ω is

$$\left[m \left(\frac{1-\theta}{1+\theta} \right)^2 + \frac{F''}{4} \right] (ds)^2 = -a'(ds)^2.$$

We have that

$$u' = m \frac{1}{\cosh^2 2s} + 4^{-1} F''$$

and therefore

$$u = 4^{-1}(F' + 2m \tanh 2s)$$

by all $u, F', \tanh 2s$ being odd functions. Let $\Gamma = F + m \log(\cosh 2s)$, then $4u = \Gamma'$. Similar to [Gu3 p.274], the geodesic equation is

$$\ddot{\Gamma} = \ddot{F} = \frac{1}{2}|d\dot{F}|^2 = (\partial \dot{F}, \bar{\partial} \dot{F}) = \frac{(\dot{F}')^2}{4u'} = \frac{(\dot{F}')^2}{\Gamma''},$$

that is,

$$\ddot{\Gamma}\Gamma'' = (\dot{\Gamma}')^2.$$

We obtain the smooth geodesics as in [Gu3,7] and so the uniqueness for $G = S = SO(n+1, \mathbf{C})$. The situation of G bigger than S is the same since $dF = F'ds$ lies only in the dual of T with respect to the Kähler metrics. The situation for $S = Spin(7, \mathbf{C})$ is similar.

If $S = C_k$, $F = Gr(2k, 2)$. As in [Gu1] we let (z, w) be the local coordinate with $z_1 = w_{k+1} = 1$ and $z_{k+1} = w_1 = 0$. We embed $Gr(2k, 2)$ into $\mathbf{C}P^{\frac{k(k-1)}{2}-1}$ by $(z, w) \to Z = z \wedge w$, that is $Z_{ij} = z_i w_j - z_j w_i$ $i < j$. Then, the symplectic form

$$\beta(z, w) = \sum_{i=1}^{k} (z_i w_{k+i} - z_{k+i} w_i)$$

becomes the linear form

$$\beta(Z) = \sum Z_{i(i+k)} = \beta_{ij} Z_{ij}.$$

One can actually check that the exceptional divisor is exactly the intersection of our manifold $Gr(2k, 2)$ with the hyperplane $\beta = 0$. We let

$$\theta_1 = \frac{\beta(Z)\beta(\bar{Z})}{(Z, \bar{Z})}.$$

As above, we have that

$$\partial \log \theta_1 = \frac{\beta_{ij} dZ_{ij}}{\beta(Z)} - \frac{\bar{Z}_{ij} dZ_{ij}}{(Z, \bar{Z})}.$$

At p_t, we have that

$$\theta_1 = \frac{4}{e^{4s} + e^{-4s} + 2} = (\cosh^2 2s)^{-1}, \quad \theta_1' = -\frac{4 \sinh 2s}{\cosh^3 2s},$$

$$4^{-1}F'' + \frac{2m}{\cosh^2 2s} = u'.$$

Therefore, we have that

$$u = 4^{-1}F' + m \tanh 2s.$$

Let

$$\Gamma = F + 2m \log\cosh 2s,$$

and as above we obtain the uniqueness. So, the same argument also works for the case $S = C_k$ and G bigger than S.

Therefore, we have the uniqueness for type I case. The proof of the type II case is more complicate and we need some new technic which will be done in [Gu4]. This also leads to the equivalence of the existence and the geodesic stability. The proof is a little bit tedious, we leave it to [Gu4] and the third part of this paper.

3. Higher Codimensional End Cases

Now, we consider the case of higher codimension of the end submanifold, i.e., the case in which $M = O \cup A$ with an open orbit O and a closed orbit A such that $\operatorname{codim} A > 1$.

In that case, the only difference is that the factor $a_i - u$ in [Gu1] might be zero, i.e., there are some $a_i = -l$. The argument before the Theorem 3 there tell us that we also have $a_{\rho,i} = -l_\rho$. We shall see later that this will also give a necessary and sufficient criterion with which a manifold with a hypersurface exceptional divisor can be blown down to an almost homogeneous manifold with a higher codimensional end. Let m be the codimension, then as before we have

$$l_\rho = -(n + m + m_2)c$$

if $F = \mathbf{C}P^n$ or $Gr(2k, 2)$ and

$$-(n - 1 + m)c$$

if $F = Q^n$. The only thing which we need to take care of in the estimate in [Gu1] is that we can use a function

$$V(u) = \int_0^u u^{n-1}(-l - u)^m du$$

in the place of V after (13) therein. Then every thing goes through except that we let

$$U_i(1 - \beta_i) = l^2 - \beta_i$$

to obtain the estimates for the global solutions.

We also have:

Theorem 1. *If the codimension m of the closed orbit is bigger than 1 and the second Betti number is 1, then the manifold M and the corresponding manifold $\tilde{M}$ with hypersurface end are both Fano. Moreover, if the corresponding manifold $\tilde{M}$ with hypersurface end has the fiber $F = Q^n$, then the corresponding manifold N with* *hypersurface end which is a double branch quotient of $\tilde{M}$ (with $F = \mathbf{C}P^n$) is Fano if $m > 2$ and has a nef anti-canonical line bundle but not Fano if $m = 2$.*

This is also true if M is Fano. It is also true for the type II manifolds. In that case, we have $l_\rho = -2M(k+m)$ for the affine case and $l_\rho = -2(2+m)$ for the case $S = A_1$ which is type II but not affine. One very interesting series of examples are the two series examples in [GC], [Gu2,3]. Since M_n there are the blowing ups of $\mathbf{C}P^n \times \mathbf{C}P^n$, therefore all M_n are Fano. We see that in those cases $m = n$, Therefore, N_n is Fano if $n > 2$ and has a nef anti-canonical line bundle but not Fano if $n = 2$. These manifolds appear as triads in many cases.

Now, by using [Nk] and [FN] we shall give a criterion with which a manifold with a hypersurface end can be blown down to an almost homogeneous manifold with a higher codimensional end.

From the proof of the Lemma 6 in [Gu1], we see that the eignvalues of the Ricci curvature of the exceptional divisor in the direction other than the 1-strings of roots with zero eigenvalues is represented as

$$Ric_s = n - 1 + m_2, c^{-1}a_{\rho,i} \pm (n - 1 + m_2).$$

In the same way, the eignvalues of the restriction of the Ricci curvature of the whole manifold in those direction is represented by

$$\tilde{Ric}_s = -l_\rho, c^{-1}a_{\rho,i} \mp l_\rho.$$

The curvature of the normal bundle of the exceptional divisor is just represented by their differences. Therefore, they are $2, \pm 2$ if $F = \mathbf{C}P^n$ (or $Gr(2k, 2)$) and $1, \pm 1$ if $F = Q^n$. We denote the first number by $D(F)$, i.e.,

$$D(\mathbf{C}P^n) = D(Gr(2k, 2)) = 2$$

and

$$D(Q^n) = 1.$$

By the Main Theorem of [Nk] and [FN] the manifold can be blown down if and only if (1) the exceptional divisor is a $C = \mathbf{C}P^{m-1}$ bundle over a manifold N and (2)

$$mC_1(N)|_C = -C_1(C)$$

. This means that at the directions of contractions we have that the eignvalue of the Ricci curvature of the $\mathbf{C}P^{m-1}$ should be the $-m$ multiple of that the curvature of N.

Now, if the exceptional divisor is a $\mathbf{C}P^{m-1}$ bundle over a homogeneous space $G/P_{1,\infty}$, then

$$P_{1,\infty}/P_\infty = S_{1,\infty}/(S_{1,\infty} \cap P_\infty) = \mathbf{C}P^{m-1}.$$

The root corresponding to a direction in the fiber has its root eigenvector in $P_{1,\infty}$ but not in P_∞. Therefore, the Ricci curvature of C comes from $c^{-1}a_{\rho,i} - Ric_s$, which is $m+1$ for the case $F = \mathbf{C}P^n$ and m for the case $F = Q^n$. Therefore, when $F = Q^n$ this is exactly the Fujiki-Nakano condition, we can always blow down if $a_{\rho,i} = c(n+m-1)$. If $F = \mathbf{C}P^n$ (or $Gr(2k,2)$), then we need $2m = m+1$, i.e., $m = 1$. Therefore, the exceptional divisor can be blown down if and only if $F = Q^n$ and $a_{\rho,i} = c(n+m-1)$. Similarly for the type II case. We have that the exceptional divisor can be blown down if and only if the manifold is affine in [Gu8] and $a_{\rho,s} = 2M(k+m) - B_\rho$.

Therefore, we have:

Theorem 2. *The hypersurface orbit can be contracted if and only if (1) $F = Q^n$ (or the manifold is affine), (2) the divisor has a $\mathbf{C}P^{m-1}$ bundle structure over a rational homogeneous space Q_1, (3) the corresponding roots in the fiber has $a_{\rho,i} = c(n+m-1)$ (or $a_{\rho,s} = 2M(k+m) - B_\rho$). Moreover, if the manifold is Kähler Einstein, so are the contraction and double branched quotient (if any of them exists).*

One series of examples which we already knew are the product P_n of two copies of $\mathbf{C}P^n$ and M_n, N_n in [GC], [Gu2,3]. Since M_n are Kähler Einstein, so are P_n and N_n.

In this way, we can also classify all the compact almost homogeneous manifolds of cohomogeneity one with a higher codimensional end and a semisimple Lie group action.

The process is: (1) Pick up a type of S with either type I or type II, actually either with $F = Q^n$ or affine. (2) Pick up a G with a parabolic subgroup P such that S is in the semisimple part. (3) Obtain the P_∞, it might be related to a system of root with a different order. (4) Find those $P_{1,\infty}$ containing P_∞ such that $P_{1,\infty}/P_\infty = \mathbf{C}P^{m-1}$. (5) Check the $a_{\rho,s}$. Or we do: (3)' Find those $a_{\rho,i} = c(n+m-1)$ (or $2M(k+m) - B_\rho$) with a positive integer m and multiplicity $m-1$. (4)' Check the possible corresponding $P_{1,\infty}$. We might do this in a different paper.

According to [Ss p.427], for the manifold $\mathbf{C}P^n$ the only posssible transitive group actions are: (1) A_n with the parabolic subgroup generated by simple roots other than α_1. (2) If $n = 2k+1$, then C_{k+1} acts transitively with the parabolic subgroup generated by simple roots other than α_1.

Here, however, we shall deal with the case of $G = S = G_2$ in [Gu5]. We have $B_\rho = -4$ and $l_\rho = -2(k+1) = -6$ since $k = 2$. $M = 1$. We have $Q(U) = (U - B_\rho^2)(9B_\rho^2 - U)$ and $a_{\rho,1} = -3B_\rho = 12 = 2(k+2) - B_\rho$. Therefore, there is a contraction along the root $\alpha_1 + \alpha_2$. It has a Kähler Einstein metric. Actually, afier we had found that the blow down variety is a manifold we found that it is actually a homogeneous manifold in [Gu5]. This also give a solution to the equation in [Gu5] (although it is not the one we need there).

It is also easy to check that the $M_{n,m}$ and $N_{n,m}$ in [Gu8 section 9] can be contracted.

4. The General Situation

Now, by [HS Corollary 4.4] every compact almost homogeneous manifold of cohomogeneity one has a double unbranched covering which is a product of the product of a torus T and an almost homogeneous manifold M with simply connected components and M is connected only when $F = Q^n$. Then, we can apply our arguments to the covering and reduce it to that on M. The double involution induces an involution on T and Q^n. It induces an involution on $H^{1,1}(T)$ which is nontrivial and an involution on $H^{1,1}(Q^n)$ which is nontrivial only if $n = 2$ in which $Q^2 = \mathbf{C}P^1 \times \mathbf{C}P^1$ and the involution just exchanges these two copies of $\mathbf{C}P^1$. The induced Kähler class on the covering is invariant under this involution. By the uniqueness of the extremal metrics, which we proved in this paper for the type I case and it also works for Q^2 with the invariant Kähler classes (see [Gu3], also for the cases with two ends there), and is enough for this situation, we have that the extremal metrics are invariant under this involution. Therefore, the extremal metrics on the covering come down to be extremal metrics on the quotient.

5. Kähler-Einstein Metrics

If the Kähler class is the Ricci class, we have $\alpha = \frac{u}{c}$, $l = l_\rho$,

$$m(u) = 2Q_1(u).$$

Therefore,

$$f_l = [n - 1 + m_2 - c^{-1}\sqrt{U}]U^{\frac{n-2}{2}}Q_1(U).$$

In this section, we shall check case by case on the type of the groups (S, G).

We say that a manifold is *nef* if the anti-canonical line bundle is nef. We say that a manifold is *Fano* if the anti-canonical line bundle is positive.

First, if $S = G = B_k$ $k \geq 2$, we have $n = 2k$, Q_1 is a constant, $l_\rho = -(n + 1)$ if $F = \mathbf{C}P^n$ or $-n$ if $F = Q^n$. Then,

$$\begin{aligned} C_n = \int_0^{l_\rho^2} f_{l_\rho} dU &= \int_0^{(n+1)^2} (n - 1 - \sqrt{U})U^{\frac{n-2}{2}} dU \\ = \frac{2(n-1)}{n}(n+1)^n &- 2(n+1)^n = -\frac{2(n+1)^n}{n} < 0 \end{aligned}$$

for the case in which $F = \mathbf{C}P^n$ and

$$C'_n = -\frac{n^{n-1}}{n+1} < 0$$

for the case in which $F = Q^n$. Therefore, there is a Kähler Einstein metric. Again, this is actually known since the manifolds are homogeneous. The same formula is true for $G = S = D_k, n = 2k - 1$.

Now, we consider the situation in which $G = B_{k+1}$ and $S = B_k$. We have

$$a_{\rho,1} = 1 + 2k = n + 1 = -l_\rho$$

if $F = \mathbf{C}P^n$ or

$$a_{\rho,l} = n + 1 = -l_\rho + 1$$

if $F = Q^n$. The manifolds are nef but not Fano (or are Fano). The same thing is true for $G = D_{k+1}$ and $S = D_k$. We only need to consider the case in which $F = Q^n$. The integral is

$$\begin{aligned} I_n = & \int_0^n (n-1-v)v^{n-1}((n+1)^2 - v^2)dv \\ = & \frac{n^{n-1}}{(n+2)(n+3)}(-(n+1)(n+2)(n+3) + 3n^2) \\ = & \frac{n^{n-1}}{(n+2)(n+3)}(2n^3 - 6n^2 - 11n - 6) > 0 \end{aligned}$$

if $n \geq 5$. Otherwise, $I_n < 0$. Therefore, the corresponding manifolds M_n are Kähler Einstein for $n \leq 4$. Others are non-Kähler Einstein Fano manifolds. Each M_n is a Q^n bundle over Q^{n+1}.

Now, we notice that the semisimple part of P_∞ is generated by $e_i \pm e_j$ (and e_i) $2 < i < j$. Let $P_{1,\infty}$ generated by P_∞ and the $sl(2)$ generated by $e_1 - e_2$, we have $P_{1,\infty}/P_\infty = \mathbf{C}P^1$. We can check that all the conditions in the Theorem 2 are satisfied. So, we can contract the exceptional divisor to a codimensional 2 submanifold (a codimensional 2 end). The corresponding integrals are:

$$\begin{aligned} I'_n = & \int_0^{n+1} (n-1-v)v^{n-1}((n+1)^2 - v^2)dv \\ = & C(n)(2(n-1)(n+3) - 2n(n+2)) = -6C(n) < 0 \end{aligned}$$

with a positive number $C(n)$. Therefore, the corresponding manifolds M'_n are always Kähler Einstein.

Now, we consider the general situation in which $S = SO(n+1, \mathbf{C})$ and $G = SO(2m+n+1, \mathbf{C})$, P be the smallest parabolic subgroup of G containing S as a simple factor. In this case,

$$Q_1(v) = \prod_{j=0}^{m-1} ((n+2j+1)^2 - v^2).$$

They are nef but not Fano (or are Fano). We have the integrals:

$$I_{n,m} = \int_0^n (n-1-v)v^{n-1} \prod_0^{m-1} ((n+2j+1)^2 - v^2)dv.$$

For $m = 2$ we can use Mathematica to check $I_{n,2}$ with

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Integrate[(n-1-v)v^(n-1)((n+m)^2 - v^2)((n+m+2)^2 - v^2), {v, 0, n}]
```

and have $I_{4,2} > 0$ but $I_{3,2} < 0$. In the same way, we can use Mathematica and check that $I_{3,m} < 0$ for $m \leq 7$ but $I_{3,8} > 0$.

Similarly, we also notice that we can blow down the hypersurface orbit as before by a $P_{1,\infty}$ which is generated by P_∞ and the $sl(2)$ corresponding to $e_m - e_{m+1}$. *Other $a_{\rho,s}$ do not lead to any contraction.* The integral is

$$I'_{n,m} = \int_0^{n+1} (n-1-v)v^{n-1}Q_1(v)dv.$$

We have

$$I'_{n,2} = C(n)(n^4 - 2n^3 - 49n^2 - 14n - 135)$$

with a positive $C(n)$. Therefore, $I'_{n,2} < 0$ if and only if $n \leq 8$. We also have $I'_{n,m} > 0$ for $(n,m) = (8,3), (7,3), (6,4), (5,6), (4,86)$ and < 0 for $(n,m) = (6,3), (5,5), (4,85), (3,800)$.

We therefore expect that $I'_{3,m} < 0$ always. To see this, we apply (24) in [Af p.197]:

$$\prod_{i=1}^{\infty}(1 - \frac{x^2}{i^2}) = \frac{\sin \pi x}{\pi x}.$$

Then, we let $x = 2t$ and only need to check the integral:

$$\int_0^2 (1-t)t^2 \sin \pi t(\pi t(1-t^2))^{-1} dt = (\pi)^{-1} \int_0^2 \frac{t \sin \pi t}{1+t} dt = -0.177175 < 0.$$

We denote the corresponding manifolds by $M_{n,m}$ and $M'_{n,m}$. Therefore, we have:

Theorem 3. *$M_{n,0}$ are homogeneous Kähler Einstein manifolds.*

$$M_{3,1}, M_{4,1}, M_{3,2}, M_{3,3}, M_{3,4}, M_{3,5}, M_{3,6}, M_{3,7}$$

are non homogeneous Kähler Einstein manifolds. Other $M_{n,m}$ are Fano manifolds without Kähler Einstein metric.

Theorem 4. *$M'_{3,m}$ and $M'_{n,1}$ are Kähler Einstein manifolds for all m and n. And $M'_{n,m}$ are Kähler Einstein for $(n,m) = (n,2)$ with $n \leq 8$ and $1 < m \leq N_n$ for $n \leq 6$ with $N_6 = 3, N_5 = 5, N_4 = 85$. Otherwise, $M'_{n,m}$ does not admit any Kähler Einstein metric.*

Next, we consider the case in which $S = SO(n+1, \mathbf{C}), G = SO(2m+n+1, \mathbf{C})$ and S_1 in the section 3 of [Gu1] is maximal. In this case, we have

$$Q_1(v) = ((n+m)^2 - v^2)^m$$

with $m > 1$. When $m > 1$, they are all Fano. The integral is

$$J_{n,m} = \int_0^{n+1} (n-1-v)v^{n-1}((n+m)^2 - v^2)^m dv$$

if $F = \mathbf{C}P^n$ or

$$J'_{n,m} = \int_0^{n} (n-1-v)v^{n-1}((n+m)^2 - v^2)^m dv$$

if $F = Q^n$, and

$$m^{-2m} J_{n,m} \to e^{2(n-1)} C_n < 0$$

or

$$m^{-2m} J'_{n,m} \to e^{2(n-1)} C'_n.$$

Therefore, $J_{n.m} < 0$ (or $J'_{n,m} < 0$) when m is big enough.

Also, we can compare the change rate of the factor

$$h(v) = ((n+m)^2 - v^2)^m$$

for different m and n. We let

$$t(m) = (\log h)' = m(\frac{1}{n+m+v} - \frac{1}{n+m-v}) = \frac{-2mv}{(n+m)^2 - v^2}.$$

Then,

$$t(m+1) - t(m) = \frac{-2v[n^2 - m(m+1) - v^2]}{((n+m)^2 - v^2)((n+m+1)^2 - v^2)} > 0$$

if $m \geq n$. Therefore, if $J_{n,m} \leq 0$ with $m \geq n$ then $J_{n,m+1} < 0$. The same thing is also true for $J'_{n,m}$.

Now, we can use Mathematica to check $J'_{n,n}$ with

```
Integrate[(n-1-v) v^(n-1) ((2n)^2 -v^2 )^(n) , {v, 0, n}]
```

we get $J'_{n,n} < 0$ when $n = 3, 4$ but $J'_{5,5} > 0$.

We then use Mathematica to check $J'_{5,10}$ with

```
Integrate[(5-1-v)v^4 (225 -v^2 )^(10) , {v, 0, 5}]
```

and have $J'_{5,10} > 0$.

Similarly, by using Mathematica we have $J'_{5,20} < 0$ and $J'_{5,m} > 0$ if $2 \leq m \leq 13$ and $J'_{5,14} < 0$. Therefore, when $m \geq 14$ $J'_{5,m} < 0$, otherwise $J'_{5,m} > 0$. We can also check that $J_{5,m} < 0$ for $m > 1$.

Similarly, we use Mathematica to check $J'_{4,m}$ for $m = 2, 3$ and $J'_{3,m}$ for $m = 2$. We find that all of them < 0. Therefore, $J'_{3,k}, J'_{4,k} < 0$ if $2 \leq k$. So are $J_{3,k}, J_{4,k}$.

In general, we expect that if

$$m > \frac{n^2(n-1)}{e},$$

then $J'_{n,m} < 0$. For example, if $n = 6$ we expect $J'_{6,60} < 0$. We check it with Mathematica and get

$$J'_{6,60}, J'_{6,30}, J'_{6,27} < 0, \quad J'_{6,20}, J'_{6,25}, J'_{6,26} > 0.$$

In the same way, we find $J'_{6,k} > 0$ for $2 \leq k \leq 5$. Therefore, $J'_{6,k} > 0$ if $2 \leq 26$, otherwise $J'_{6,k} < 0$. We can also check that $J_{6,k} < 0$ for all $k > 1$. One might expect that $J_{n,m} < 0$ always. However, we have $J_{n,n} > 0$ for $n = 101, 51, 26, 25$, $J_{n,n} < 0$ for $n = 11, 21, 24$. Then, one might expect that $J_{n,m} < 0$ for $n \leq 24$. However, we have $J_{n,m} > 0$ for $(n,m) = (24,12), (18,9), (16,8)$ etc.. One can check that $J_{n,m} < 0$ for $n \leq 15$. And $J_{16,m} > 0$ if $5 \leq m \leq 8$, otherwise $J_{16,m} < 0$. We can also check that $J_{n,2} < 0$. And

$$\begin{aligned} J_{n,3} &= C(n)(8n^8 + 6n^7 - 1534n^6 - 16019n^5 \\ &- \quad 75163n^4 - 194786n^3 - 263486n^2 - 216981n - 76545) < 0 \end{aligned}$$

if and only if $n \leq 17$. We can check that $J_{17,m} < 0$ if and only if $m \leq 3$ or ≥ 11.

Similarly, we see that if $F = Q^n$ we can blow down the manifold at the hypersurface orbit and obtain a manifold with a $m+1$ codimensional end. Let

$$K_{i,j} = \int_0^{n+m} v^i((n+m)^2 - v^2)^j dx,$$

then we have that our integral is

$$J''_{n,m} = (n-1)K_{n-1,m} - K_{n,m}.$$

We also have that

$$\begin{aligned} K_{i,j} &= \int_0^{n+m} v^i((n+m)^2 - v^2)^j dv \\ &= \frac{1}{i+1}\int ((n+m)^2 - v^2)^j d(v^{i+1})|_0^{n+m} = \frac{2j}{i+1}K_{i+2,j-1}. \end{aligned}$$

Therefore,

$$K_{n-1,m} = \frac{2^m m!(n+m)^{n+2m}}{\prod_{i=0}^m (n+2i)}, \quad K_{n,m} = \frac{2^m m!(n+m)^{n+2m+1}}{\prod_{i=0}^m (n+2i+1)}.$$

Therefore, we only need to prove that

$$1 - a_{n,m} = 1 - \frac{(n+m)\prod_{i=0}^m (n+2i)}{\prod_{i=0}^{m+1}(n-1+2i)} < 0,$$

i.e., $a_{n,m} > 1$. Now,

$$\frac{a_{n,m+1}}{a_{n,m}} = \frac{(n+m+1)(n+2m+2)}{(n+m)(n+2m+3)} > 1,$$

we have that

$$a_{n,m} > a_{n,0} = \frac{n^2}{n^2-1} > 1$$

as desired. Therefore, we have $J''_{n,m} < 0$ always.

Therefore, we obtain that if we denote the corresponding manifolds by $N_{n,m}$ (or $N'_{n.m}, N''_{n,m}$), then:

Theorem 5. *$N_{n,m}$ $3 \leq n \leq 15$, and $N'_{3,m}, N'_{4,m}$, $N''_{n,m}$ admit Kähler-Einstein metric for all $m > 1$. $N'_{5,m}$ admit Kähler-Einstein metric if and only if $m > 13$. $N'_{6,m}$ admit Kähler Einstein metric if and only if $m > 26$. $N_{16,m}$ admit Kähler Einstein metric if and only if $m > 8$ or $2 \leq m < 5$. $N_{17,m}$ admit Kähler Einstein metric if and only if $m > 10$ or $2 \leq m < 4$. $N_{n,2}$ admit Kähler Einstein metric for any n. $N_{n,3}$ admit Kähler Einstein metric if and only if $n \leq 17$. In general, $N_{n,m}$ (or $N'_{n,m}$) admit Kähler-Einstein metric when m big enough, i.e., there is an integer $N(n)$ (or $N'(n)$) such that if $m > N(n)$ (or $> N'(n)$) then $N_{n,m}$ (or $N'_{n,m}$) admit Kähler-Einstein metric. Moreover, if $m \geq n$ and $N_{n,m}$ (or $N'_{n,m}$) admit a Kähler-Einstein metric, so does $N_{n,m+1}$ (or so does $N'_{n,m}$).*

One more observation: If $S = SO(n+1, \mathbf{C})$ and $G = SO(2m+n+1, \mathbf{C})$, then any this kind of manifold with a higher codimensional end is a G equivariant $N''_{n,m}$ bundle over a projective rational homogeneous manifold. One might conjecture that this is in general true for the type I case. We shall see that this is true if S is simple.

We also observe that all the Kähler Einstein manifolds $N''_{n,m}$ are actually homogeneous with the group $G_1 = SO(2m+n+2, \mathbf{C})$. It is the rational homogeneous manifold G_1/P_1 where P_1 is the parabolic subgroup generated by all the simple roots other than α_{m+1}. It is the orbit of G_1 on

$$B = \begin{bmatrix} 1 & i & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & i & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & & \cdots & & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & i & 0 & \cdots & 0 & 0 \end{bmatrix}^T_{(m+1)\times(2m+2+n)}$$

which represents the complex $m+1$ dimensional space generated by the column vectors of B. The G action comes from the classical G action on the first $2m+n+1$ rows and the trivial action on the last row. The close orbit is just GB and the open orbit is GB_1 where

$$B_1 = \begin{bmatrix} 1 & i & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & i & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & & \cdots & & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & i \end{bmatrix}^T_{(m+1)\times(2m+2+n)}.$$

Now, we consider the cases in which $S = D_k$ $5 \le k \le 7$ and $G = E_{k+1}$. And we denote the corresponding manifolds by EQ_{k+1} with $F = Q^{2k-1}$ or EP_{k+1} for $F = \mathbf{C}P^{2k-1}$. We consider the case $G = E_8$ first. We have the simple roots:

$$\alpha_i = e_{i+1} - e_{i+2}, i \le 6; \alpha_7 = e_7 + e_8, \alpha_8 = \frac{1}{2}(e_1 + e_8 - \sum_{2 \le i \le 7} e_i).$$

The subgroups D_s is generated by $\alpha_i, 7-s < i \le 7$ and E_r is generated by α_i $8-r < i \le 8$. E_8 has roots of the form (A) $\pm e_i \pm e_j$ and (B)

$$\frac{1}{2}\sum_{i=1}^{8}(-1)^{v_i} e_i$$

with $\sum v_i$ even. In all of these cases, we have $H = ie_{9-s}$. We can write E_8 as E_{m+s} with $m+s=8$. Then

$$B_{\rho,1} = 14 + 32 = 46, \quad B_{\rho,i} = 16 - l_1 - l_2 \ 1 < i \le m, \quad B_{\rho,l} = 0 \ m < l$$

where l_1, l_2 determinate the $A_{l_2 - l_1}$ factor in S_1 which contains i, i.e., $l_1 \le i \le l_2$. If $s < 7$, we let P_0 be the proper parabolic subgroup containning D_7. Then for any parabolic subgroup P, P_0/H is a manifold with $G = D_7$ and $S = D_s$. Therefore, the manifold can be contracted and the contracted manifold is an E_8 equivariant

$N''_{2s-1,m}$ bundle over a projective rational homogeneous manifold. Since P_0 is the maximal parabolic subgroup, every possible contraction is inside P_0.

In the case of $s = 7$, we have that

$$Q_1(x) = ((46)^2 - x^2)^{33}.$$

Then, we have the integral

$$Q_8 = \int_0^{13} (12 - x)x^{12} Q_1(x)dx > 0$$

for the case $F = Q^{13}$ and

$$P_8 = \int_0^{14} (12 - x)x^{12} Q_1(x)dx < 0$$

for the case $F = \mathbf{C}P^{13}$. We also see that the hypersurface orbit can not be blown down, otherwise $P_{1,\infty}$ should have all the type (B) root vectors.

If $G = E_7$, we write it as E_{m+s} with $m + s = 7$. It is a subgroup generated by α_i with $2 \leq i \leq 8$. Then, E_7 has positive roots $e_i \pm e_j$ with $3 \leq i < j$, $e_1 - e_2$ and

$$\frac{1}{2}(e_1 - e_2 + \sum_{3 \leq i \leq 8} (-1)^{v_i} e_i)$$

with $\sum_{3 \leq i \leq 8} v_i$ odd. If $s < 6$, we can also see as above that the manifold can be contracted to an E_7 equivariant $N''_{2s-1,m}$ bundle over a rational homogeneous manifold.

If $S = D_6$, then

$$B_{\rho,1} = 1 + 16 = 17, B_{\rho,2} = -17, B_{\rho,l} = 0 \ 2 < l.$$

Therefore,

$$Q_1(x) = ((34)^2 - x^2)^{16}.$$

We have that

$$Q_7 = \int_0^{11} (10 - x)x^{10}((34)^2 - x^2)^{16} dx > 0$$

if $F = Q^{11}$,

$$P_7 = \int_0^{12} (10 - x)x^{10}((34)^2 - x^2)^{16} dx < 0$$

if $F = \mathbf{C}P^{11}$. The hypersurface orbit does not contract.

If $G = E_6$, we write $E_6 = E_{m+s}$ with $m+s = 6$. E_6 has positive roots $e_i \pm e_j, 4 \leq i \leq j$ and

$$\frac{1}{2}(e_1 - e_2 - e_3 + \sum_{4 \leq i \leq 8} (-1)^{v_i} e_i)$$

with $\sum_{4 \leq i \leq 8}$ even. If $s < 5$, the manifold can be contracted to a $N''_{2s-1,m}$ fibration.

If $S = D_5$, we have that

$$B_{\rho,1} = 8, B_{\rho,2} = B_{\rho,3} = -8, B_{\rho,l} = 0 \ 4 \leq l.$$

Therefore,

$$Q_1(x) = ((24)^2 - x^2)^8.$$

We have that

$$Q_6 = \int_0^9 (8-x)x^8((24)^2 - x^2)^8 dx > 0$$

for the case with $F = Q^9$, and

$$P_6 = \int_0^{10} (8-x)x^8((24)^2 - x^2)^8 dx < 0$$

if $F = \mathbf{C}P^9$. The hypersurface orbit does not contract.

Theorem 6. *EP_m admit Kähler Einstein metrics but EQ_m are Fano manifolds without Kähler-Einstein metric.*

Now, we consider the case in which $S = D_3$, then $S = A_3$. If $G = A_{m+3+k}$ and S is generated by $e_{m+1} - e_{m+2}$, $e_{m+2} - e_{m+3}$, $e_{m+3} - e_{m+4}$, we have that these three roots should correspond to $f_2 + f_3$, $f_1 - f_2$, $f_2 - f_3$ in the root system of D_3. Therefore,

$$2f_2 = e_{m+1} - e_{m+2} + e_{m+3} - e_{m+4}$$

and

$$\begin{aligned} f_1 &= e_{m+2} - e_{m+3} + \frac{1}{2}(e_{m+1} - e_{m+2} + e_{m+3} - e_{m+4}) \\ &= \frac{1}{2}(e_{m+1} + e_{m+2} - e_{m+3} - e_{m+4}) = -iH. \end{aligned}$$

If the parabolic subgroup P is maximal, we have that

$$Q_1(u) = ((2(4+m)^2 - v^2)^{2m}((2(4+k))^2 - v^2)^{2k}.$$

The corresponding integral is

$$C_{m,k} = \int_0^6 (4-v)v^4((2(4+m))^2 - v^2)^{2m}((2(4+k))^2 - v^2)^{2k} dv$$

if $F = \mathbf{C}P^5$ or

$$C'_{m,k} = \int_0^5 (4-v)v^4((2(4+m))^2 - v^2)^{2m}((2(4+k))^2 - v^2)^{2k} dv$$

if $F = Q^5$. We also see that if $k \geq 4$, then $C_{m,k} < 0$ (or $C'_{m,k} < 0$) implies $C_{m,k+1} < 0$ (or $C'_{m,k+1} < 0$). Using Mathematica we check that $C'_{0,k} < 0$ for $k \leq 4$. Therefore, $C'_{0,k} < 0$ for all k. So are $C_{0,k}$.

$$C'_{4,4} > 0, C'_{8,8} > 0, C'_{16,16} < 0.$$

In the same way we check that $C'_{m,m} > 0$ if $1 \leq m \leq 15$. We also check that

$$C'_{16,15}, C'_{4,3}, C'_{4,2}, C'_{4,1}, C'_{3,2}, C'_{3,1}, C'_{2,1} > 0.$$

Therefore, $C'_{m,k} > 0$ if $1 \leq m, k \leq 15$. Now, we also check that

$$C'_{15,17} < 0, C'_{14,17} > 0, C'_{14,18} < 0, C'_{13,18} > 0, C'_{13,19} < 0.$$

Then, we have that

$$C'_{12,19}, C'_{12,20} > 0, C'_{12,21} < 0.$$

We then check that

$$C'_{1,40}, C'_{1,25}, C'_{1,22} < 0, C'_{1,20} > 0, C'_{1,21} < 0.$$

Therefore, $C'_{1,k} > 0$ if $k \leq 20$, otherwise $C'_{1,k} < 0$. Then, we check that $C'_{2,k} > 0$ for $k = 21, 41$ and $C'_{2,k} < 0$ if $k = 42$. Now, we check $C'_{7,k} < 0$ for $k = 41, 38, 37$ and > 0 if $k = 31, 36$. Next, we check that $C'_{4,k} > 0$ if $k = 40, 50, 55$ and < 0 if $k = 80, 60, 57, 56$. And

$$C'_{3,56}, C'_{5,50}, C'_{6,43} < 0, \quad C'_{3,55}, C'_{5,49}, C'_{6,42} > 0.$$

Then,

$$C'_{10,25}, C'_{11,23}, C'_{9,28}, C'_{8,32} < 0, \quad C'_{10,24}, C'_{11,22}, C'_{9,27}, C'_{8,31} > 0.$$

This is quite complicate. However, we can check that $C_{m,k} < 0$ for $1 \leq m, k \leq 4$, so $C_{m,k} < 0$ for all k.

Therefore, if we denote the corresponding manifolds by $CM_{m,k}$ or $CM'_{m,k}$ and we always assume that $m \leq k$. Then, we have:

Theorem 7. *$CM_{m,k}$ are always Kähler Einstein. $CM'_{m,k}$ are Kähler Einstein if and only if $m, k \geq 16$ or $m < 16, k \geq k_m$, where*

$$k_m = 0, 21, 42, 56.56, 50, 43, 37, 32, 28, 25, 23, 21, 19, 18, 17$$

for $m = 0, 1, \cdots, 15$.

We also notice that the manifold $C'_{m,k}$ can not be contracted. Actually, by some argument in [Gu6], we can prove that if $S = D_3$ and the manifold can be contracted then the contracted manifold is a $N''_{3,m}$ bundle over a projective rational homogeneous space.

Now, we consider the case in which $S = D_2$, then $S = A_1 \times A_1$. If

$$G = A_{m_1+1+k_1} \times A_{m_2+1+k_1}$$

and S is generated by $e_{m_1+1} - e_{m_1+2}$ and $e'_{m_2+1} - e'_{m_2+2}$, then these two roots should correspond to $f_1 + f_2$ and $f_1 - f_2$ in the root system of D_2. Therefore,

$$f_1 = \frac{1}{2}(e_{m_1+1} - e_{m_1+2} + e'_{m_2+1} - e'_{m_2+2}) = -iH.$$

If the parabolic subgroup P is maximal, we have that

$$\begin{aligned} Q_1(u) &= ((2(2+m_1))^2 - v^2)^{m_1}((2(2+k_1))^2 - v^2)^{k_1} \\ &\cdot \quad ((2(2+m_2))^2 - v^2)^{m_2}((2(2+k_2))^2 - v^2)^{k_2}. \end{aligned}$$

The integrals are:

$$C_{m_1,k_1,m_2,k_2} = \int_0^4 (2-v)v^2 Q_1(v)dv \quad \text{or} \quad C'_{m_1,k_1,m_2,k_2} = \int_0^3 (2-v)v^2 Q_1(v)dv.$$

We also have that if $m_1 \geq 2$, then $C_{m_1,k_1,m_2,k_2} < 0$ (or $C'_{m_1,k_1,m_2,k_2} < 0$) implies $C_{m_1+1,k_1,m_2,k_2} < 0$ (or $C_{m_1+1,k_1,m_2,k_2} < 0$). We can use Mathematica and check that

$$C'_{2,2,2,2}, C'_{2,2,2,1}, C'_{2,2,1,1}, C'_{2,1,1,1}, C'_{1,1,1,1} < 0.$$

Therefore, by comparison we have that

$$C_{m_1,k_1,m_2,k_2} < 0 \quad \text{and} \quad C'_{m_1,k_1,m_2,k_2} < 0.$$

If we denote the corresponding manifolds by $CM_{i,j;k,l}$ or $CM'_{i,j;k,l}$. We notice that the hypersurface orbit of CM'_{m_1,k_1,m_2,k_2} can not be contracted.

We then have:

Theorem 8. *All $CM_{i,j;k,l}$ and $CM'_{i,j;k,l}$ admit Kähler-Einstein metrics.*

We now leave other examples for the case in which $S = SO(n+1, \mathbf{C})$ to the readers since Theorems 4, 5, 5, 7, 8 give us enough new Kähler-Einstein manifolds and Theorems 3, 4 give us a large class of Fano manifolds which do not admit any Kähler-Einstein metric.

Now, we consider the cases in which $S = C_k$, we have that

$$n = 2(2k) - 4 = 4(k-1),$$

$$l_\rho = -(n+1+m_2) = -(4k-4+1+3) = -4k.$$

If $G = C_k$ also, then the integral is

$$\begin{aligned} \int_0^{4k} (n-1+m_2-x)x^{n-1}dx &= \int_0^{4k} (4k-2-x)x^{4k-5}dx \\ &= -\frac{2(2k-3)(4k)^{4(k-1)}}{4(k-1)(4k-3)} < 0 \end{aligned}$$

for $k > 1$. Again, this is known since it is homogeneous.

If $G = C_{k+1}$, we have $H = i(e_2 - e_3)$ and

$$B_{\rho,1} = -(2+2k) = -2(k+1), B_{\rho,l} = 0 \;\; 2 \leq l.$$

There are four positive roots which will produce a_i. They are $e_1 \pm e_i$ $i = 2, 3$. We have, for example,

$$e_1 + e_2 = e_1 + \frac{1}{2}(e_2 - e_3) + \frac{1}{2}(e_2 + e_3).$$

They come in 2 pairs. Therefore,

$$Q_1 = ((4(k+1))^2 - x^2)^2.$$

The integral is

$$\begin{aligned}&\int_0^{4k} (4k-2-x)x^{4k-5}((4(k+1))^2 - x^2)^2 dx \\ = \;& C(k)(-112k^5 + 108k^4 + 61k^3 + 52k^2 - 22k - 5) < 0\end{aligned}$$

with a positive constant $C(k)$ for $k > 1$. Therefore, the corresponding manifolds GrP_k are Kähler Einstein.

Theorem 9. *The Fano manifolds GrP_k are all Kähler Einstein with $k > 1$.*

One might wonder what happens when $G = C_{m+k}$. We have that

$$B_{\rho,i} = -(2 - l_1 - l_2 + 2(m+k)) \; i \leq m; B_{\rho,l} = 0 \; m < l.$$

If P is minimal, we have the integral

$$G_{k,m} = \int_0^{4k} (4k-2-x)x^{4k-5} \prod_1^m ((4(k+m))^2 - x^2)^{2m} dx.$$

We have $G_{k,2} < 0$ for $k = 1000, 100, 10, 5, 4$ and $G_{k,2} > 0$ for $k = 2, 3$. Similarly, if P is maximal we have integral

$$H_{k,m} = \int_0^{4k} (4k-2-x)x^{4k-5}((2(m+1+2k))^2 - x^2)^{2m} dx.$$

We have $H_{k,2k-1} > 0$ for $k = 100, 10, 5, 3, 2$. In both cases we have quite different picture from those in the case of $S = SO(n+1, \mathbf{C})$.

Now, we consider the cases in which $S = Spin(7, \mathbf{C})$. When $G = S$ we have the manifold $\mathbf{C}P^7$ or Q^7 and the integrals are proportional to those of $G = S = SO(8, \mathbf{C})$.

If $G = Spin(9, \mathbf{C})$, the Lie algebra is B_4. H is proportional to $e_2 + e_3 + e_4$. The numbers a_i come from the positive roots $e_1 \pm e_i$. They come in 3 pairs. We have, for example,

$$H_{e_1+e_2} = ie_1 + \frac{i}{3}(e_2 + e_3 + e_4) + \frac{i}{3}(2e_2 - e_3 - e_4) = ie_1 + \frac{1}{\sqrt{3}}H + \frac{1}{2}(2e_2 - e_3 - e_4).$$

Therefore,

$$Q_1(u) = ((\sqrt{3} \cdot 7)^2 - u^2)^3.$$

By replaceing u with cx we have the integral

$$\int_0^7 (6-x)x^6((14)^2 - x^2)^3 dx > 0$$

if $F = Q^7$, and

$$\int_0^8 (6-x)x^6((14)^2 - x^2)^3 dx < 0$$

if $F = \mathbf{C}P^7$.

If $G = Spin(2m+7)$, we have that

$$B_{\rho,i} = -l_1 - l_2 + 2m + 7 \ i \leq m, B_{\rho,l} = 0 \ m < l.$$

If P is minimal,

$$Q_1(x) = \prod_0^{m-1} ((2(7+2i))^2 - x^2)^3.$$

All of them are Fano. When $F = Q^7$, they are not Kähler Einstein and the hypersurface orbit can not be contracted. If $F = \mathbf{C}P^7$, we denote the manifolds by $SpinM_m$. The integrals are

$$\int_0^8 (6-x)x^6 Q_1(x) dx.$$

It is < 0 if $m \leq 100$. One might expect that it is < 0 always. Therefore, we need to check the integral

$$\int_0^8 (6-x)x^6 \prod_0^\infty (1 - \frac{x^2}{(2(7+2i))^2})^3 dx \leq 0.$$

To do this, we need understand the infinite product. By the formula (24) in [Af p.197], we have that

$$2\cos\frac{\pi x}{2} = \frac{\sin \pi x}{\sin \frac{\pi x}{2}} = 2\prod_0^\infty (1 - \frac{x^2}{(1+2i)^2}).$$

Therefore,

$$\prod_0^\infty (1 - \frac{x^2}{(2(7+2i))^2})^3 = \left(\frac{4 \cdot 36 \cdot 100 \cos\frac{\pi x}{4}}{(4-x^2)(36-x^2)(100-x^2)}\right)^3.$$

We only need to prove that

$$\int_0^8 (6-x)x^6 \frac{\cos^3 \frac{\pi x}{4} dx}{(4-x^2)^3(36-x^2)^3(100-x^2)^3} \leq 0.$$

Again, we can check with a computer that this integral is

$$-1.90429 \times 10^{-10} < 0.$$

Let me explain how I got this number. One might go to the website and find a numerical integrator engine. Type in the function and use different rules. To avoid the singularity one might choose a step (or terms) 9998. We can also use Mathematica to find three integrals of type

```
NIntegrate[(6-x)x^6 (Cos[Pi x/4])^3 (4-x^2 )^(-3)
        (36-x^2 )^(-3) (100-x^2 )^(-3) , {x, a, b}]
```

for

$$(a,b) = (0, 1.999), (2.001, 5.999), (6.001, 8)$$

and we get

$$2.145 \times 10^{-11}, 2.654^{-9}, -2.866 \times 10^{-9}.$$

By summing them together we obtain a similar negative number. We intentionally avoid the number 2 and 6 since the integrand has two removable singularities there and the computer does not know how to handle it.

Now, it is very easy to check that $(\log(m^2 - x^2))'$ is a concave (down) function of m. If $G = B_{m+3}$ and $F = \mathbf{C}P^7$ then for any possible parabolic subgroup P containing S, we have that

$$Q_1(x) = \prod_{i=1}^{k}((2(6 + \sum_{j=1}^{i} m_j))^2 - x^2)^{3m_i}.$$

Therefore, by comparing the change rate we see that the manifolds are Kähler Einstein.

Theorem 10. *$SpinM_m$ is Kähler Einstein for all m. Moreover, whenever $G = D_{m+3}$ and $F = \mathbf{C}P^7$ the $Spin(7)$ manifolds are Kähler Einstein.*

If P is maximal,

$$Q_1(x) = ((2(m + 6))^2 - x^2)^{3m}.$$

We can check that for the case $F = \mathbf{C}P^7$

$$\int_0^8 (6 - x)x^6 Q_1(x)dx < 0$$

for $2 \leq m \leq 6$. Therefore, all of them are Kähler Einstein for $m > 1$. We denote the corresponding manifolds by $SpinP_m$. We can also check that

$$\int_0^7 (6 - x)x^6 Q_1(x)dx < 0$$

if $m > 30$. We denote the corresponding manifolds by $SpinQ_m$. We also notice that the hypersurface orbit can not be contracted if $m \neq 4$. Even if $m = 4$, the hypersurface can not be contracted. And this is also true for any parabolic subgroup P.

Theorem 11. *$SpinP_m$ are all Kähler Einstein. $SpinQ_m$ is Kähler Einstein if and only if $m > 30$.*

The first part of Theorem 11 also follows from the second part of Theorem 10.

If $G = F_4$, then G has roots $\pm e_i$ $i \leq 4$, $\pm e_i \pm e_j$ and

$$\frac{1}{2}(\sum_{1\leq i\leq 4}(-1)^{v_i}e_i).$$

F_4 has simple roots

$$e_2 - e_3, e_3 - e_4, e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4).$$

S has simple roots $e_2 - e_3, e_3 - e_4, e_4$. $H = \frac{i}{\sqrt{3}}(e_2 + e_3 + e_4)$. We have that

$$B_1 = -11,\ B_i = 0\ i > 1.$$

a_i come from $e_1 \pm e_i$ $i > 1$ and

$$\frac{1}{2}(e_1 + \sum_{2\leq i\leq 4}(-1)^{v_i}e_i).$$

$$Q_1(x) = ((22)^2 - x^2)^6((22)^2 - 9x^2).$$

Therefore, if $F = \mathbf{C}P^7$, it is not Fano. We denote it by $FSpinP$. If $F = Q^7$, it is Kähler Einstein. We denote it by $FSpinQ$. The hypersurface orbit can not be contracted.

When $G = F_4$, we can also consider the case of $S = B_3$ (or B_2, in this case the manifold can be contracted to an $N''_{4,1}$ bundle). In this case the difference comes from $H = ie_2$. And therefore,

$$Q_1(x) = ((22)^2 - x^2)^5.$$

The hypersurface orbit does not contract.

Theorem 12. *$FSpinP$ is not Fano, while $FSpinQ$ is Kähler Einstein.*

In particular, all the $Spin(7, \mathbf{C})$ Fano manifolds with $F = \mathbf{C}P^7$ admit Kähler Einstein metrics. We call them the $SpinP$ Fano manifolds. All the $Spin(7, \mathbf{C})$ manifolds are Fano except the $FSpinP$ in the Theorem 12.

Therefore, we have many more Kähler Einstein manifolds in Theorems 9, 10, 11, 12. By our methods the reader could get many more.

We also see that if $S = Spin(7, \mathbf{C})$ the hypersurface orbit can not be contracted. We have

Theorem 13. *The type I manifold can be contracted only if $S = SO(n + 1, \mathbf{C})$ and $F = Q^n$.*

One can also check easily that if $S = B_2$ and $G = C_n$ the manifold does not contract. Therefore, if the manifold can be contracted for $S = SO(n + 1, \mathbf{C})$ with $n > 3$ then the contracted manifold is a $N''_{n,m}$ fiber bundle over a projective rational homogeneous manifold. One can actually see that $N''_{n,m}$ are just the so called isotropic Grassmannian manifolds. Therefore, we have:

Theorem 14. *Whenever S is simple, any type I manifold which can be contracted is a blow up of an isotropic grassmannian fiber bundle over a projective rational homogeneous manifold.*

We have one more observation: As in section 3, we see that l_ρ is related to the canonical class of the fiber F, we denote it by $K(F)$. Then $K(\mathbf{C}P^n) = -n - 1, K(Gr(2k,2)) = -4k, K(Q^n) = -n$ and they are negative. We have:

Theorem 15. *A type I cohomogeneity one Fano manifold with an exceptional orbit of codimension m admits a Kähler-Einstein metric if and only if the integral*

$$\int_0^{-K(F)+m-1} (-K(F) - D(F) - x)Q(x)dx$$

is negative, where $D(F)$ is defined in the section 3.

This also makes more precise for the last statement in the Theorem 2.

Also, as an application, by considering the canonical circle bundle we obtain Sasakian manifolds with and without Sasakian-Einstein metrics (see [BG Theorem 2.4 (iv)], also [Kb], [WZ]).

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